computational definition
large

Chapter 18

Large, Medium And Small

Arithmetic Of *large*, *small* And *medium*, 1.

18.1 Arithmetic Of *large*, *small* And *medium*

We now come to something that is absolutely and totally crucial in mathematics and which will be at the heart of all that we will be doing from now on.

1. While the concepts of *absolutely-larger* and *larger-in-size* are both quite clear, the apparently simpler concept of "large" is in fact not at all that simple to pin down. Before anything else, though, let us say that, from now on, by "large" we will mean "large-in-size", that is we will be ignoring the *sign*.

a. Up to a point, *large* is a "relative concept".

EXAMPLE 1. A million dollars is probably not a *large* amount of money for people like Bill Gates or George W. Bush but for "the rest of us" a million dollars is most probably a *large* amount of money.

EXAMPLE 2. Nobody likes losing a *large* amount of money: Bill Gates and George W. Bush would not like to lose a billion dollars any more than "the rest of us" would like to lose a thousand dollars.

In other words, "large" has a meaning which is the same for everybody and it is only the *cutoff point* that changes from people to people.

b. However, we will need a **computational definition**, that is a definition that we can work with and we will say that a number is **large** when any number of copies of that number will multiply to a number that small

is *larger-in-size* than the number itself.

EXAMPLE 3. We will say that: -1.1 is *large* because, say, three copies of -1.1 multiply to -1.331 which is *larger-in-size* than -1.1.

c. In practice, though, whenever we will have to pick a *large* number to test what happens with *large inputs*, we will leave ourselves a "safety margin" and we will pick $+10$ or -10 or any number that is *larger-in-size than* 10.

EXAMPLE 4. The number $10 + u$ where u is any *positive* number is *large* because, say, three copies of $10 + u$ multiply to $(10 + u) \cdot (10 + u) \cdot (10 + u)$ which works out to a number *larger-in-size* than $10 + u$.
The details of the computation are quite easy and the reader really ought to do the computation to see *precisely* why

$$(10 + u) \cdot (10 + u) \cdot (10 + u) > 10 + u$$

no matter what the positive number u is.

2. Similarly, while the concepts of "absolutely-smaller" and "smaller in size" are both quite clear, the apparently simpler concept of "small" is in fact not at all that simple to pin down. Before anything els, though, let us say that by "small" we will mean "small in size", that is we will be ignoring the *sign*.

a. Up to a point, *small* is a "relative concept".

EXAMPLE 5. While, for "the rest of us" a thousand dollars is not something to be sneered at, for people like Bill Gates or George W. Bush a thousand dollars is "small", not even a drop in a bucket.

Nevertheless, the word has a meaning which is the same for everybody and it is only the *cutoff point* that changes from people to people.

EXAMPLE 6. Nobody likes to work for a *small* amount of money: Bill Gates and George W. Bush would not like to work a whole day for a million dollars any more than "the rest of us" would like to work a whole day for one dollar.

b. However, we will need a *computational definition*, that is a definition that we can work with, and we will say that a number is **small** when any number of copies of that number will multiply to a number that is *smaller-in-size* than the number itself.

EXAMPLE 7. We will say that: -0.2 is *small* because, say, three copies of -0.2 multiply to -0.008 which is *smaller-in-size* than -0.2.

c. In practice, though, whenever we will have to pick a *small* number to test what happens with *small inputs*, we will leave ourselves a "safety margin" and we will pick $+0.1$ or -0.1 or a number that is *smaller-in-size than* 0.1.

EXAMPLE 8. The number $\frac{1}{10+u}$ where u is any *positive* number must be *small* because, say, three copies of that number multiply to $(\frac{1}{10+u}) \cdot (\frac{1}{10+u}) \cdot (\frac{1}{10+u})$ which works

out to a number smaller than $\frac{1}{10+u}$. medium
The details of the computation are quite easy and the reader really ought to do the computation to see *precisely* why

$$(\frac{1}{10+u}) \cdot (\frac{1}{10+u}) \cdot (\frac{1}{10+u}) < \frac{1}{10+u}$$

no matter what the positive number u is.

3. There are of course numbers that are neither *large* nor *small*.

Example 9. $+1$ is neither *large* nor *small* because any number of copies of $+1$ will multiply to $+1$ whose size is 1, the same as the size of $+1$.

Example 10. -1 is neither *large* nor *small* because any number of copies of -1 will multiply to $+1$ or -1 depending on whether the number of copies is even or odd but in both cases, the size of the result will be 1, the same as the size of -1.

Beyond $+1$ and -1, though, there are

- numbers there are not "very-large" in that, when we multiply copies of the given number, the result is not "very-much-larger-in-size" than the given number,
- numbers that are *not* "very-small" in that, when we multiply copies of the given number, the result is nor "very-much-smaller-in-size" than the given number.

Example 11. Strictly speaking, the given number $+1.1$ is *large* since $(+1.1)\cdot(+1.1) = +1.21$ which is *larger-in-size* than the given number $(+1.1)$. But not by that much as $+1.21$ is not "very-much-larger-in-size" than $+1.1$.

