cancel
two-way collections
action
step

Chapter 13

Signed Number-Phrases

13.1 Two-way Collections

Quite often we don't deal with items that are all of the same kind but with items of two different kinds and a special case of this is when two items of different kinds cannot be together as they somehow **cancel** each other. As a result, we will now consider what we shall call **two-way collections**, that is collections of items that are all of one kind or all of another kind with items of different kinds canceling each other.

1. In the real-world, *two-way collections* come up very frequently and in many different types of situations but they generally fall in either one of two types:

- In one type of two-way collections, called **actions**, the items are **steps** in either *this-direction* or *that-direction*.

  Example 1. In fact, we already encountered in the previous chapter this kind of items: counting *up* and counting *down*. Of course, the situation there was not symmetrical: we could always count steps *up* but we could not always count steps *down*. But there would have been no point counting at the same time three steps up and five steps down since steps up would cancel out steps down and this would have just amounted to counting two steps down.

  Example 2.

  – Actions that a businesswoman may take on a bank account are to *deposit*

state
degree
benchmark
nature (of an action)
nature (of a state)
extent (of an action)
size (of a state)
direction (of an action)
side (of a state)
plain number-phrases

three thousand dollars, *withdraw* two thousand dollars, etc
  - Actions that a gambler may take are to *win* fifty-eight dollars, *lose* sixty-two dollars, etc
  - Actions that a mark may take on a horizontal line include moving two feet *leftward*, five feet *rightward*, etc.
  - Actions that a mark may take on a vertical line include moving five inches *upward*, five inches *downward*, etc.
- In the other type of two-way collections, called **states**, the items are **degrees** of one kind or another but they have to be either on *this-side* or *that-side* of some **benchmark**.

  Example 3.
  - States that a business may be in include being three thousand dollars *in the red*, being seven thousand dollars *in the black*, etc.
  - States that a gambler may be in include being sixty-two dollars *ahead of the game*, being thirty-seven dollars *in the hole*, etc.
  - States that a mark may be in on a horizontal line with some benchmark include being two feet *to the left* of the benchmark, being nine feet *to the right* of the benchmark, etc.
  - States that a mark may be in on a vertical line with some benchmark include being five inches *above* the benchmark, being three inches *below* the benchmark, etc.

2. Since all the items in a given two-way collection are of the same kind, a two-way collection is essentially a collection with a twist. So, just as we said that, in the real world,

- the *nature of a collection* is the *kind* of items in the collection,
- the *extent of a collection* is the *number* of items in the collection,

we shall now say that:

- the **nature of an action** is the *kind* of steps in the action and the **nature of a state** is the *kind* of degrees in which the state can be
- the **extent of an action** is the *number* of steps in the action and the **size of a state** is the *number* of degrees of the state.
- the **direction of an action** is the *direction* of the steps in the action and the **side of a state** is the *side* of the degrees in the state.

  Example 4. When a person climbs up and down a ladder, an *action* may be climbing up seven rungs. Then,
  - the *nature* of the action is *climbing rungs*
  - the *size* of the action is *seven*
  - the *direction* is *up*

We have seen in Chapter 1 that we can use **plain number-phrases**, that is either *counting* number-phrases or *decimal* number-phrases, only in situations where the items are all of the same *one* kind. We shall now introduce and discuss a *new type* of number-phrase that we shall use in a type of situations that occurs frequently in which the items are all of *either one of two* kinds.

initial state
final state
change
gain
loss

Just as we did for *plain* number-phrases in Chapters 2, 3, and 4, we will have to *define* for this *new type* of number-phrase what we mean by:

i. To "compare" two number-phrases,

ii. To "add" a second number-phrase to a first number-phrase,

iii. To "subtract" a second number-phrase from a first number-phrase.

and in particular to develop the corresponding *procedures.*

What will complicate matters a little bit, though, is that the procedures for the *new type* of number-phrases will involve the procedure that we developed for *plain* number-phrases. So, until we feel completely comfortable with the distinction, we shall use new symbols for "comparison", "addition" and "subtraction" for the new kind of number-phrases [1].

13.2 Effect Of An Action On A State

We now look at the connection between *states* and *actions.*

1. A *state* does not exist in isolation but is always one of many.

EXAMPLE 5. The *state* of an account is usually different on different days.

