How Mathematics Could Make Sense To Lots Of People And Why It Does Not: The Case Against Educology

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Preface

The reason I have long *wanted* to write this book stems from four observations:

1. The abysmal level in mathematics of most students, from the lower elementary grades up to and including the first two years of college as witnessed, among others, the rather disgraceful place of the U.S.A. on the Trends In Mathematics and Science Survey (TIMSS)[1] in general and by the high number of Students In Remedial Mathematics (SIRM) in particular. Moreover, not only do few SIRM pass these "remedial" courses, but, of those who do, only a vanishingly small number pass any mathematics course thereafter.

2. The fact that this has been going on for at least thirty some years, the bewildering number and variety of the "approaches" that were tried notwithstanding. While it is not clear that things have really gotten worse, they certainly haven't gotten any better at any time.

3. The belief in most of those involved, SIRM as well as we, college teachers, that this somehow is the *result* of the students' own inadequacy. A case of "blaming the victims" if you wish. And, if not *their* fault, then it is their elementary and/or secondary teachers whom we blame. Of course it never occurs to us to wonder about *who* taught said elementary and/or secondary teachers. Presumably, that would be cutting it too close to the bone.

4. The fact that, for anyone with a genuine interest in learning mathematics and minimal honesty, current textbooks at the elementary, secondary, and first two years undergraduate levels, from "remedial arithmetic[2]" to calculus, are impossible to read without getting a case of screaming meemies. Here, the situation has steadily deteriorated during that same thirty-some-year period.

The reason I finally *wrote* the book was the feeling that some awareness might be beginning to appear of the *real* cause of the disaster, namely the lack of a Coherent View of Mathematics (CVM) in those purporting to teach mathematics and, consequently, their failure to impart a Profound Understanding of Fundamental Mathematics (PUFM) to their students[3]. And this refers as much to Full Professors of Mathematics Education as to elementary and/or secondary school teachers, if not more.

Essentially, I take CVM to refer to an architecture of the contents that permits a presentation whose logical flow makes sense to SIRM and from which *they* will be able to derive a PUFM. In the CVM *this* book advocates, students ought eventually to arrive at the point where they will be able to analyze "any" given function and construct a qualitative graph that embodies its features, in other words, to Differential Calculus. In the course of achieving this goal, though, nothing ought to be justified by recourse to the usual "you will see later what this is good for". From the point of view of SIRM, this kind of "investment" is mere "pie in the sky". Moreover, SIRM ought not to have to depend on

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[1] Formerly known as Third International Mathematics and Science Survey. By itself, the change is revealing.

[2] Educologists now refer to this as "pre-algebra".

[3] I did not coin either phrase but was very glad to find out that they had been, apparently by Ma.

something they saw way back because this has very bad connotations for them. Thus, at least for me, the main problem in creating a CVM was to design a content architecture that would ensure that everything would arrive on an "as needed basis" like on a Japanese assembly line. No parts inventory. [4]

But, rather than giving a detailed description of the features CVM and PUFM ought to have, see Appendix II for several instances, I chose to develop a text embodying these features and thus to write what is essentially the bare bones of a *textbook* and so one with neither “worked out examples” nor “exercises”. (It is, however, interspersed with “Inflammatory” Footnotes to the particular attention of Educologists.) It is this core/scenario/script, entitled *Mathematics for Candide* after Voltaire’s character, which makes up the bulk of the book you have in your hands. The actual textbook(s), I shall leave to others to write.

Historically, mathematical texts were written for a tiny, self-selected minority, namely those willing, for whatever reason, to put the time and effort necessary to read them and, thus, writers paid no attention whatsoever to their readers’ woes. Moreover and *ipso facto*, this tiny minority got to be considered as having a special talent rather than, say, merely being obstinate, or having the linguistic ability to make sense of a priori hermetic texts, or, after mathematics became, of late, the supreme tool for screening students, being endowed with a photographic memory.

