What is important is the real world, that is physics, but it can be explained only in mathematical terms.

Dennis Serre[1]

[...] mathematics as a precise language for expressing relationships among quantities in the real world [...].

Carver Mead[2]

real world
entity
communicate
information
investigate
relationship
plan
act
language
word
dictionary

Chapter 1

Languages

What Languages Are Made Of, 1 – Learning Languages, 3 – In This Book, 4 – Natural Languages, 5 – Specialized Languages, 5 – Symbolic Systems, 6.

Contrary to what most people think, MATHEMATICS originates in the **real world** and deals with real-world **entities**. However, in order to:

- **Communicate** with others about the real-world, so as to share **information** about real-world entities,
- **Investigate** the **relationships** that exist among real-world entities so as to understand how the real world works,
- **Plan** how to **act** on the real world because acting on the real world without thinking ahead usually has very unfortunate consequences.

we always need a **language**.

There are all sorts of languages: anywhere between 3000 and 8000 spoken languages, hundreds of sign-languages, etc. There are written languages, pictorial languages, secret and non-secret codes, bar codes, computer languages, etc. (See `http://en.wikipedia.org/wiki/Language`.) And, as we will see, MATHEMATICS requires languages of a special kind.

1.1 What Languages Are Made Of

Very, very roughly, most languages are constructed as follows:

1. The building blocks of the language are **words**. Words must be taken from a **dictionary** available to all users of the language.

[1] Bulletin of the AMS, Vol 47 Number 1 Pages 139-144

[2] Foreword to *Street-Fighting Mathematics* by Sanjoy Mahajan, The MIT Press.

noun
verb
sentence
grammatical rules
grammar
state
statement
meaningless
TRUE
to be the case
FALSE

EXAMPLE 1. Of the following:
- word, wood
- woord, wod

the first two are English words but the other two are not.

The two main kinds of *words* that we will be using are:
- **Nouns** to represent real-world *entities*,
- **Verbs** to represent *relationships* among real-word entities.

EXAMPLE 2. Of the following:
- apple, one-dollar-bill
- is-larger-than, eats

the first two are *nouns* and the other two are *verbs*.

2. In order to communicate we use **sentences**. Sentences are constructed from *words* according to **grammatical rules** listed in a **grammar** available to all users of the language.

EXAMPLE 3. Of the following:
- The hamburger paints the ocean with the mountains.
- Dog fox lazy quick brown the jumps the over.

the first is a *sentence* but the other is not a sentence.

3. A *sentence* may or may not be **stating** something about the real world. A *sentence* that states something about the real world is called a **statement** (about the real world). A sentence that does not state something about the real world is said to be **meaningless**.

EXAMPLE 4. Of the following two *sentences*:
- The spaghetti sauce is too hot to be eaten with a spoon.
- The rose understands the navy with a spoon.

the first is a *statement* (about the real world) but the second is *meaningless* since it is not a statement (about the real world).

4. A statement (about the real world) is then:
- **TRUE** if what it states (about the real world) **is the case** in the real world,
- **FALSE** if what it states (about the real world) is *not the case* in the real world.

EXAMPLE 5. The following are both (famous) *statements* (about the real world):
- The moon is made of green cheese.
- Humans are the only featherless bipeds.

However,

> The first statement is FALSE because, as Armstrong checked on July 20, 1969, it is not the case that the moon is made of green cheese.

while

The second statement is TRUE because, even though there are other species that are featherless and other species that are bipedal, it happens that humans are the only ones to be both featherless and bipedal.

procedure
simulate
process
native language
object language
metalanguage

5. Our main goal in this text will be, using *sentences*, to develop **procedures** to **simulate** on paper real-world **processes**.

EXAMPLE 6. Suppose we have to carry out the real-world *process* of handing out seventeen dollars to each one of two thousand six hundred and forty eight people. A preliminary question then is whether we have the cash and thus how much this would amount to.
To find out, we *represent* on paper both how many one-dollar bills per persons and how many persons and we then use the procedure called *multiplication* to figure out on paper how many one-dollar bills we will need in the real world.
We then use another procedure to compare what we have with what we will need.

1.2 Learning Languages

Learning a language is nowhere as simple a thing as one might think.

