[...] mathematics as a precise language for expressing relationships among quantities in the real world [...].

Carver Mead[1]

numbering system
kind
collect
collection
bunch

Chapter 2

Tally Mark Systems

There is no such thing as a universal symbolic system and, in order to deal with different aspects of the real world, we need different symbolic systems.

EXAMPLE 1. There are (at least) two very different aspects of the real world:

- One aspect of the real world has to do with *numbers*. The corresponding symbolic systems are called **numbering systems** and are what we deal with in ARITHMETIC.
- Another aspect of the real world has to do with *shapes*. The corresponding symbolic systems are what we deal with in GEOMETRY.

In this text, we will deal exclusively with just the aspects of the real world that have to do with *numbers* and therefore with just ARITHMETIC.

2.1 Number-Phrases

The real-world items that we want to represent can either be all of the same **kind**, in which case we can **collect** them into a **collection**, or they can be of different kinds in which case we will just say that they make up a **bunch**.

EXAMPLE 2. Given the following real-world items,

[1]Foreword to *Street-Fighting Mathematics* by Sanjoy Mahajan, The MIT Press.

curly brackets
{ }
number
denominator
numerator
number-phrase
tally mark system
tally mark
I

since they are not all of the *same kind* they make up a bunch.

1. In order readily to distinguish collections from bunches, in the case of a *collection* we will:

- Draw a line around the substitutes when the substitutes are *pictures*,

  EXAMPLE 3. The real-world items  make up the collection 

- Enclose the substitutes between **curly brackets**, **{ }**, when the substitutes are *English words*.

  EXAMPLE 4. The real-world items ***apple***, ***apple***, ***apple*** make up the collection {***apple***, ***apple***, ***apple***}

2. The reason collections are easier to deal with than bunches is that:

- In the case of a *bunch* of items. that is when the items are of *different kinds*, the information that we will need to encode is:
  i. The *different kinds* of items that are in the bunch,
  ii. The **number** of items of *each* kind.
- In the case of a *collection*, that is when all the items are of the same kind, we won't need to encode as much information because, since all the items are of the same kind, we will just need to encode what that one kind is—instead of the many kinds in a bunch—and the total number of items in the collection. So we will need only:
  i. A **denominator** to encode the *kind* of the real-world items that are in the collection. The denominator will just be a *noun*.
  ii. A **numerator** to encode the *number* of real-world items that are in the collection. What to use for the numerator is what we are going to have to work on.

So, any *collection* of real-world items can be represented by a **number-phrase**, that is by a *denominator* together with a *numerator*.

EXAMPLE 5. In Thailand, the collection {} could be represented by the *number-phrase* ๓ แอปเปิล where แอปเปิล is the *denominator* and ๓ is the *numerator*.

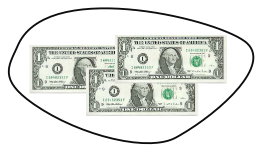
NOTE. We will always write the numerator first and the denominator second as when we write 3 **Dollars** and as is usual. And, even though it is also usual to write **$**3, we will not do so here.

2.2 Tally Mark Systems

The simplest systems for coding numerators are **tally mark systems**:

1. Given a collection of real-world items, in order to get a numerator, we write on paper a **tally mark**, say **I**, for each and every real-world item in the collection[2]. So, there will be as many tally marks on paper as there are real-world items in the collection.

EXAMPLE 6. Given the following collection of real-world items,



i. We code the *kind* of items, for instance with the name of the President whose picture is on each item:

Washington

ii. We code the *number* of items by writing a tally mark for each item in the collection:

III

iii. So the number-phrase that represents the collection in the tally mark system is

III **Washington**

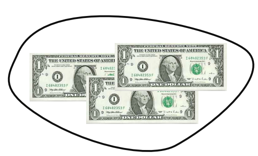
2. There is no loss of information as all we did was to separate the information about the *number* of items—which is encoded by the *numerator*—from the information about the *kind* of items—which is encoded by the *denominator*[3]. Indeed, from the number-phrase on paper, we can reconstitute the collection in the real world.