If I wrote *Mathematics for Candide* for SIRM, it is primarily because, for all these years, they have been *some* of my students in the strict sense of “remedial” and *all* of my students in only a slightly more relaxed sense of the term. Indeed, the traditional exposition works only for people ready and capable to "believe" or, at least, to defer their doubts until better times, and, in the meantime, capable of memorizing what they don’t understand. In other words, the people, in Jacques Brel's words, with "l'assurance de ceux qui ont un papa—the self-confidence of those who have a daddy"! SIRM, though, have little reason to think that society has their best interest at heart and, therefore, no reason to "trust" their teachers—which is what it comes down to when all is said and done in the traditional exposition. This a priori suspicion is one of many reasons I find SIRM by far the more interesting students.

As for you, my improbable reader, I do *not* assume that you have *any* “particular mathematical knowledge” either[5]. However, I *do* hope that, Candide-like, you will not take anything for granted and will question anything you find here. Better a question ultimately left unanswered than a question not asked or an issue not examined: To use John Holt’s immortal words, better be “question oriented” than “answer oriented”.

To summarize: Either the great majority of SIRM are congenital cretins (Would you believe Johnny can’t even add?) or the Teaching Industry is *not* doing what it should (Teaching them at least how to add)[6]. But, should there be a CVM that helps SIRM acquire a PUFM, an interesting question then would arise as to *why* the Teaching Industry should be sticking to what is clearly not working. It is worth noting in this regard that, from a probabilistic point of view, for the great majority of SIRM to be congenital cretins is not even remotely possible. Yet, the great majority of them fail.

Beyond that, since Elementary and Secondary Schools Mathematics do not work *for essentially the same reasons* as Remedial Mathematics, if a CVM should be found to work in Two-Year Colleges, it, or suitably modified versions, ought to work in Elementary and Secondary Schools.

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[4] The way this particular CVM was developed was by iterating back from the ultimate goal, to find and develop what was needed and only what was needed, and forth to make sure that, at any moment, the stuff, as given, was going to make sense to the students on its own merits and only on its own merit.

[5] This is a joke referring to Nicolas Bourbaki’s famous opening of *Elements of Mathematics*: “The treatise takes mathematics at its beginning and gives complete demonstrations. Reading it therefore does not presuppose, in principle, any particular mathematical knowledge“. The treatise, though, is generally *not* recommended for beginners and the sentence ends with “but only some familiarity with mathematical reasoning and some power of abstraction.” A colossal understatement!

[6] Of course, all the current dummming down makes it a lot easier for the teachers and all that recourse to extraneous gadgetry makes them feel “up to date” as well as, possibly, “supporters of the economy”.

Of course, even though, as mentioned above, *Mathematics for Candide* is by no means intended as a textbook, it is sure to raise mathematical hackles or, at the very least, many questions. However, other than in the occasional Inflammatory Footnotes, I shall not attempt to justify it preemptively. Instead, I would hope that *Mathematics for Candide* might serve as a basis for discussions about emendations, explorations, alternatives, etc and, above all, that it might eventually lead to the "open source" development, *on the web*, of a CVM embodied in a complete, *student-friendly*, downloadable textbook[7].

The book ends with six Appendices the last one of which is *The Case Against Educology*. That the case thus comes almost as an afterthought is because: (*a*) for there to be a case at all, there first had to be shown that things need not be the way they are, and (*b*) should *Mathematics for Candide* provide a convincing example of what PUFM a CVM can offer to "just plain folks", that is to people with little time for, and even less of an *a priori* interest in, mathematics, then the case will *ipso facto* be an open and shut one. In the meantime, the reader might consider the following

Fiction?

Imagine, if you will, a culture in which **Biochemistry** is as advanced as it is in our own culture but where the problem of its applications to the health of the population is of no interest to any but an extremely few biochemists and where a **Pill Industry** produces, more or less randomly, an infinite variety of **pills** for **pill-pushers** to prescribe as remedies for various ailments.