1. The language that each one of us first learned as a child, our **native language**, we learned by directly associating *nouns* with the real-world *entities* that the nouns represent. But how children learn *verbs* to represent real-world *relationships* among entities is already much more complicated.

2. A "second language" is usually not learned the same way as a native language, i.e. not by direct association with the real world, but through a first language that we already know. So, learning a second language automatically involves *two* languages which we need to distinguish:

- The **object language**, which is the language we want to learn,
- The **metalanguage**, which is the language we already know and which we use to learn the object language.

EXAMPLE 7. For an American learning Spanish, Spanish is the *object language* and English is the *metalanguage* but for a Spaniard learning English, the situation is the other way around.

Very often, the metalanguage we use in learning a second language is our native language but not always.

3. In a book, the *metalanguage* is used in several different ways:

- To describe the real-world *entities* that we want to represent on paper because real-world entities usually cannot be exhibited in a book,
- To describe *relationships* among real-world entities because relationships among real-world entities are not always obvious,

but also

pictures
substitute
shorthand
boldfaced
margin
index

– To lay down the *grammar* of the object language.

1.3 In This Book

We will be using English as our metalanguage.

1. When dealing with real-world items that we cannot exhibit, though, we will usually let **pictures** stand as **substitutes** for the real-world entities that we cannot exhibit. However, as even using pictures will not always be possible, we will also have to use English words which will then be printed in ***bold-faced italics***. So, in this book, words printed in bold-faced italics belong to the metalanguage.

EXAMPLE 8. Say we want to represent *real-world* one dollar bills. The difficulty is that we cannot exhibit in this book the *real-world* one dollar bills that we want to represent on paper because we are already on paper. So, as a *substitute* for real-world one dollar bills, we use the *pictures*

However, we can also use as a substitute for the real-world one dollar-bills, the English words (in bold-faced italics),

one dollar bill ***one dollar bill*** ***one dollar bill***

NOTE. It is very important to distinguish *nouns* from *substitutes*: while they both stand for real-world items, *nouns* belong to the *object language* that we develop while *substitutes* belong to the *metalanguage* and will be used only inasmuch as we are unable to exhibit the real-world items that the *nouns* are to represent on paper.

2. When there will be too many real-world entities for us to substitute with picture or even with English words, we will use English numbers in small caps like THIS as a **shorthand**. So, in this book, numbers spelled out in small caps belong to the metalanguage.

EXAMPLE 9. We will use FIVE ***apples*** as a shorthand for ***apple apple apple apple apple*** as well as FIVE 🍎 as a shorthand for 🍎🍎🍎🍎🍎.

3. In order to make it easier to find where the words, both in the object language and in the metalanguage, appear for the first time and are explained, this book uses a standard method: these words are:

– **boldfaced** the first time they appear and are being explained,
– printed in the **margin** of the page where they first appear and are being explained,
– listed in the **index** at the end of the book along with the number of the page where they first appear and are being explained.

4. Very occasionally, when announcing things to come, *and only then*, we will use words without explaining them and leave their "official" explanation

for later on. In that case, the words will appear between single 'quotation marks' and will appear neither in the index nor in the margin. Double quotation marks will remain used as usual.

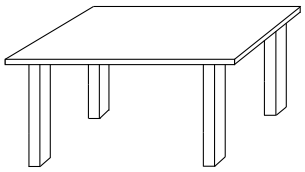
natural language
specialized language

1.4 Natural Languages

A **natural language** is the kind of language we use most of the time, routinely, and rather loosely. Natural languages depend on place and time.

EXAMPLE 10.

The real-world entity whose picture is



is represented by different words depending on the natural language we are using:

Language	Word
Chinese	表
English	table
Finnish	taulukko
French	table
German	Tisch
Greek	τραπέζι
Italian	tavola
Japanese	食卓
Latin	tabula
Russian	стол
Spanish	mesa
Vietnamese	bảng

How natural languages were born and how they then evolved and interacted with one another is buried in history and investigated in ETYMOLOGY.

1.5 Specialized Languages

While natural languages usually serve us fairly well for *communicating*, when it comes to *planning* they have severe limitations.

