initial collection
attach
resulting collection

Chapter 8

Additions & Translations

Up until now, we were concerned only with the development of symbolic systems to *represent* collections of real-world items. We will now take our first step towards the use of symbolic systems to *simulate* actions on collections of real-world items.

We investigate the *second* of the three fundamental processes involving two collections. But while the result of a comparison of two collections is a sentence that describes a relationship, the result of an action is a number-phrase that represents a collection.

We will introduce the procedure in the case of *basic* collections using *basic counting* number-phrases and we will then extend the procedure to *extended* collections using *decimal-number phrases*.

augend addend

8.1 Addition To

We begin begin by describing the real-world *process* that we want to simulate with a paper procedure.

1. Given an **initial collection** of real-world items, we want to **attach** another collection of same kind real-world items to get a **resulting collection**.

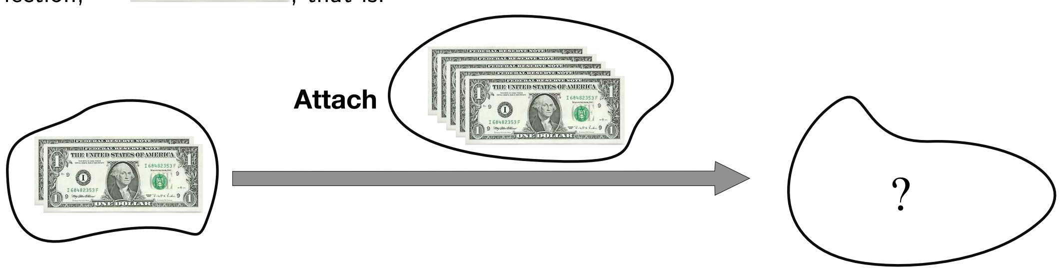
To get the resulting collection:

i. We set the attached collection next to the initial collection

ii. We move the collection to be attached against the initial collection

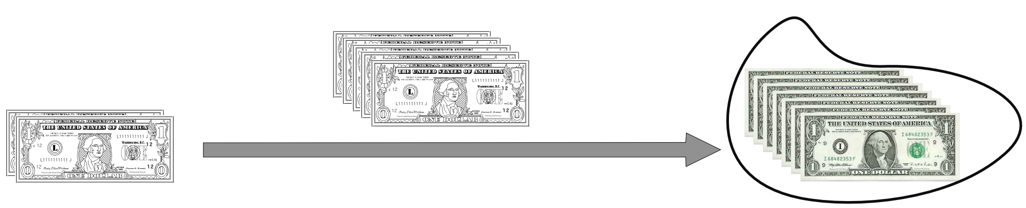
iii. The *resulting collection* is made of all the items in the initial collection as well as the items of the attached collection.

Example 1. Given Jack's collection, , we want to attach Jill's collection, , that is:



i. We set Jill's collection next to Jack's collection:

ii. We move Jill's collection against Jack's collection



operator
+
specifying-phrase
bar

iii. The resulting collection is: 

2. We now develop the paper *procedure* that simulates the above real-world *process*, that is the procedure that:

i. Given the number-phrase that represents the initial collection ,

ii. Given the number-phrase that represents the collection to be attached

iii. Gives the number-phrase of the resulting collection .

a. Since the items in the two collections are of the same kind, the denominator of the resulting number-phrase will be the same as the denominator common to the two collections.

b. We get the numerator of the resulting number-phrase as follows:

- **i.** We take as start-digit the numerator of the initial number-phrase
- **ii.** We take as end-digit the numerator of the second number-phrase
- **iii.** We count up from the start

3.

8.2 Language For Addition

In order to represent on paper the result of an *operation*, such as attaching a second collection to a first collection, we need to expand again our mathematical language.