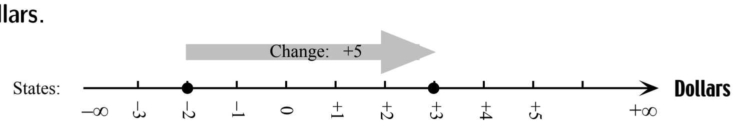
Given two states, we shall refer to the first one as the **initial state** and to the second one as the **final state**. The **change** from the *initial* state to the *final* state can be *up* in which case we shall call the change a **gain** or can be *down* in which case we shall call the change a **loss**.

On paper, we shall use + for a *gain* and we shall use − for a *loss.*

EXAMPLE 6.

- At the *beginning* of a month, Jill's account was two dollars in-the-red
- At the *end* of the month, Jill's account was three dollars in-the-black

So, during that month Jill's account went *up* by five dollars and we shall write the *gain* as $+5$ **Dollars**.

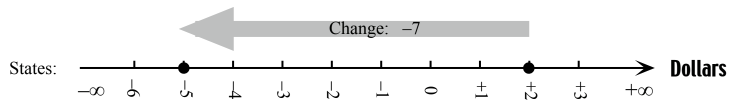


EXAMPLE 7.

- At the *beginning* of a month, Jack's account was two dollars in-the-black
- At the *end* of the month, Jack's account was five dollars in-the-red

So, during that month Jack's account went *down* by seven dollars and we shall write the *loss* as -7 **Dollars**.

[1] One can only wonder as to how Educologists can let their students use, without warning, the same symbols in these rather different situations.



THEOREM 1. *Regardless of what the sign of the initial state and the sign of the final state are, we have that*

$$\textit{change} = \textit{final state} \ominus \textit{initial state}$$

EXAMPLE 8.

- At the *beginning* of a month, Jill's account was two dollars in-the-red
- At the *end* of the month, Jill's account was three dollars in-the-black

$$\begin{aligned} \text{change} &= +3 \textbf{ Dollars } \ominus\ -2 \textbf{ Dollars} \\ &= +3 \textbf{ Dollars } \oplus\ +2 \textbf{ Dollars} \\ &= +5 \textbf{ Dollars} \end{aligned}$$

EXAMPLE 9.

- At the beginning of a month, Jack's account was two dollars in-the-black
- At the end of the month, Jack's account was five dollars in-the-red

$$\begin{aligned} \text{change} &= -5 \textbf{ Dollars } \ominus\ +2 \textbf{ Dollars} \\ &= -5 \textbf{ Dollars } \oplus\ -2 \textbf{ Dollars} \\ &= -7 \textbf{ Dollars} \end{aligned}$$

2. A *change* always happens as the result of an *action*.

EXAMPLE 10. On an account,

- A *deposit* results in a *gain*,
- A *withdrawal* results in a *loss*.

In fact, we have exactly

$$\text{action} = \text{change}$$

so that, as a consequence of the previous THEOREM, *actions* and *states* are related as follows:

THEOREM 2 (Conservation Theorem)**.**

$$\textit{action} = \textit{final state} \ominus \textit{initial state}$$

EXAMPLE 11.

signed number-phrase

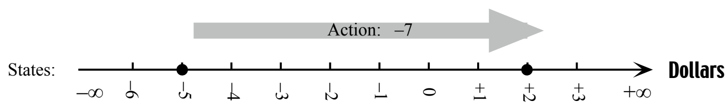
- On Monday, Jill's account was five dollars *in-the-red*,
- On Tuesday, Jill *deposits* seven dollars.

So, we have:

i.

$$\textbf{Action} = +7 \textbf{ Dollars}$$

ii.



So, on Wednesday, Jill's account is two dollars *in-the-black*

iii. Then we compute the change:

$$\begin{aligned}\textbf{Change} &= \textbf{Final State} \ominus \textbf{Initial State}\\ &= +2 \textbf{ Dollars} \ominus -5 \textbf{ Dollars}\\ &= +2 \textbf{ Dollars} \oplus +5 \textbf{ Dollars}\\ &= +7 \textbf{ Dollars}\end{aligned}$$

And we have indeed that

$$\text{action} = \text{final state} - \text{initial state}$$

What happened is that each state is the result of *all prior* actions. So, by subtracting the *initial* state from the final state, we eliminate the effect of all the actions that resulted in the *initial* state, that is the effect of all the actions except the effect of the last one, namely the seven dollars deposit.

13.3 Signed Number-Phrases

Plain number-phrases are not sufficient to represent on paper either *actions* or *states* because they do not indicate the *direction* of the action or the *side* of the state.

Example 12.