Now, some fifty years ago, a few of these very few biochemists had been able to convince the Pill Industry that biochemical theory could provide the *composition* of pills appropriate for specific ailments. This was called the New Pill Theory. In its haste to capitalize on it, though, the Pill Industry grabbed the theory and ran away with it. While, in itself, this need not have been a disaster, the pill-pushers who didn't have a clue as to what the theory was or implied had no choice but to dispense the new pills exactly as they had always done the old ones, namely without any regard for their composition. The results were of course quite predictable: whatever empirically obtained remedies there might have been with the old pills, the New Pill Theory wasn't working at all. The composition of the pills had thus been definitively "shown" to be entirely irrelevant to the health of the people and the New Pill Theory was abandoned.

On the other hand, the Pill Industry could not just say "Oops, we goofed" and return to the *status quo ante* of the Old Pills. In fact, they had discovered the advantages of a steady an income based on a steady introduction of *new* pills and in fact started to introduce them at an ever faster pace. Under these conditions, it is not surprising that a new science, **pillology**, should have been born, completely independent of Biochemistry, whose practitioners would theorize endlessly as to what would and wouldn't work and in *any* case formulate new pills for the Pill Industry to sell.

Since the one thing **pillologists** could *not* take into account was the *composition* of the pills, they turned to a bewildering variety of "possible" factors. There were pillologists who thought that it was the *look* of the pills that mattered, shape, color, texture, etc. Others, though, thought that it was the *way* it was administered that was important: before or after or between meals, with proteins or with carbohydrates, etc. Others yet thought that it was the *frequency* that was determining: there were endless debates on the merits of a once-a-day pill versus the one-every-hour pill with regard to *retention*. Still others presented papers at conferences showing that it was the *size* of pills that was crucial and while some held that *evanescent* pills were better at cheating the body's natural defenses, others preferred to squash these with the use of mega pills. Eventually, it even became fashionable (and lucrative for its

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[7] Unfortunately, while *programmers* realized a long time ago that they cannot work alone, it is only recently that *research* mathematicians have begun to work jointly other than occasionally. But, ironically, in expository matters where they seem capable of only minor tweaks on standard expositions, they are absolutely opposed to any collaboration. This seems to be particularly the case with *teaching* mathematicians.

makers) to hold that an *electronic* dispenser was an absolute necessity. In any case, an immense advantage of pillology was that it left pillologists completely free to hold *any* opinion they wished and that made *them* feel good.

Of course, Pillology was not accepted as a *science* by other scientists. Then pillologists decided that they wanted full scientist status and thought of turning their opinions into "scientific theses". For that purpose, though, they had to "support" their opinions in *some* manner. Of course, since they lacked the barest inkling of Biochemistry, pillologists were quite incapable even to *dream* of *causal chains* that would indicate what would work for what ailment. Instead, they turned to *statistical studies* that they could publish in unbounded numbers, a great career advantage, by the way, for the practitioners of the new science. But since their knowledge of Statistics was on a par with their knowledge of Biochemistry, they could blissfully ignore the inconvenient issue of statistical significance and would more often than not come up with stunning conclusions such as "46.27% of 53 patients who had been given big blue square tablets once a day did better than average while only 38.81% of 47 patients who had been given small yellow round capsules every hour did".

And, last but certainly not least, there was always the ultimate excuse for all involved, namely that, after all, people are *not* created equally healthy: Some are born to live long lives while some are born to be ill and to die young. In other words, Biochemists, the pill industry, pill-pushers and pillologists alike, could all blame the victims and rest content.

Agenda

There has been considerable talk and research about what *learning* is all about. A significant amount of this "research" has been rather a blessing to Educologists in that it has allowed them to conclude that we really do not know enough about learning to do anything other than to "show and tell and drill" a small number of "topics" carefully selected for the "target audience"[8].

Yet we *do* know that learning depends at least on how much *sense* things make. It has been pointed out, for instance, that it takes French children longer than Italian children to learn how to read and British children even longer. Is then one to infer nationality-based levels of intelligence among children, or of efficiency among their teachers, or should one merely note that: (a) in Italian, when you can speak it you can write it and, conversely, when you can write it, you can speak it while, (b) in French, even though when you can write it you can speak it, to be able to speak it does not guarantee at all that you can write it and finally, (c) in English, you can neither write it when you can speak it nor speak it when you can write it as with Bernard Shaw's famous 'ghoti' to be read 'fish' with 'gh' read 'fff' as in 'enough', 'o' read 'eee' as in 'women' and 'ti' read 'shhh' as in 'nation'. Thus, the amount of time needed by children to learn how to read would seem rather to correlate with the extent to which what is written and what is spoken are *systematically* related. It would certainly appear to be easier to retain things that make sense, at least in the way that they are organized, than when they appear to be random and have to be "just" memorized. It is certainly easier to find a book in a library organized according to *some* system than by "just" memorizing its place in an unorganized library.