EXAMPLE 7. Given the number-phrase III **Washington**, we can reconstitute the real-world collection that is being represented:

i. The denominator **Washington** tells us to use the following kind of item:

ii. The numerator III tells us how many of this kind of items there has to be in the collection:

iii. So, from the number-phrase III **Washington** we can reconstitute the collection of real-world items:

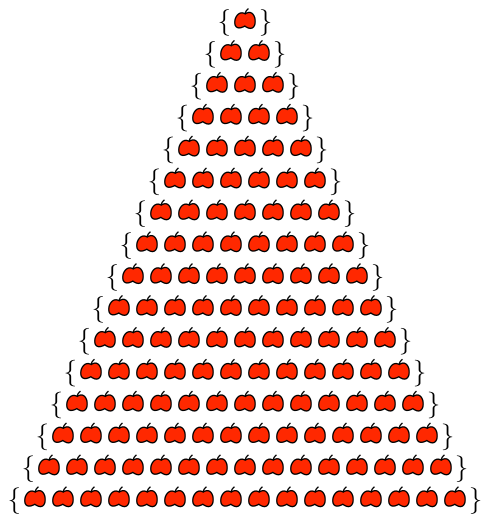


[2]Educologists will certainly speak of a 1-to-1 correspondence and thus realize that this is the *cardinal* approach to the natural numbers.

[3]In spite of which this is precisely the point where, in the name of "abstraction", Educologists cut their students away from *denominators* without noticing, of course, that this is exactly the point where they start losing their students.

3. As long as the number of real-world items does not get too large, a tally mark system is very economical in that it involves only one symbol to write the numerators and one noun to write the denominator.

EXAMPLE 8. In a tally mark system we can represent collections of apples as follows:

Collections of apples	Number-phrases
	**I Apple**
	II Apple
	III Apple
	IIII Apple
	IIIII Apple
	IIIIII Apple
	IIIIIII Apple
	IIIIIIII Apple
	IIIIIIIII Apple
	IIIIIIIIII Apple
	IIIIIIIIIII Apple
	IIIIIIIIIIII Apple
	IIIIIIIIIIIII Apple
	IIIIIIIIIIIIII Apple
	IIIIIIIIIIIIIII Apple
	IIIIIIIIIIIIIIII Apple

4. As a matter of fact, a tally mark system is most likely how, several thousands years ago, ARITHMETIC got started when, one may imagine, Sumerian merchants, faced with the problem of having more goods in the warehouse and/or money in the safe than they could handle directly, decided to have both the goods and the money represented by tally marks scratched on clay tablets so that they could see from these clay tablets the situation their business was in without the inconvenience of having to go to the warehouse andIor to open the safe.

5. One advantage of tally mark systems is in tallying ongoing results such as the score in a game of sports since no iintermediate results need to be erased or discarded. (See Wikipedia, Unary numeral systems)

6. Nevertheless, tally mark systems have two major flaws:

Flaw A. When the number of items in the collection gets to be large, so does the number of tally marks and it gets more and more difficult to see, at a glance, how many items a long string of tally marks represents.

EXAMPLE 9. It is not immdediately obvious how many items there are in the collection represented by the number-phrase

IIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII Apples

clustered tally mark system
string
cluster

as compared to the collection represented by the number-phrase

IIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII Apples

and, even more importantly,

Flaw B. Tally mark systems do not lend themselves easily to the development of *operations*.

EXAMPLE 10. Try to multiply

IIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII Feet

by

IIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIIII Feet

2.3 Clustered Tally Mark Systems

The simplest attempt at remedying **Flaw A**, namely that numerators for large collections are hard to read, are **Clustered Tally Mark System**:

1. Instead of writing all the tally marks in a **string**, we can **cluster** the *tally marks* to make the string easier to read.

EXAMPLE 11. Children spontanesously tend to write four tally marks and then cluster them by writing the fifth tally mark across the first four tally mark. In other words, children might represent collections of apples as follows:

Collections of apples	Number-phrases
{🍎}	**I Apple**
{🍎🍎}	**II Apple**
{🍎🍎🍎}	**III Apple**
{🍎🍎🍎🍎}	**IIII Apple**
{🍎🍎🍎🍎🍎}	**𝍸 Apple**
{🍎🍎🍎🍎🍎🍎}	**𝍸 I Apple**
{🍎🍎🍎🍎🍎🍎🍎}	**𝍸 II Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 III Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 IIII Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 𝍸 Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 𝍸 I Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 𝍸 II Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 𝍸 III Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 𝍸 IIII Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 𝍸 𝍸 Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 𝍸 𝍸 I Apple**
{🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎🍎}	**𝍸 𝍸 𝍸 II Apple**

Roman system
shorthand symbol
positional

2. There are many other ways to cluster the tally marks so as to make it easier to read the numerators.

EXAMPLE 12. Instead of writing 𝍸 𝍸 III **Apple**, the following are also used to cluster the slashes and make numerators easier to read:

- **IIIII IIIII III Apple**
- **I|||| I|||| I|| Apple**
- ⧅ ⧅ ⊓ **Apple**

3. But, while clustered tally marks systems do address **Flaw A**, at least to an extent, they still cannot represent very large collections in a readable manner. And they sure do not lend themselves to easy computing.