– 3000 **Dollars** does not say if the businesswoman made a deposit or a withdrawal or if the business is in the red or in the black.
– 62 **Dollars** does not say if the gambler is ahead of the game or in the hole.
– 2 **Feet** does not say if the mark is to the left or to the right of the benchmark.
– 5 **Inches** does not say if the mark is moving up or down.

1. Since a two-way collection is just a collection with a *direction* or a *side*, we will represent on paper a two-way collection by a **signed number-phrase** that will consist of:

- a *denominator* to represent on paper the *nature* of the action (that is the *kind* of the steps in the action) or of the state (that is the *kind* of the degrees in the state).

record
standard direction
opposite direction
standard side
opposite side
sign
+
positive
−
negative
context
signed-numerator
positive numerators
negative numerators

- a *numerator* to represent on paper the *extent* of the action (that is the *number* of steps in the action) or the *extent* of the state (that is the *number* of degrees in the state),
- a *sign* to represent on paper the *direction* of the action (that is the *direction* of the steps in the action) or the *side* of the state (that is the *side* of the benchmark that the degrees of the state are on.)

2. However, in order to say what direction the action or what side the state, we must always begin by **recording** for future reference:

- which direction is to be the **standard direction** and which direction is therefore to be the **opposite direction**,
- which side of the benchmark is going to be the **standard side** and which side is therefore to be the **opposite side**,

NOTE. Historically, it has long gone without saying that *standard* was what was "good" and *opposite* what was "bad".

EXAMPLE 13.

- To *deposit* money is usually considered to be "good" as it goes with *saving* while to *withdraw* money is usually considered to be "bad" as it goes with *spending*.
- To *win* is usually considered to be "good" while to *lose* is considered to be "bad".
- To go *up* is usually considered to be "good" while to go *down* is usually considered to be "bad".

3. Once we have recorded what is *standard* and therefore what is *opposite*, we can use a **sign** to represent on paper the *direction* of the action (that is the direction of the steps in the action) or the *side* of the state (that is the side of the benchmark that the degrees of the state are on):

- we will use the sign $+$, read here as **positive**, to represent on paper whatever is *standard*, whether an action or a state.
- we will use the sign $-$, read here as **negative**, to represent on paper whatever is *opposite*, whether an action or a state.

NOTE. This use of the symbols $+$ and $-$ is entirely different from their use in Chapter 1 where they denoted *addition* and *subtraction*. This complicates *reading* the symbol as we need to rely on the **context**, that is the text that is around the symbol, to decide what the symbol stands for.

4. However, because this will make developing and using *procedures* a lot easier, we will lump the *sign* together with the *numerator* and call the result a **signed-numerator**. Signed-numerator with a $+$ are said to be **positive numerators** and signed-numerators with a $-$ are said to be **negative numerators**. **NOTE.** Historically, just as with standard and opposite and perhaps as a result, *positive* has been identified with "good" and *negative* with "bad".

So, altogether, a *signed* number-phrase will consist of:

- a signed-numerator
- a denominator

sign, of the numerator
size

EXAMPLE 14. Say that we have put on record that the *standard* direction is to *win* money so that to *lose* money is the *opposite* direction. Then,

When a *real-world* gambler:	We write on *paper*:
• *wins* forty-seven dollars	$+47$ **Dollars**
• *loses* sixty-two dollars	-62 **Dollars**

EXAMPLE 15. Say we have put on record that the *standard* side is *in-the-black* so that *in-the-red* is the *opposite* side. Then,

When a *real-world* business is:	We write on *paper*:
• three thousand dollars *in-the-black*	$+3000$ **Dollars**
• seven hundred dollars *in-the-red*	-700 **Dollars**

5. We are using the same symbol, 0, both for

- the counting numerator that is left of the succession of counting numerators $1, 2, 3, 4, \ldots$
- the signed numerator which is inbetween the succession of positive numerators $+1, +2, +3, +4, \ldots$ and the recession of negative numerators $-1, -2, -3, -4, \ldots$.

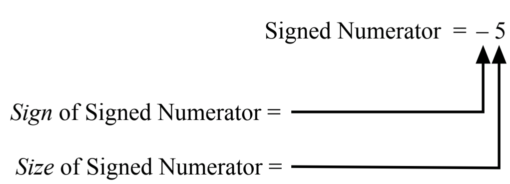
In this case, we shall have to live with the ambiguiity and decide each time, according to the context, which one the numerator 0 really is.