1. Every trade has its **specialized language** because people working in a trade:

– Need to know exactly what they are to do,

EXAMPLE 11. We cannot just tell an architect "Design a house." Architects need specifications. In turn, architects cannot just give pictures to construction workers. Construction workers need blueprints.

– Need to communicate with each other while doing the work.

EXAMPLE 12. Imagine two persons working on some dangerous machinery and one telling the other: "Move the gizmo which is on the whatchamacallit next to the doodad on the other side of the doohickey."

make the case
correct
compute
systematic
symbol
symbolic system
specified
noun symbol
verb symbol
logical symbol
symbolic sentence
code

2. Sometimes, specialized languages use specially made-up words but most often they use common words with a meaning special to the trade.

EXAMPLE 13. When an *electrician* asks for a pancake, that's not because s/he is hungry but because s/he needs a junction box the thickness of drywall. When a carpenter talks about drywall, s/he is not likely to be talking about a wall that is not wet. When a *mason* talks about a hawk, s/he is not likely to be talking about a bird.

EXAMPLE 14. "[The actor] must learn the theatre's special vocabulary. Partly technical, partly slang, much of it is standardized on the English-speaking stage. As a working actor, you must be familiar with this language, just as a mechanic must know the names of his tools or a surgeon the names of her instruments." (From Charles McGaw's *Acting is Believing*.)

1.6 Symbolic Systems

In the mathematical trade, the situation is exactly the same as in any other trade except a lot more so. The reason is because, in mathematics, in addition to communicating and planning, we *always* have to **make the case**, just as if we were in a court of law, that:

- The *sentences* we write are TRUE, that is make statements that are the case in the real-world,
- The *procedures* we develop are **correct**, that is simulate the real-world processes.

Because of these two goals, we will have to develop an object language that makes it as easy as possible for us to **compute**, that is to manipulate the object language **systematically**. So, the main feature we will require from this object language is that it itself be *systematic*.

In order to stress the *systematic* aspect of the object language that we will develop, we will use the word **symbol** instead of the word *word* and we will use the word **symbolic system** instead of the word *language*. More precisely, a symbolic system is **specified** by a dictionary of symbols:

- **Noun symbols** to represent real-world *entities*,
- **Verb symbols** to represent *relationships* among real-world entities,

together with

- **Logical symbols** to represent in the symbolic system English words such as "and", "or", "not".

Then we will assemble these symbols to write **symbolic sentences** which will be TRUE or FALSE depending on whether the statements they make about the real world are or are not the case and we will say that we have **coded** the information about the real world into the *symbolic system*.

EXAMPLE 15. Let the *real-world items* be persons named in the real world ***Andy***, ***Beth*** and ***Cathy***, and let the *real-world relationship* be that: &

↑	***Andy***	***Beth***	***Cathy***
Andy	***likes***	***likes***	***likes***
Beth	***likes***	***does not like***	***likes***
Cathy	***does not like***	***likes***	***does not like***

So, for instance, according to the above table, it is the case in the real world that ***Andy likes Cathy*** and it is also the case in the real world that ***Cathy does not like Andy***.

In order to deal on paper with these real-world entities, we may, for instance, use the symbolic system *specified* by the following *dictionary*:

- Noun symbols: a to represent ***Andy***, b to represent ***Beth***, c to represent ***Cathy***.
- Verb symbol: L to represent ***likes***,
- Logical symbol: $\&$ to represent "and".

Then, for instance, we can write:

- The symbolic sentence
$$bLa$$
which states that "Beth likes Andy" and, since the real-world fact according to the above table is that ***Beth likes Andy***, the symbolic sentence bLa is TRUE.
- The symbolic sentence
$$cLa$$
which states that "Cathy likes Andy" and, since the real-world fact according to the above table is that ***Cathy does not like Andy***, the symbolic sentence cLa is FALSE.
- The symbolic sentence
$$cLb \,\&\, cLa$$
which states that "Cathy likes Beth and Cathy likes Andy" and, since the real-world facts according to the above table are that ***Cathy likes Beth*** but that ***Cathy does not like Andy***, the symbolic sentence $cLb \,\&\, cLa$ is FALSE.
- However, we would not be able to write in this symbolic system a symbolic sentence to state that "Cathy does not like Andy" because we did not include in the dictionary of our symbolic system a logical symbol to represent "not".