1. The first thing we need is a symbol, called **operator**, to represent the *operation*. In the case of *attaching* a second collection to a first collection, we will of course use the *operator* $+$, read as "plus". **NOTE.** It should be stated right away, though, that this use of the symbol $+$ is only one among very many different uses of the symbol $+$ and that this will create in turn many difficulties. We shall deal with these difficulties one at a time, as we encounter each new use of the symbol $+$.

2. Given two collections represented by number-phrases, we will represent f attaching the second collection to the first by a **specifying-phrase** that we write as follows:

i. We write the first number phrase:

first number phrase

ii. We write the symbol for adding:

first number phrase $\underline{+\qquad\qquad\qquad\qquad}$

iii. We write the second number-phrase over the **bar**:

first number phrase $\underline{+ \text{ second number phrase}}$

identify
identification-sentence
arrowhead

Altogether then, the *specifying-phrase* that corresponds to attaching to a first collection a second collection is:

$$\text{first number phrase } \underline{\;+ \text{ second number phrase}\;}$$

Example 2. In order to say that we want to add to the first number-phrase 5 **Washingtons** the second number-phrase 3 **Washingtons** we write the *specifying phrase*:

$$5 \textbf{ Washingtons } \underline{\;+\, 3 \textbf{ Washingtons}\;}$$

3. This language gives us a lot of flexibility:

- *Before* we count the result of attaching a second collection to a first collection, we can already represent the *result* by using a *specifying-phrase*.
- *After* we have found the result of attaching a second collection to a first collection, we can represent the result by a *number-phrase*.
- Altogether, to summarize the whole *process*, we can **identify** the *specifying phrase* with an **identification-sentence** which we write as follows
  i. We write the specifying phrase
  ii. We lengthen the *bar* with an **arrowhead**
  iii. We write the number-phrase that represents the result.

Example 3.

i. *Before* we attach to Jack's  Jill's , we can already represent the *result* by the *specifying-phrase*

$$6 \textbf{ Washingtons } \underline{\;+\, 3 \textbf{ Washingtons}\;}$$

ii. *After* we have found that the result of attaching to Jack's  Jill's  is  we can represent the result by

$$9 \textbf{ Washingtons}$$

iii. Altogether, to summarize the whole *process* with an identification-sentence we lengthen the bar with an arrowhead and we write the number-phrase that represents the result of the attachment.

$$6 \textbf{ Washingtons} \xrightarrow{+\, 3 \textbf{ Washingtons}} 9 \textbf{ Washingtons}$$

4. Usually, though, we will not write things this way and we only did it above to show how the mathematical language represented the reality. As usual, some of it "goes without saying":

- In the *specifying phrase*, the *bar* goes without saying

- In the identification sentence, the arrowhead is replaced by the symbol = = addition

Example 4. Instead of writing the specifying phrase

$$6 \textsf{ Washingtons} \xrightarrow{+\,3 \textsf{ Washingtons}}$$

we shall write

$$6 \textsf{ Washingtons} + 3 \textsf{ Washingtons}$$

and instead of writing the identification sentence

$$6 \textsf{ Washingtons} \xrightarrow{+\,3 \textsf{ Washingtons}} 9 \textsf{ Washingtons}$$

we shall write

$$6 \textsf{ Washingtons} + 3 \textsf{ Washingtons} = 9 \textsf{ Washingtons}$$

8.3 Procedure For Adding A Number-Phrase

Given two collections, the paper procedure that gives (the *numerator* of) the number-phrase that represents the result of attaching the second collection to the first collection is called **addition** and depends on whether the two number-phrases are *basic counting* number-phrases or *decimal* number-phrases.

In order to *add* a second *basic* collection to a first *basic* collection, we count *up* from the numerator of the first collection by a length equal to the numerator of the second collection.

There are then two cases depending on whether, when we count up from the numerator of the first number-phrase by a length equal to the second numerator, we need to end up *past* 9 or not.

- If we do not need to end up *past* 9, the result of the addition is just the end-digit.

