If you cannot count properly,
you had better not go to
market—or to war.

Gore Vidal[1]

basic counting
basic succession
basic precession

Chapter 5

Counting

So far, the *Hindu-Arabic system* is only a—very good—shorthand system for *representing* collections of items. But, basically, it is still a tally mark system where we get the numerators by writing one tally mark for each item. As such it is ill-suited to the development of paper procedures.

What we want to develop now is a procedure that bypasses entirely the tally marks and that allows us to find *directly* the numerator in the metric-exponential system of the number-phrase that represents a given collection of real-world items.

5.1 Basic Counting

We begin with the case of a *basic collection*, that is a collection with no more *items* than we have *shorthands* available which, in our case, is NINE.

1. The basic tool for this procedure, which we will call **basic counting**, is to memorize the shorthands $1, 2, 3, 4, 5, 6, 7, 8, 9$ as a **basic succession**, that is in the *order*[2]:

"ONE, TWO, THREE, FOUR, FIVE, SIX, SEVEN, EIGHT, NINE"

While we are at it, it will turn out, even though this is not commonly done in elementary schools, that it will be a very good investiment also to memorize the digits as a **basic precession**

"NINE, EIGHT, SEVEN, SIX, FIVE, FOUR, THREE, TWO, ONE"

[1] *Creation*, Ballantine Fiction, Page 200

[2] Here Educologists will of course recognize that, in contradistinction to the way we initially defined numerators, this is the *ordinal* approach to the natural numbers.

end-digit

2. We now proceed with the *procedures* which we will use for going:

- From a given real-world basic collection to the paper-world basic number-phrase,
- From a given paper-world number-phrase back to the real world basic collection.

a. Given a *basic collection*, the procedure for getting the *numerator* of the number-phrase is:

i. We point successively at each and every item in the collection while reciting the digits in the *basic succession* that we memorized.

ii. We write the **end-digit**, that is the *last* digit we recited in the succession.

EXAMPLE 1. Given the *collection*

In order to get the *numerator* of the basic number-phrase that represents it, we *count* the collection, that is:

In the real world		In the paper world
i. We point at the first item:		
	while saying ONE,	
We point at the second item:		
	while saying TWO,	
We point at the third item:		
	while saying THREE,	
		ii. We write the end-digit 3

Then, as usual, using as *denominator* the name of the President whose picture is on the items, that is **Washington**, the *collection*

is represented by the *basic number-phrase*

3 **Washingtons**

.

b. Conversely, given a *basic number-phrase*, to get the *basic collection* that it represents:

i. We get one *item*—of the kind specified by the *denominator*—each and every time we say a *shorthand* in the basic succession

ii. We stop after we have picked the item for the numeral in the *numerator*

EXAMPLE 2. Given the *basic counting number-phrase*

5 **Washingtons**

to get the *basic collection* that it represents:

i. The *denominator* **Washingtons** tells us that the items must be of the same *kind* as 

ii. The *numerator* 5 tell us to pick an item each and every time we say a digit in the succession; we stop after we have picked the item for the *end-digit*:

In the paper world		In the real world	
5 tells us to *recite*:	ONE,	we get a 	and we now have 
	TWO	we get a 	and we now have 
	THREE	we get a 	and we now have 
	FOUR	we get a 	and we now have 
	FIVE	we get a 	and we now have 
		We stop	

Altogether, the *basic number-phrase*

5 **Washingtons**

represents the *basic collection*

c. The sticklers among us, though, will have rightfully observed that, strictly speaking, *basic counting* is neither a paper *procedure*, since it involves the real-world items, nor a real-world *process*, since it involves the shorthands we write on paper. Indeed, *basic counting* is a bridge from the real-world to the paper-world.

extended counting
extended succession
endless

5.2 Extended Counting

We now come to the issue of how to deal with *extended collections*, that is collections with more real-world *items* than we have *shorthands* on paper. In our case, an extended collection will be a collection with more than nine items.

1. There are two ways to get the combination-phrase for an extended collection.

- The way we learned as a child, namely **extended counting**, which is based on the memorization of the **extended succession**

  "TEN, ELEVEN, TWELVE, THIRTEEN, FOURTEEN, . . . "

  which it took us quite a while to memorize.

  EXAMPLE 3. There was the time when we could recite "ONE, TWO, THREE" but didn't know what came after. Later, there was a time when we could recite up to, say, twelve, but then would say, for instance, SEVENTEEN. Etc

  However, and this is possibly the single most important fact about ARITHMETIC, while there are only so many shorthands in the *basic* succession—NINE in our case, the *extended* succession is **endless**. This usually comes to children as a "revelation": we can always go one further.
- The other way is to find directly the numerators that make up the array-phrase that represents the extended collection in the Hindu-Arabic system. What we will do is to use basic counting *as we bundle*, first the real-world items in the extended collection, then the collections of bundled items, etc.