I would thus argue that, precisely because of our by now total reliance on senseless "show and tell and drill", what we don't know about learning hardly matters anymore and that the root cause of the much-lamented state of mathematics in the U. S. A. is that students have been relentlessly driven to the absolute identification of learning with memorizing and thus to complete stupefaction. As for the much-invoked Math Anxiety, it is merely that which derives universally from being forced to operate without understanding what one is doing. In other words, the so-called Math Anxiety has little to do with mathematics and is merely a by-product of the way we teach it, an ironical result of Educology's "it isn't *what* you teach, it is *how* you teach it".

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[8] Educologists call them CLO, etc.

A very difficult question though is how much of a PUFM is necessary for “just plain folks”. The question of course begs another question: “necessary for *what*?” On a short-term basis, one might respond “enough to be able to operate on a daily basis at an acceptable level”. But an even more important question is what would be necessary on a more general, longer-term basis. The following, from an article by Colin McGinn, *Homage to Education*, in the August 16, 1990 issue of the London Review of Books, addresses this almost completely neglected issue. The article is a review of a book of, and of a book about, R. G. Collingwood. The part that is directly relevant is where McGinn "*spell*[*s*] *in* [*his*] *own way what* [*he*] *thinks Collingwood is getting at here.*"

> "*Democratic States are constitutively committed to ensuring and furthering the intellectual health of the citizens who compose them: indeed, they are only possible at all if people reach a certain cognitive level..* [...]. *Democracy and education (in the widest sense) are thus as conceptually inseparable as individual rational action and knowledge of the world.* [...]." But what is education? "*Plainly, it involves the transmission of knowledge from teacher to taught. But what exactly is knowledge?* [...]. [*It*] *is true justified belief that has been arrived at by rational means.*" [...]. *Thus the norms governing political action incorporate or embed norms appropriate to rational belief formation.* [...]. *The educational system of schools and universities is one central element in this cognitive health service* [...].
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> "*It would be a mistake to suppose that the educational duties of democratic state extended only to political education, leaving other kinds to their own devices.* [...]. *How do we bring about the cognitive health required by democratic government? A basic requirement is to cultivate in the populace a respect for intellectual values, an intolerance of intellectual vices or shortcomings.* [...]. *The forces of cretinisation are, and have always been, the biggest threat to the success of democracy as a way of allocating political power: this is the fundamental conceptual truth, as well as a lamentable fact of history.*"
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> However, "*people do not really like the truth; they feel coerced by reason, bullied by fact. In a certain sense, this is not irrational, since a commitment to believe only what is true implies a willingness to detach your beliefs from your desires.* [...]. *Truth limits your freedom, in a way, because it reduces your belief-options; it is quite capable of forcing your mind to go against its natural inclination. This, I suspect, is the root psychological cause of the relativistic view of truth, for that view gives me license to believe whatever it pleases me to believe.* [...]. ***One of the central aims of education, as a preparation for political democracy, should be to enable people to get on better terms with reason – to learn to live with the truth***."(Emphasis added.)

So what better way to do so than first to help “just plain folks” reason their way through *mathematics* which, in any case, is a lot simpler than the real world it represents? However, the question about “how much of a PUFM” in turn leads us to the question: a Profound Understanding of “*What* mathematics?”