EXAMPLE 13. Try to multiply 𝍸 III **Feet** by 𝍸 𝍸 IIII **Feet**.

2.4 The Roman System

A different way to improve the tally system is the **Roman system**.

1. The Roman system uses **shorthand symbols**. In other words, the Roman system trades *long* strings of a *single* symbol for *short* strings of *several* shorthand symbols.

Instead of the longhand	We write the shorthand
IIIII	V
VV	X
XXXXX	L
LL	C
CCCCC	D
DD	M

2. The Roman system is partly **positional** in that, for instance,

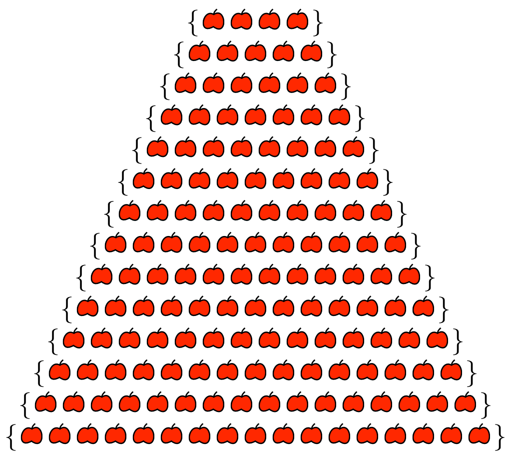
	The string	is a shorthand for
	VI	IIIIII
but	IV	IIII

3. The Roman system allows us to write number-phrases that represent collections with fairly large numbers of items with a fairly small number of shorthand symbols.

EXAMPLE 14. In the Roman system we represent collections of ***apples*** as follows:

Collections of apples	Number-phrases
{🍎}	**I Apple**
{🍎🍎}	**II Apple**
{🍎🍎🍎}	**III Apple**

gain



IV **Apple**
V **Apple**
VI **Apple**
VII **Apple**
VIII **Apple**
IX **Apple**
X **Apple**
XI **Apple**
XII **Apple**
XIII **Apple**
XIV **Apple**
XV **Apple**
XVI **Apple**
XVII **Apple**

4. However, the Roman system has three major flaws:

Flaw C. The *gains* provided by the shorthands are all different.

EXAMPLE 15. When we use V to shorthand the longhand IIIII, the gain is 1 symbol instead of 5 symbols because the shorthand symbol V replaces the *five* symbols IIIII but when we use X to shorthand the longhand VV, the shorthand gain is only 1 symbol for 2 symbols because the shorthand symbol X replaces the *two* symbols VV.

This essentially made it impossible to develop *procedures*.

Flaw D. As the collections get larger and larger, the Roman system requires more and more shorthand symbols to keep the numerators short enough to remain readable, namely

L, C, D, M.

and the flaw is that the Roman system has no mechanism for automatically providing shorthand symbols beyond M.

EXAMPLE 16. The Roman system has no symbol to represent a MILLION.

Flaw E. While partly positional, the Roman system is not *systematically* positional.

EXAMPLE 17. We use IV instead of IIII, XL instead of XXXX and CM instead of DCCCC, but we cannot use VL instead of XLV.

All of which make the Roman system totally unfit for the development of *computational procedures*. (Which makes sense since the Romans were militaries rather than merchants.)

EXAMPLE 18. Try to multiply MCMLXXXIV **Feet** by CXVIII **Feet**.

sort
combination
square brackets
[ ]
combination-phrase

2.5 Combination-Phrases

Inasmuch as a bunch is *not* a collection, that is involves more than one kind of real-world items, we need more than just one denominator to represent the bunch and so we cannot use just a number-phrase.

EXAMPLE 19. Given the following bunch of real-world items,

since they are *not* all of the same kind (they do *not* make up a *collection*) there is no single President whose name we can use as a *denominator*.

