follow up
merge

Chapter 15

Signed Addition & Subtraction

Adding a Signed Number-Phrase, 1 – Subtracting a Signed Number-Phrase, 4.

15.1 Adding a Signed Number-Phrase

We investigate the *second* fundamental process involving *actions* and *states*.

1. Just as in in the case of *collections* we could *attach* a *second* collection to a *first* collection, here we can

- **follow up** a *first* action with a *second* action.
  EXAMPLE 1.
  - a gambler may *win* forty-five dollars and then follow up with *winning* sixty-two dollars.
  - a gambler may *win* thirty-one dollars and then follow up with *losing* forty-four dollars.
  - a gambler may *lose* twenty-one dollars and then follow up with *winning* fifty-seven dollars.
  - a gambler may *lose* seventy-eight dollars and then follow up with *losing* thirty-four dollars.
- **merge** a *first* state with a *second* state
  EXAMPLE 2.
  - a business that is three thousand dollars *in the black* may merge with a business that is six hundred dollars *in the black*.

adding
$\oplus$

- a business that is three hundred dollars *in the black* may merge with a business that is five hundred dollars *in the red*.
- a business that is two thousand dollars *in the red* may merge with a business that is seven hundred dollars *in the black*.
- a business that is seven hundred dollars *in the red* may merge with a business that is two hundred dollars *in the red*.

NOTE. English forces us to use a different word order here: while we attached a *second* collection *to* a *first* collection, here we must say that we follow up a *first* action *with* a *second* action. In order to be consistent, and although it is not necessary, we will also say that we merge a *first* state *with* a *second* state.

2. Then, just like *adding a counting-number-phrases* was the paper procedure to get the result of *attaching* a collection, **adding** a *signed number-phrase* will be the paper procedure to get the *result* of *following up* an action and/or *merging* a state.
In order to distinguish adding *signed* number-phrases from adding *counting* number-phrases as we develop the procedure, we shall use for a while the symbol $\oplus$. Later, we will just use $+$ and learn to rely on the *context*.

3. Just like, in Chapter 1, we introduced *counting* number-phrases with *slashes*, /, to discuss addition of *signed* number-phrases, we will use temporarily *arrows* of two kinds, $\leftarrow$ and $\rightarrow$.

EXAMPLE 3. We will use temporarily

$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ **Dollars** instead of $+5$ **Dollars**

and

$\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$ **Dollars** instead of -5 **Dollars**.

When adding a signed number-phrase, we must distinguish two cases.

a. The second signed number-phrase has the *same* sign as the first signed number-phrase. Then, all the items are of the *same kind* and so *following up* is the same as *attaching*. So, in that case, to get the *size* of the result, we add the *sizes* of the two signed number-phrases.

EXAMPLE 4.

In the *real-world*, when we:	We write on *paper*:
deposit five dollars	$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ **Dollars**
and then	$\oplus$
deposit three dollars,	$\rightarrow \rightarrow \rightarrow$ **Dollars**
altogether	$=$
this	$[\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \oplus \rightarrow \rightarrow \rightarrow]$ **Dollars**
is the same as	$=$
when	$\rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow \rightarrow$ **Dollars**
we	$=$
deposit eight dollars	$+8$ **Dollars**

or

EXAMPLE 5.

In the *real-world*, when we	We write on *paper*:
withdraw five dollars	$\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$ **Dollars**
and then	$\oplus$
withdraw three dollars,	$\leftarrow \leftarrow \leftarrow$ **Dollars**
altogether	$=$
this	$[\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \oplus \leftarrow \leftarrow \leftarrow]$ **Dollars**
is the same as	$=$
when	$\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$ **Dollars**
we	$=$
withdraw eight dollars	-8 **Dollars**

b. The second signed number-phrase has the *opposite* sign from the first signed number-phrase. Then, the items are of the *same kind* and so *following up* is the same as *attaching*. So, in that case, to get the *size* of the result, we add the *sizes* of the two signed number-phrases.

EXAMPLE 6.