Now, even though Remedial Mathematics traditionally covers only Arithmetic and Basic Algebra, *Mathematics for Candide* goes all the way up to the Fundamental Theorem of the Calculus:

SYSTEMATIC ARITHMETIC

- Accounting For Bundles of Money
- Addition Leads to Large Bundles
- From Combinations To Number-Phrases
- Accounting For Goods: Counting and Adding Counts.
- Systematic Notations
- The Matter of Ever-Larger Denominators.
- Subtraction Of Counts Doesn’t Always Work!
- Accounting For Transactions: Addition; Subtraction of Entries Always Works!
- Multiplication Of Counts of *Certain* Goods
- Evaluating Counts With Co-counts: Values.
- Evaluating Entries With Co-entries: Gains and Losses.
- Division Of Counts. With and Without Exchange Facilities.

The Matter of Ever-Smaller Denominators
Fractions
Multiplicative) Powers
Roots

Real Numbers

Real Real Numbers = Decimal Numbers
Rulers: Their Extent, Scale and Resolution
Windows and Grids

First Degree Algebra

Language
The Four Operations
Comparisons Involving *One* Kind Of Goods and Money
Comparisons Involving *Two* Kinds of Goods and Money
Double Comparisons Involving *Two* Kinds of Goods and Money

Polynomial Algebra

Language
Powers of x
(Laurent) Polynomials
The Four Operations
Factoring.

Functions Specified By An Input-Output Rule

The First Gauges: Power Functions
Constant Functions
Affine Functions
Quadratic Functions
Cubic Functions
Polynomial Functions of degree >3
Rational Functions
Local Continuity as Approximability by a Constant Function
Local Differentiability as Approximability by an Affine Function
Derivatives
Sided-Limits

Functions Specified By An Equation

Root Functions
Transcendental Functions
Exponential Functions
Winding Functions
Composite Functions

The above being obviously and extravagantly unconventional, I adduce the following in its favor:

1. That *Mathematics for Candide* goes *beyond* traditional Remedial Mathematics is because:

- **a.** Traditional Remedial Mathematics is essentially a trap as, even in the best case, the passage to traditional courses is so abrupt as to be all but impossible for SIRM. Which is why very few of them ever attempt anything beyond Algebra and, of those rare brave souls who try Precalculus, only a vanishingly small number succeeds. Almost none even *dream* of Calculus, the “royal road to science and technology”. See Appendix VII.
- **b.** There is no good reason for such a trap as, *up to and including* Differential Calculus, also know as “the mathematics of change”, there is really nothing inherently difficult. To demonstrate this, in fact, is the main goal of *Mathematics for Candide*. As I shall emphasize in the Inflammatory Notes, it is only the Teaching Industry that makes mathematics difficult.

2. That *Mathematics for Candide* essentially *ends* with *Differential* Calculus and does *not* include the customary *Integral* Calculus is because the latter is truly a much more technical subject and thus genuinely more difficult. Furthermore, it is of use and interest mostly in specialized subjects such as Physics, Electronics, etc. In fact, the natural continuation of Differential Calculus and the one needed for the comprehension of evolutionary systems is Dynamical Systems.

3. That Dynamical Systems, a specialized version of Differential Equations created a century ago by Henri Poincaré, is *not* included in *Mathematics for Candide* is mostly due to the fact that a computer is *needed* to produce the non-trivial *phase portraits* that are at the heart of Dynamical Systems. Moreover, there *are* a few nice books on the subject that would be fairly accessible to readers of this book.

4. Why Linear Algebra should appear in Appendix I rather than, say, immediately after Systems of Two First Degree Equations, or even earlier, is a more complicated issue. See Appendix I. In any case, the relevant "hooks" are built-in in *Mathematics for Candide* in a manner that leads directly to a "Profound Understanding of Fundamental *Linear Algebra*".

5. That the Multi-Dimensional Calculus does not appear at all owes more to my own physical limitations than to anything else but *Mathematics for Candide* is readily compatible with large parts of Robert Osserman's *Two-Dimensional Calculus* which, therefore, ought to be fairly accessible to readers of this book.

6. Starting with sets, groups and their graphs rather than with numbers would have made great sense but was out of the question with SIRM for whom time is of the essence. Moreover, even these days, anything smacking of the New Math is still anathema. Yet, Z. P. Dienes did *show* how much such a start facilitates much of what we will be doing here, unfortunately without the benefit of such a preparation.