13.4 Size And Sign

On the other hand, given a *signed numerator*, we shall say that:

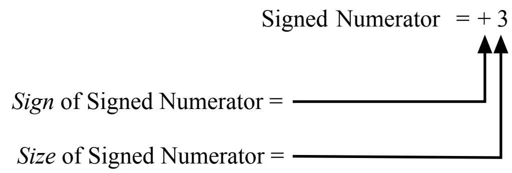
- the **sign of the numerator** is the sign which was put in front of the plain numerator to make the signed numerator
- the **size** of the numerator is the plain numerator from which the signed numerator was made.

EXAMPLE 16.



In other words, -5 is a signed-numerator whose *size* is 5 and whose *sign* is $-$.

EXAMPLE 17.



In other words, $+3$ is a signed-numerator whose *size* is 3 and whose *sign* is $+$.

Indeed, signed number-phrases can contain more information than is necessary for a particular purpose and then all we need is either the *sign* or the *size* of the signed number-phrase.

1. In many circumstances, what matters is only the *size* of the signed number-phrases and not the *sign*.

EXAMPLE 18. Say we are told that

- Jill's balance is $+70,000,000$ **Dollars**
- Jack's balance is $-70,000,000$ **Dollars**.

We can safely conclude that neither Jack nor Jill belongs to "the rest of us".

EXAMPLE 19. If we are stopped on the turnpike doing $+100$ $\frac{\textbf{Miles}}{\textbf{Hour}}$, that is while driving from Philadelphia to New York, or doing -100 $\frac{\textbf{Miles}}{\textbf{Hour}}$ that is while driving back from New York to Philadelphia , it does not matter which way we were going: regardless of the *direction*, we are going to get into big trouble.

So, in such cases, it is the *size* of the given *signed numerator* that matters.

EXAMPLE 20. The *size* of Jill's $+70,000,000$ **Dollars** is $70,000.000$ and the *size* of Jack's $-70,000,000$ **Dollars** is also $70,000,000$ **Dollars**.

So, what makes Jack and Jill different from "the rest of us" is the *size* of their balance and not its *sign*.

EXAMPLE 21. The *size* of our speed when we are going $+100$ $\frac{\textbf{Miles}}{\textbf{Hour}}$ (that is from Philadelphia to New York) is 100 $\frac{\textbf{Miles}}{\textbf{Hour}}$ and the *size* of our speed when we are going -100 $\frac{\textbf{Miles}}{\textbf{Hour}}$ (that is from New York to Philadelphia) is also 100 $\frac{\textbf{Miles}}{\textbf{Hour}}$.

So, what gets us into trouble is the *size* of our speed.

2. In many other circumstances, what matters is only the *sign* of the signed number-phrase and not the *numerator*.

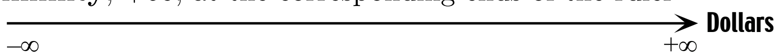
EXAMPLE 22. Usually, banks do not accept *negative* balances, regardless of their *size*. In other words, all bank care about is the *sign* of the balance.

EXAMPLE 23. If we are stopped going the wrong way on a one way street, it won't matter if we were well under the speed limit. In other words, what gets us into trouble is the *sign* of our speed and not its *size*.

13.5 Graphic Illustrations

To *graph* a *two-way collection* represented on paper by a *signed number-phrase*, we proceed essentially just as with counting number-phrases and/or decimal number-phrases. The only differences are that on a **signed ruler**:

- we shall have the symbol for **minus infinity**, $-\infty$, and the symbol for **plus infinity**, $+\infty$, at the corresponding ends of the ruler

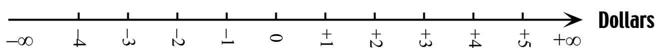


- the tick-marks, if any, are labeled with *signed number-phrases*.

signed ruler
minus infinity
$-\infty$
plus infinity
$+\infty$

As with all rulers and depending on the circumstances, 0 may or may not appear.

EXAMPLE 24.



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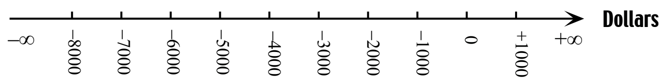

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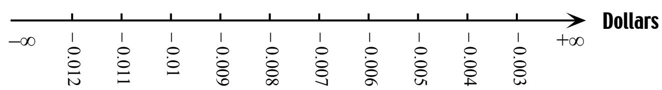
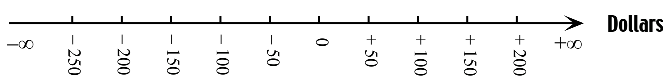


EXAMPLE 27.



EXAMPLE 28.