In the *real-world*, when we	We write on *paper*:
deposit three dollars	$\rightarrow \rightarrow \rightarrow$ **Dollars**
and then	$\oplus$
withdraw five dollars,	$\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$ **Dollars**
altogether	$=$
this	$[\rightarrow \rightarrow \rightarrow \oplus \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow]$ **Dollars**
is	$[\rightarrow \rightarrow \rightarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow]$ **Dollars**
the same	$[\rightarrow \rightarrow \not\!/\!/ \not\!/\!/ \leftarrow \leftarrow \leftarrow \leftarrow]$ **Dollars**
as	$[\rightarrow \not\!/\!/ \quad \not\!/\!/ \leftarrow \leftarrow \leftarrow]$ **Dollars**
when	$[\not\!/\!/ \quad \not\!/\!/ \leftarrow \leftarrow]$ **Dollars**
we just	$\leftarrow \leftarrow$ **Dollars**
withdraw two dollars	-2 **Dollars**

or

EXAMPLE 7.

In the *real-world*, when we	We write on *paper*:
deposit three dollars	$\rightarrow \rightarrow \rightarrow$ **Dollars**
and then	$\oplus$
withdraw five dollars,	$\leftarrow \leftarrow \leftarrow \leftarrow \leftarrow$ **Dollars**
altogether	$=$
this	$[\rightarrow \rightarrow \rightarrow \oplus \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow]$ **Dollars**
is	$[\rightarrow \rightarrow \rightarrow \leftarrow \leftarrow \leftarrow \leftarrow \leftarrow]$ **Dollars**
the same	$[\rightarrow \rightarrow \not\!/\!/ \not\!/\!/ \leftarrow \leftarrow \leftarrow \leftarrow]$ **Dollars**
as	$[\rightarrow \not\!/\!/ \qquad \not\!/\!/ \leftarrow \leftarrow \leftarrow]$ **Dollars**
when	$[\not\!/\!/ \qquad \not\!/\!/ \leftarrow \leftarrow]$ **Dollars**
we just	$\leftarrow \leftarrow$ **Dollars**
withdraw two dollars	-2 **Dollars**

THEOREM 1. *To add signed-numerators:*

- *When the two signed number-phrases have the* same *sign,*
  - *We get the* sign *of the result by taking the common sign*
  - *We get the* size *of the result by* adding *the two sizes.*
- *When the two signed number-phrase have* opposite *signs, we must first* compare *the* sizes *of the two signed number-phrases and then*
  - *We get the* sign *of the result by taking the sign of the signed number-phrase whose* size *is* larger*,*
  - *We get the* size *of the result by* subtracting *the smaller size from the larger size.*

EXAMPLE 8. To identify the specifying-phrase $(+3)\oplus(+5)$ and since $(+3)$ and $(+5)$ have the *same* sign, we *proceed* as follows:

- We get the *sign* of the result by taking the common sign which gives us $+$
- We get the *size* of the result by *adding* the sizes 3 and 8 which gives us 8

In symbols,

$$\begin{aligned}(+3)\oplus(+5) &= (+[3+5])\\ &= (+8)\end{aligned}$$

EXAMPLE 9. To identify the specifying-phrase $(+3)\oplus(-5)$ and since $(+3)$ and (-5) have *opposite* signs, we must compare the *sizes*. Since $3<5$,

- We get the *sign* of the result by taking the sign of the number-phrase with the larger *size* which gives us $-$
- We get the *size* of the result by subtracting the smaller size, 3, from the larger size, 5 which gives us 2

In symbols,

$$(+3)\oplus(-5) = (-[5-3])$$

$$= (-2)$$

15.2 Subtracting a Signed Number-Phrase

We investigate the *third* fundamental process involving *actions* and *states*.

While, in the case of *collections*, *detaching* a collection made immediate sense as "un-attaching", in the case of actions "un-following up" and in the case of states "un-merging" do not make immediate sense. So, instead, we shall look at subtraction from the point of view of *correction* after we have done a long string of signed-additions and realized that there is an *incorrect entry*, that is a signed number-phrase that we shouldn't have added (for whatever reason), so that the *total* is *incorrect*.