  Example 5. To add Jill's 5 **Washingtons** to Jack's 3 **Washingtons**, that is, to *identify* the *specifying-phrase*

  $$3 \textsf{ Washingtons} + 5 \textsf{ Washingtons}$$

  i. Starting from 3, we count *up* by a length equal to 5:

  $$\xrightarrow{4,\ 5,\ 6,\ 7,\ 8}$$

  ii. The end-digit is 8.
  iii. We write the *identification-sentence*:

  $$3 \textsf{ Washingtons} + 5 \textsf{ Washingtons} = 8 \textsf{ Washingtons}$$

- If we need to end up *past* 9, then we must bundle and change TEN of the items.

EXAMPLE 6. To add Jill's 8 **Washingtons** to Jack's 5 **Washingtons**, that is to identify the *specifying-phrase*

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons}$$

i. Starting from 5, we count *up* by a length equal to 8 but stop after TEN:

$$\xrightarrow{4,\ 5,\ 6,\ 7,\ 8,\ 9,\ \text{TEN}}$$

ii. We bundle TEN **Washingtons** and change for a 1 **DEKAWashingtons** and count the rest

$$\xrightarrow{1,\ 2,\ 3}$$

iii. We write the *identification-sentence*:

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 1 \textbf{ DEKAWashingtons} \ \& \ 3 \textbf{ Washingtons}$$

which of course we could also write

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 1.3 \textbf{ DEKAWashingtons}$$

or

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons} = 13. \textbf{ Washingtons}$$

or ...

Actually, we usually do the latter a bit differently, that is, instead of *basic* counting up just past 9, interrupt ourselves to bundle and change, and then start *basic* counting again, it is easier to use some *extended* counting and count all the way and *then* bundle and change what we must and count the rest.

EXAMPLE 7. To add Jill's 8 **Washingtons** to Jack's 5 **Washingtons**, that is to identify the *specifying-phrase*

$$5 \textbf{ Washingtons} + 8 \textbf{ Washingtons}$$

i. We count up from 5 by a length equal to 8 using *extended-counting*:

$$\xrightarrow{4,\ 5,\ 6,\ 7,\ 8,\ 9,\ \text{TEN, ELEVEN, TWELVE, THIRTEEN}}$$

ii. Then we say that we can't *write* THIRTEEN **Washingtons** since we only have digits up to 9 so that we should bundle TEN **Washingtons** and change for a 1 **DEKAWashingtons** with 3 **Washingtons** left

graph
ruler, quantitative
number lines
tick-marks
labeled
in order
equally spaced

iii. We write the *identification-sentence*:

5 **Washingtons** + 8 **Washingtons** = 1 **DEKAWashingtons** & 3 **Washingtons**

that is, using a decimal number-phrase,

5 **Washingtons** + 8 **Washingtons** = 1.3 **DEKAWashingtons**

or, if we prefer,

5 **Washingtons** + 8 **Washingtons** = 13. **Washingtons**

or . . .

The difference is of course not a great one. It is only that we said that we would deal with *extended* collections using only *basic* counting and indeed, in the second example, we fudged a bit when, after having counted to THIRTEEN, we said that after bundling and changing we had 3 left: officially, we cannot do so since we have not yet introduced *subtraction*.

However, if the first example illustrates the fact that, when needed, we can indeed do things "cleanly", the second example illustrates the fact that, while we are usually not willing to count *very* far, a bit of (extended) counting beyond 9 makes life easier.

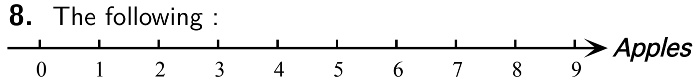
8.4 Quantitative Rulers

Very often, we will want to visualize things by **graphing** number-phrases and we will do that with **quantitative rulers** which are indeed essentially what goes in the real world by the name of "ruler".

NOTE. In high school, *quantitative rulers* usually go by the name of **number lines**, a term we will not use in this text[1].