2. The general idea will be systematically to reduce the number of objects in the collection which we have to represent until we need only represent combinations of *basic* collections. More precisely, the counting procedure in the case of an *extended collection* is:

i. We use the basic succession to count items up to NINE.

ii. We get one more item so that now we have a collection that we cannot represent but that we can bundle.

iii. We use the basic succession to count the unbundled items up to NINE.

iv. Get one more item so that now we have another collection that we cannot represent but that we can bundle (and possibly exchange for an equivalent new item).

v. Keep going until either

- All the items have been counted in which case we can represent the combination by a combination phrase

or

- We have a collection of TEN bundles of TEN items in which case we bundle that collection into a super collection (and possibly exchange for an equivalent new item) and continue.

count
start-digit
end-digit
count from ... to ...
direction
count-up
count-down
precession

5.3 Counting From A Number-Phrase To A Number-phrase

Before we can develop procedures, though, we must make the concept of *counting* more flexible by allowing a **count**

- to start with *any* digit which we will call the **start-digit**. (So, the start-digit doesn't have anymore to be 1 as it always did above.)
- to end with *any* digit which we will call the **end-digit**. (So, the end-digit may be "before" the start digit as well as "after" the start digit.)

Specifically, when we **count from** the start-digit **to** the end-digit:

i. We start (just) *after* the start-digit

ii. We stop (just) *after* the end-digit.

However, given a *start-digit* and a *end-digit*, we may have to count in either one of two possible **directions**:

- We may have to **count-up**, that is we may have to use the *succession*

$$\xrightarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9}$$

which we read along the arrow, that is *from left to right*.

EXAMPLE 4. To count from the start-digit 3 to the end-digit 7:

i. We must count *up*, that is we must use the *succession*

$$\xrightarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9}$$

ii. We *start* counting *up* in the succession *after* the start-digit 3, so that 4 is the first digit we say,

$$\xrightarrow{4,\ \dots}$$

iii. We *stop* counting *up* in the succession *after* the end-digit 7 so that 7 is the last digit we say

$$\xrightarrow{\dots 7}$$

Altogether, the count from the start-digit 3 to the end-digit 7 is

$$\xrightarrow{4,\ 5,\ 6,\ 7}$$

- We may have to **count-down**, that is we may have to use the **precession**

$$\xleftarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9}$$

length (of a count)

which we read along the arrow, that is *from right to left*.

NOTE. If we prefer to read *from left to right*, we may also write the *precession* as

$$\xrightarrow{9,\ 8,\ 7,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1}$$

which we read along the arrow, that is *from left to right*.

EXAMPLE 5. To count from the start-digit 6 to the end-digit 2:

i. We must count *down*, that is we must use the *precession*

$$\xrightarrow{9,\ 8,\ 7,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1}$$

ii. We *start* counting *down* in the precession *after* the start-digit 6 so that 5 is the *first* digit we say

$$\xrightarrow{5,\ \ldots}$$

iii. We *stop* counting *down* in the precession *after* the end-digit 2 so that 2 is the *last* digit we say.

$$\xrightarrow{\ldots\ 2}$$

Altogether, the count from the start-digit 6 to the end-digit 2 is

$$\xrightarrow{5,4,3,2}$$

NOTE. Memorizing the *precession* $\xrightarrow{9,\ 8,\ 7,\ 6,\ 5,\ 4,\ 3,\ 2,\ 1}$ just like we memorized the *succession* $\xrightarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9}$ makes life a lot easier.

Finally, the **length of a count** from a start-digit to an end-digit is how many digits we say regardless of the direction, that is whether up in the succession or down in the precession.

EXAMPLE 6. When we count from the start-digit 3 to the end-digit 7, the *length* of the count is 4.

EXAMPLE 7. When we count from the start-digit 6 to the end-digit 2, the *length* of the count is 4.

What that does, as in Chapter 1, is again to separate *quality*—represented by the *direction* of the count, up or down, from *quantity*— represented by the *length* of the count, how many digits we count.

NOTE. As already mentioned, we will only use *basic* counting, whether up or down, but *extended* counting would work essentially the same way.

5.4 Counting A Number-phrase From A Number-Phrase

Up

Down

5.5 Quantitative Rulers

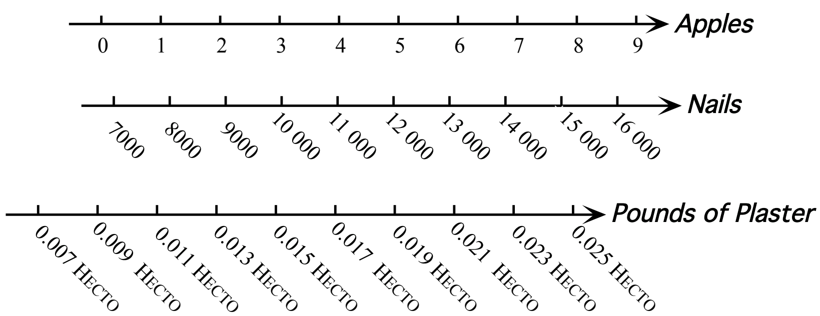
graph
ruler, quantitative
number lines
tick-marks
labeled
in order
equally spaced

Very often, we will want to visualize things by **graphing** number-phrases and we will do that with **quantitative rulers** which are indeed essentially what goes in the real world by the name of "ruler".

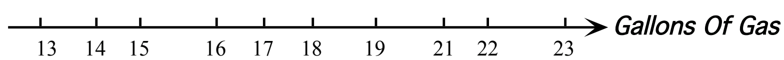
NOTE. In high school, *quantitative rulers* usually go by the name of **number lines**, a term we will not use in this text[2].

More precisely, the **tick-marks** on a *quantitative ruler* must be **labeled**, **in order**, and **equally spaced**.

EXAMPLE 8. The following :



are all *quantitative* rulers but the following is *not* a quantitative ruler:



[2] It would be interesting to trace the origin of this remarkably un-enlightening term. But then, it is probably due to Educologists' well known craving for the esoteric.