1. Up front, things would seem to work out exactly as in the case of un-signed number-phrases.

EXAMPLE 10. Suppose that we work in a bank and that we had added transactions as the day went which gave us the following specifying phrase

-2 **Dollars** $\oplus$ -7 **Dollars** $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars** and that at the end of day we identified the specifying-phrase which gave us

$$-132 \textbf{ Dollars}$$

but that we then realized that -7 **Dollars** was an *outcast* (it was not for a *transaction* but for money involved in some other matter) with the consequence that -132 **Dollars** is *incorrect* in that it is not the sum total of the *transaction* for the day.

2. To get the correct total, we have the following two choices for the procedure:

- **Procedure A** would be to *strike out* the incorrect signed number-phrase and *redo* the entire addition:

  EXAMPLE 11. In the above example, we would *strike out* the *incorrect entry* -7 **Dollars**

  -2 **Dollars** $\oplus$ ~~-7 Dollars~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

  Of course, since Procedure A is going to involve a lot of unnecessary work redoing all that had been done correctly, it is very inefficient.
- **Procedure B** would be to *cancel out* the *effect* of the incorrect entry on the incorrect total by *subtracting* the incorrect entry from the incorrect total.

  EXAMPLE 12. In the above example, we would *subtract* the incorrect entry -7 **Dollars** from the incorrect total -132 **Dollars**

  -132 **Dollars** $\ominus$ -7 **Dollars**

$\ominus$
add the opposite
subtract

except that, at this point, we have no *procedure* for $\ominus$! Indeed, at this point, the only procedure we have for subtracting is for subtracting *unsigned* number-phrases.

On the other hand, the obvious way to *cancel out* the *effect* of the incorrect entry on the incorrect total and that it is by **adding the opposite** of the incorrect entry to the incorrect total. (Accountants call this "entering an *adjustment*".)

EXAMPLE 13. In the above example, we would *add the opposite* of the incorrect entry -7 **Dollars**, that is we would add -7 **Dollars** to the incorrect total -132 **Dollars**

-132 **Dollars** $\oplus$ $+7$ **Dollars**

3. We now want to *see* that the two procedures *must* give us the same result either way. For that, we place the specifying-phrases in the two procedures side by side and we see that that the remaining number-phrases are the same either way.

EXAMPLE 14. In the above example, we place the specifying-phrases in the two procedures side by side:

- The specifying-phrase in **Procedure A** is:

  -2 **Dollars** $\oplus$ ~~$+7$ Dollars~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

- The specifying-phrase in **Procedure B** is:

  -2 **Dollars** $\oplus$ ~~-7 Dollars~~ $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars** $\oplus$ ~~$+7$ Dollars~~

We see that, either way, the remaining number-phrases are:

-2 **Dollars** $\oplus$ $+5$ **Dollars** $\oplus$ ... $\oplus$ $+3$ **Dollars**

4. Altogether then:

- *Adding the opposite* of the incorrect entry (**Procedure B**):

$$-132 \text{ Dollars} \quad \oplus \quad +7 \text{ Dollars}$$

necessarily amounts to exactly the same as

- *Striking out* the incorrect entry (**Procedure A**):

$$-132 \text{ Dollars} \quad \ominus \quad -7 \text{ Dollars}$$

Since **Procedure B** is much faster than **Procedure A**, we say that the procedure for **subtracting** a signed number-phrase will be to *add its opposite*.