The way out is, rather obviously, as follows:

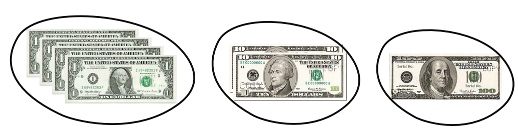
i. We **sort** the real-world items in the bunch by *kind* so that we now have a **combination** of collections,

ii. We represent each collection by a number-phrase,

iii. We write the number-phrases, separated by commas, within **square brackets**, [ ], to get a **combination-phrase**[4].

EXAMPLE 20. Given the bunch of real-world items

i. We sort the bills into the following *combination* of collections:



ii. We represent each collection by a number phrase. In the tally mark system, we get:

IIII Washington **II Hamiltons** **I Franklin**

iii. We write the combination-phrase that represents the given *combination* of collections:

$\Big[$ **IIII Washington**, **II Hamiltons**, **I Franklin** $\Big]$

[4]No doubt, Educologists will realize that we are nearly dealing with *vectors*.

2.6 Bundling Systems

bundle
bundling rate
dozenal system
dozen
gross
great gross
virtual
sexagesimal system

In the course of history, the collections that humankind had to deal with got larger and larger and the symbolic systems necessary to code them had to keep up. It wasn't easy.

The first idea to deal with collections of large numbers of real-world items was to **bundle** the real-world items at a set **bundling rate** into collections and then to represent the bunch consisting of:

- the collections of bundled real-world items

and

- the collection of remaining unbundled real-world items

by a combination-phrase. Of course, there was no reason not to bundle the bundles into "super bundles".

Example 21. Possibly one of the first implementations of this idea, the **dozenal system** in which the bundling rate is twelve, came up some three thousand years ago in Iraq. To this day, eggs are bundled by the **dozen** and, in fact, bundles of twelve eggs are then also bundled by the dozen which results in a **gross** of eggs. Then, bundles of twelve bundles of twelves eggs are also bundled by the dozen which results in a **great gross** of eggs.

In fact, the bundling can be **virtual** in that it just provides us with a mental image intended to help with the representation of the collection.

Example 22. In the **sexagesimal system**, which originated some five thousand years ago, also in Iraq, the bundling is done at the bundling rate of SIXTY ***seconds*** to a ***minute*** and SIXTY ***minutes*** to an ***hour***.

For centuries, bundling systems were used extensively even though they had three major flaws which made them progressively less and less adequate:

Flaw E. Bundling rates usually changed with larger and larger collections.

Flaw F. Bundling systems have no mechanism for providing more names as the collections get larger and larger.

Flaw G. Bundling rates depend on the *kind* of real-world items we are numbering.

Example 23. According to Appendix C of the NIST Handbook 44, in the English System:

- The bundling rates for lengths are:

Collection	Single item
TWELVE ***Inches***	***Foot***
THREE ***Feet***	***Yard***
SIX HUNDRED SIXTY ***Yards***	***Mile***

- The bundling rates for weights are:

Collection	Single item
SIXTEEN ***Ounces***	***Pound***
TWO THOUSAND ***Pounds***	***Ton***

- The bundling rates for liquids are

Collection	Single item
SIXTEEN ***Fluid Ounces***	***Pint***
TWO ***Pints***	***Quart***
FOUR ***Quarts***	***Gallon***

2.7 Old Systems

Thus, for many centuries, there were two symbolic systems for numbering collections:

- The Roman system which uses number-phrases which are much more compact than combination-phrases.
  However, the Roman system uses shorthand rates in the numerators that do not remain the same and there is no mechanism for creating further numerators.
- The bundling systems which, at least at the beginning, use very low numerators but at the cost of using the much more cumbersome combination-phrases instead of number-phrases.
  However, the bundling systems use bundling rates in the denominators that do not remain the same and there is no mechanism for creating systematically further denominators.

EXAMPLE 24. Compare:

Roman system	Bundling system
I Foot	**I Foot**
II Feet	**II Feet**
III Feet	**I Yard**
IV Feet	**I Yard I Foot**
V Feet	**I Yard II Feet**
VI Feet	**II Yards**
VII Feet	**II Yards I Foot**

Roman system	Bundling system
I Cup	**I Cup**
II Cups	**I Pint**
III Cups	**I Pint I Cup**
IV Cups	**I Quart**
V Cups	**I Quart I Cup**
VI Cups	**I Quart I Pint**
VII Cups	**I Quart I Pint I Cup**

Neither one being very systematic, there was room for improvement.