7. While similarly out of the question with SIRM, *starting* with even a *modest* introduction to linguistic issues in a Model Theoretic setting would have been very helpful. See Appendix III. Still, making the difference between objects and their names is *essential* to a PUFM so, for the better or for the worse, I chose a middle road that consists in making the distinction clearly but on an "as needed" basis and without making a big deal of it.

8. And, last but not least, my point was to demonstrate the *cumulative effects* of a CVM and the advantages in the long run of *constructing an organized ensemble* as opposed to just *presenting an amorphous ensemble of topics*. *Mathematics for Candide* intends to show how carefully lying out Arithmetic provides a lot of what is needed to understand Differential Calculus. Which, by the way, is the reason an *integrated sequence* is pragmatically necessary for SIRM to have a realistic chance to make it[9]. See Appendix IV. In the same spirit, even better results would presumably be achieved with an *integrated program*. See Appendix V.

If I don't know *for a fact* that this particular CVM should work, it is because: (*a*) I was able to develop the arithmetic and algebra parts only in stand-alone courses and (*b*) I was able to observe the *cumulative effect* only in a 2-semester, 8-credit precalculus-differential calculus sequence that I once developed with the support of an NSF calculus grant. However, the reason I know *logically* that the CVM that *Mathematics for Candide* illustrates should work is because:

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[9] While I believe anyone can make it, not everyone has the time and/or energy to: Some have families to take care of, some have job(s), many have both, etc. Very few have the freedom to study unhampered by the external world. See Appendix V though.

i. It relies on a *story line*[10] to organize methodically the development of the contents from the point of view of SIRM who cannot endure being jerked around from topic to topic and need to know where they are, what is going on and where they are going.

ii. It gives SIRM a plausible *framework* for them to organize their own thinking and to work things out by themselves without, it should go without saying, having first to be "shown how to do it".

iii. It presents mathematics as a faithful, if necessarily limited, *representation* of the real world and thus (re)connects it with the students' familiar experience. In particular, the necessary *procedures* are developed from generic "concrete" *examples* but without (of course) taking advantage of their specificity so as to be general enough while continuing to make concrete, familiar sense[11].

iv. It uses *language* that helps SIRM figure out, by themselves, how to operate. Even if it might sometimes appear a bit idiosyncratic, this is still a light enough price to pay for SIRM to acquire a PUFM.

v. It regards *proofs* as an argumentation to convince SIRM, much in the way that attorneys make their cases to juries[12], that things mathematical are the way they are only because common sense and the real world want it that way.

On the other hand, no "applications" will be found in this text and for good reason: In spite of what the Teaching Industry would like us to believe, mathematics up to and including Calculus doesn't have any that aren't contrived. It is only when we want to find out how things *are* at any moment in terms of how they *change* from moment to moment and how they were at some initial moment, in other words when we investigate Initial Value Problems, that we begin to have *real life* applications[13]. As for Linear Algebra, which would seem to have almost immediate applications, see Appendix I.

In any case, even if they truly existed, *afterthought* applications *to* the real world would be less than a very poor substitute for *starting from* the real world.

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[10] A story line is not necessarily a path in which each issue is dealt with *once and for all*. While this is something to be whished for because it makes for tidiness, strict adherence to such a path can have very unpleasant consequences. Moreover, even for a well-defined body of mathematics, a story line certainly need not be unique. For instance, in Boolean algebra, we can start with $\cap$ and $\cup$ and define $<$ and $>$ as well as vice-versa.

[11] As opposed to this Platonism, a Formalist viewpoint would require a detachment hardly to be expected from SIRM. The approach of Z. P. Dienes on the other hand stands somewhere in-between and indirectly inspired much of this CVM.

[12] In particular, the amount of "evidence" to be produced at any time by either side depends on the other side. After more than half of this book was written, I found out that, if the goal is to *convince* SIRM, then Stephen Toulmin's "courtroom-based" models of argumentation may be more than merely "relevant".

[13] Boundary Value problems, in which we want to find out how things are at any moment in term of how they change from place to place as well as from moment to moment and of how they were initially at the boundary of that region, are of course also *real life* problems but technically *much* more complicated.