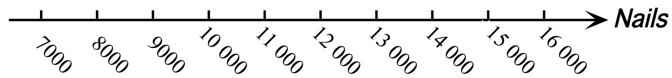
1. More precisely, the **tick-marks** on a *quantitative ruler* must be **labeled**, **in order**, and **equally spaced**.

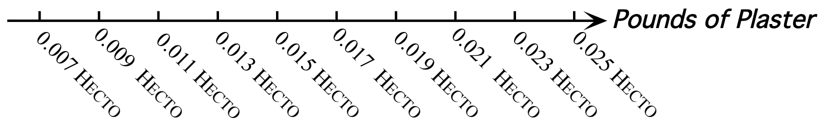
EXAMPLE 8. The following :



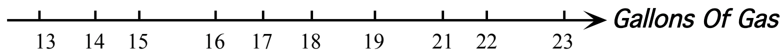
[1]It would be interesting to trace the origin of this remarkably un-enlightening term. But then, it is probably due to Educologists' well known craving for the esoteric.

extent
curly brackets
{ }
resolution
bounded numbers
encompass





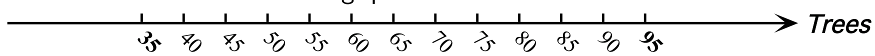
are all *quantitative* rulers but the following is *not* a quantitative ruler:



2. Quantitative rulers are specified by two things:

- The **extent** of a given *quantitative ruler* consists of both the *smallest label* and the *largest label* which we write between **curly brackets** { }.
- The **resolution** of a given *quantitative ruler* is the *space* between the labels of two consecutive tick-marks.

Example 9. Given the following quantitative ruler

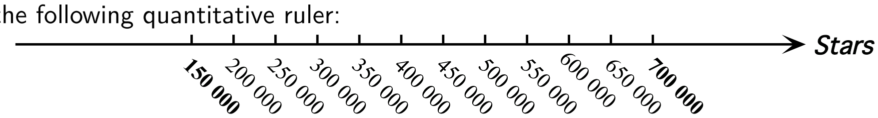


- the *extent* of the given ruler is $\{35, 95\}$
- the *resolution* of the given ruler is 5

3. Thus, given a quantitative ruler, there are going to be two kinds of number:

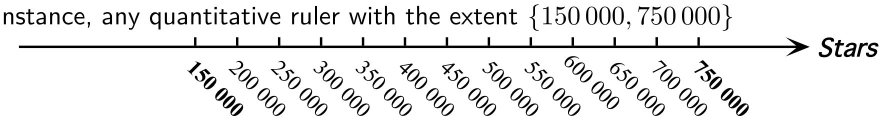
- Numbers that fall *within the extent* of the quantitative ruler which we will call **bounded numbers**. It is important to realize that any given number we happen to be interested in can always be viewed as a *bounded number* since we can always draw a quantitative ruler whose extent will **encompass** the given number.

Example 10. We can view the number 308 195 as a *bounded number* by using the following quantitative ruler:



In fact, any number of number-phrases can be viewed as *bounded numbers* since we need only use a quantitative ruler whose extent encompasses both the smallest numerator and the largest numerator.

Example 11. The numbers $176\,329, 53.78, 543\,830$ will be *bounded numbers* for any ruler with an extent that encompasses both 53.78 and 176 329 such as, for instance, any quantitative ruler with the extent $\{150\,000, 750\,000\}$



or

offscreen numbers

- Numbers that fall *beyond the extent* of the quantitative ruler which we will call **offscreen numbers**[2]. Because there is a number of difficulties with *offscreen numbers*, though, we will not deal with them right away and will return to them in due time.

===================OK SO FAR===========

8.5 States

We will want to make pictures to visualize

8.6 Actions

===================OK SO FAR===========

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We investigate the *third* of the three fundamental processes involving two collections. We will introduce the procedure in the case of *basic* collections using *basic counting* number-phrases and we will then extend the procedure to *extended* collections using *decimal-number phrases*.

[2]Educologists will surely know why we didn't use the term "unbounded".