EXAMPLE 15. In order to identify the specifying-phrase $(+3)\ominus(+5)$,

i. we identify instead the specifying-phrase $(+3)\oplus(-5)$

ii. we do the addition which gives us -2

EXAMPLE 16. In order to identify the specifying-phrase $(-3)\ominus(-5)$,

i. we identify instead the specifying-phrase $(-3)\oplus(+5)$

ii. we do the addition which gives us $+2$

EXAMPLE 17. In order to identify the specifying-phrase $(-3)\ominus(+5)$,

i. we identify instead the specifying-phrase $(-3)\oplus(-5)$

ii. we do the addition which gives us -8

EXAMPLE 18. In order to identify the specifying-phrase $(+3)\ominus(-5)$,

i. we identify instead the specifying-phrase $(+3)\oplus(+5)$

ii. we do the addition which gives us $+8$

15.3 From Plain To Positive

We now have two kinds of number-phrases: *plain* number-phrases and *signed* number-phrases. The two, though, overlap and we want to analyze the connections between the two and what is gained when we go from using *plain* number-phrases to using *signed* number-phrases.

1. We developed

- *plain* number-phrases in order to deal with collections of items that are all of *one* kind,
- *signed* number-phrases in order to deal with collections of items that are all of one kind or all of another kind—with items of different kinds canceling each other.

But then, given collections of items that are all of *one* kind, it often happens that we can eventually think of another kind of items that cancel the first kind of items.

EXAMPLE 19. We may start *counting* steps to find out *how much we walked*. But eventually, we may want to know *how far we progressed*, being that there are steps *backward* as well as step *forward* and, if it doesn't matter what kind of steps they are when it comes to *how much we walked*, it does matter very much when it comes to *how far we progressed* and so we need to keep track of the direction of the steps.

2. But then, we can represent the original collection of items in two ways:

- With a *plain* number-phrase
- With a *positive* number-phrase

EXAMPLE 20. Given a collection of seven steps (necessarily all in the same direction since all items in a collection have to be the same), we can represent the collection by:

- the plain number-phrase

  7 **Steps**

- or we can adopt that direction as *standard direction* and then represent the collection by the *positive* number-phrase

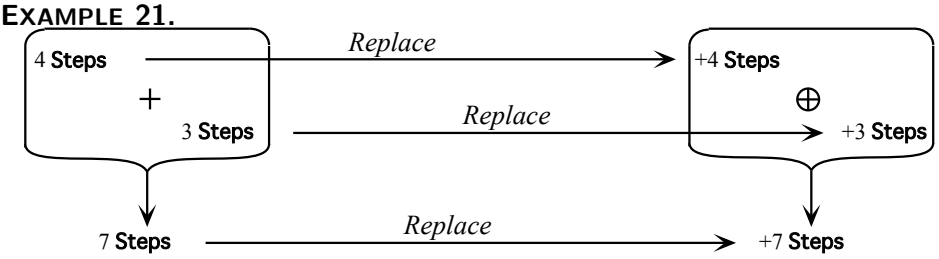
  $+7$ **Steps**

3. We now check that, when we do an addition, we can go either one of two routes:

- We can first *replace* the two *plain* number-phrases by *positive* number-phrases and then oplus the two *positive* number-phrases,
- We can add the two *plain* number-phrases and then *replace* the result of the addition by a *positive* number-phrase.

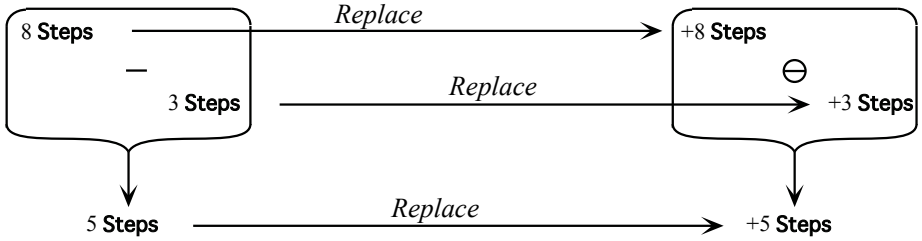
Both routes get us to the same result.

EXAMPLE 21.



This works also with *subtraction*.

EXAMPLE 22.



NOTE. The reader should check on her/his own that if, instead of *replacing plain* number-phrases by *positive* number-phrases, we were to *replace plain* number-phrases by *negative* number-phrases, then things would not always work in the sense that the two routes would not always result with the same number-phrase.