Being neither "very-large" nor "very-small" is a "relative concept" but, contrary to what we were able to do with *large* and *small*, the only computational definition we can give is too restrictive in that it allows only $+1$ and -1

But since being neither "very-much-larger-in-size" nor "very-much-smaller-in-size" than the given number does make sense and will turn out to be useful, we will say that a number is **medium** when any number of copies of that number will multiply to a number that is neither "very-much-larger-in-size" nor "very-much-smaller-in-size" than the given number.

Still, really, the proverbial question is: Where do we draw the line? This, though, is a bit tricky to elucidate and we will content ourselves with the proverbial answer which is: It depends on the circumstances.

In practice, when we will want to pick a *medium number*, we will pick a number whose *size* is between 3 and 7:

- 3 because two copies of 3 multiply to 9 which is already well away from the product of two copies of 1 which is 1m
- 7 because two copies of 7 multiply to 49 which is still well away from the product of two copies of 10 which 100

order of magnitude
Multiplication

4. Up to a certain point, we can *compare* numbers that are *large* as well as numbers that are *small*.

What will do is to compare numbers that are *large*, or numbers that are *small*, according to their **order of magnitude**, that is, very loosely speaking, according to the number of copies that are being involved.

EXAMPLE 12. While 10000 and 100 are both *large*, we will say that 10000 is "larger-in-size-by-an-order-of-magnitude" than 100 because $10000 = 10 \cdot 10 \cdot 10 \cdot 10$ while $100 = 10 \cdot 10$

EXAMPLE 13. While $\frac{1}{10000}$ and $\frac{1}{100}$ are both *small*, we will say that $\frac{1}{10000}$ is "smaller-in-size-by-an-order-of-magnitude" than $\frac{1}{100}$ because $\frac{1}{10000} = \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10} \cdot \frac{1}{10}$ while $\frac{1}{100} = \frac{1}{10} \cdot \frac{1}{10}$

5. Up to a certain point too, we can also *compute* with *large*, *small* and *medium*:

a. Multiplication. Since *large*, *small* and *medium* were defined in terms of multiplication of copies, most of the following is pretty much as we would expect:

THEOREM 1 (Multiplication Size).

$large \cdot large = large$	$large \cdot medium = large$	$large \cdot small = any\ size$
$medium \cdot large = large$	$medium \cdot medium = medium$	$medium \cdot small = small$
$small \cdot large = any\ size$	$small \cdot medium = small$	$small \cdot small = small$

$large \cdot small$ and $small \cdot large$, though, present a problem because the result could be *large*, *small* or *medium* depending on how small *small* is compared to how large *large* is and we will need to look at each case separately.

EXAMPLE 14. The following are all instances of $large \cdot small$ but turn out to be all *different-in-size*:

$$1000 \cdot \frac{1}{10} = 100$$
$$1000 \cdot \frac{1}{1000} = 1$$
$$1000 \cdot \frac{1}{100000} = \frac{1}{100}$$

EXAMPLE 15. The following are all instances of $small \cdot large$ but turn out to be all *different-in-size*: Division

$$\frac{1}{1000} \cdot 100000 = 100$$
$$\frac{1}{1000} \cdot 1000 = 1$$
$$\frac{1}{1000} \cdot 10 = \frac{1}{100}$$

b. Division. Since *division* is the same as *multiplication* by the *reciprocal*, and since *large*, *small* and *medium* were defined in terms of multiplication of copies, most of the following is pretty much as we would expect:

THEOREM 2 (Division Size).

$$\frac{large}{large} = any\ size \qquad \frac{large}{medium} = large \qquad \frac{large}{small} = large$$
$$\frac{medium}{large} = small \qquad \frac{medium}{medium} = medium \qquad \frac{medium}{small} = large$$
$$\frac{small}{large} = small \qquad \frac{small}{medium} = small \qquad \frac{small}{small} = any\ size$$

$\frac{large}{large}$ and $\frac{small}{small}$, though, present a problem because the result could be *large*, *small* or *medium* depending on how small *small* is compared to how large *large* is and we will need to look at each case separately.

EXAMPLE 16. The following are all instances of $\frac{large}{large}$ but turn out to be all *different-in-size*:

$$\frac{1\,000}{10} = 100$$
$$\frac{1\,000}{1\,000} = 1$$
$$\frac{1\,000}{100\,000} = \frac{1}{100}$$

EXAMPLE 17. The following are all instances of $\frac{small}{small}$ but turn out to be all

Addition
Subtraction

different-in-size:

$$\frac{\frac{1}{1\,000}}{\frac{1}{10}} = \frac{1}{1\,000} \cdot \frac{10}{1} = \frac{10}{1\,000} = \frac{1}{100}$$

$$\frac{\frac{1}{1\,000}}{\frac{1}{1\,000}} = 1$$

$$\frac{\frac{1}{1\,000}}{\frac{1}{100\,000}} = \frac{1}{1\,000} \cdot \frac{100\,000}{1} = \frac{100\,000}{1\,000} = \frac{100}{1} = 100$$

c. Addition and **Subtraction** are tricky to deal with.

EXAMPLE 18. The following two cases are both instances of $large - large$ but the results are *different-in-size*

$$1000.1 - 900 = 100.1$$
$$1000.1 - 1000 = 0.1$$

Fortunately, we will not have to deal with either *addition* or *subtraction* in general and, in the cases that we will deal with, later on, the result will be exactly as we would expect once we will have discussed the concept of *order of magnitude* to help us.