

## MATH 161 REVIEW II Discussions

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[ Run: 10/23/2011 at 19:56 Seed: 6544. Order of Checkable Items: List.]

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- II-1.** Given the *affine* function  $HERB$  such that  $HERB(-1) = +5$  and  $HERB(+3) = +15$

$$\text{AND} \begin{cases} \text{Height } HERB |_{-1} = +5 \\ \text{Height } HERB |_{+3} = +15 \end{cases}$$

find Slope  $HERB$ , that is the slope of the global graph that specifies  $HERB$ .

**Discussion:** Since we are being asked about the slope of a *function* this is a GLOBAL question.

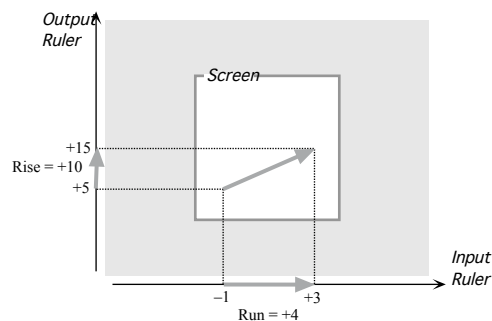
In fact, that the function  $f$  is given to be *affine* is absolutely crucial because only for *affine* functions does the graph have a *constant slope*.

a. When we go from input  $-1$  to input  $+3$ , we *run*  $+4$

b. When we go from the corresponding output  $+5$  to the corresponding output  $+15$ , we *rise*  $+10$ .

c. The slope is therefore  $\frac{+10}{+4} = +\frac{5}{2}$

Drawing a sketch helps ensuring that we count *run* and *rise* "the same way", that is from the same starting plot point to the same ending plot point.



- II-2.** Given that the function  $DONA$  is *affine* and given the *boundary value conditions*:

$$\text{AND} \begin{cases} DONA(x)|_{x \leftarrow +4} = -1 \\ DONA(x)|_{x \leftarrow -2} = +2 \end{cases}$$

find the *global input-output rule* that specifies *DONA*

**Discussion:** Since we are being asked for the *global input-output rule* that specifies *DONA*, this is a *global* question which we will try to answer using the given information.

a. Since *DONA* is given to be an *affine function*, the global input-output rule of *DONA* is of the form:

$$x \xrightarrow{DONA} DONA(x) = ax + b$$

b. We use the information we have about the two inputs:

- Using the input +4, the global input-output rule gives

$$\begin{aligned} x|_{x \leftarrow +4} \xrightarrow{DONA} DONA(x)|_{x \leftarrow +4} &= ax + b|_{x \leftarrow +4} \\ &= a(+4) + b \end{aligned}$$

that is

$$+4 \xrightarrow{DONA} +4a + b$$

and since the first boundary condition was that

$$+4 \xrightarrow{DONA} -1$$

because one input can have at most one output, we must have:

$$+4a + b = -1$$

- Using the input -2, the global input-output rule gives

$$\begin{aligned} x|_{x \leftarrow -2} \xrightarrow{DONA} DONA(x)|_{x \leftarrow -2} &= ax + b|_{x \leftarrow -2} \\ &= a(-2) + b \end{aligned}$$

that is

$$-2 \xrightarrow{DONA} -2a + b$$

and since the second boundary condition was that

$$-2 \xrightarrow{DONA} = +2$$

because one input can have at most one output, we must have:

$$-2a + b = +2$$

c. We now solve the double equation problem where the unknowns are  $a$  and  $b$ :

$$\text{BOTH} \begin{cases} +4a + b = -1 \\ -2a + b = +2 \end{cases}$$

Given that both equations involve  $b$  with the same coefficient, namely 1, subtracting one from the other will eliminate  $b$ :

$$[+4a + b] - [-2a + b] = [-1] - [+2]$$

and since subtracting means adding the opposite

$$\begin{aligned} [+4a + b] + [+2a - b] &= [-1] + [-2] \\ +4a + b + 2a - b &= -1 - 2 \\ +6a &= -3 \\ a &= \frac{-3}{+6} \\ a &= -\frac{1}{2} \end{aligned}$$

whose solution is  $-\frac{1}{2}$

To get  $b$  we can substitute  $-\frac{1}{2}$  for  $a$  in either equation. We use the first one:

$$\begin{aligned} +4a + b|_{a=-\frac{1}{2}} &= -1 \\ +4 \left[ -\frac{1}{2} \right] + b &= -1 \\ -2 + b &= -1 \\ b &= -3 \end{aligned}$$

whose solution is  $-3$

And so the global input-output rule that specifies  $DONA$  is

$$x \xrightarrow{DONA} DONA(x) = -\frac{1}{2}x - 3$$

**II-3.** Given the affine function  $JADIH$  whose global rule is

$$x \xrightarrow{JADIH} JADIH(x) = -\frac{2}{3}x + 3$$

find its global graph.

**Discussion:** Since we are being asked for the global graph of a function this is a GLOBAL question.

There are two ways we can proceed:

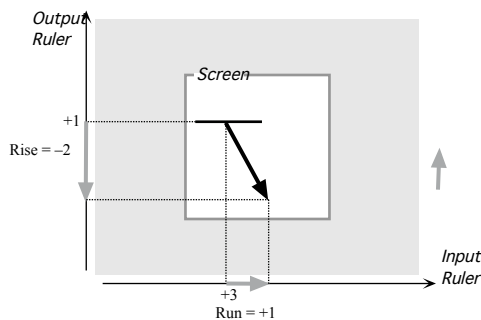
a. We can look at this as an **Initial Value Problem**, that is we can "pick" an input, say  $+3$ , localize near  $+3$  and construct the local graph:

$$+3 + h \xrightarrow{f} f(x)|_{x=+3+h} = -\frac{2}{3}x + 3 \Big|_{x=+3+h}$$

which gives after the *standard computation*:

$$= [+1]h^0 + [-2]h^1$$

and therefore the local graph where we picked the run equal to  $+1$  so that for the slope to be  $-2$  the rise had to be  $-2$ :



We can then extrapolate the local graph because of the THEOREM that says that the graph of an affine function is a straight line.

b. We can look at this as a **Boundary Value Problem**, that is:

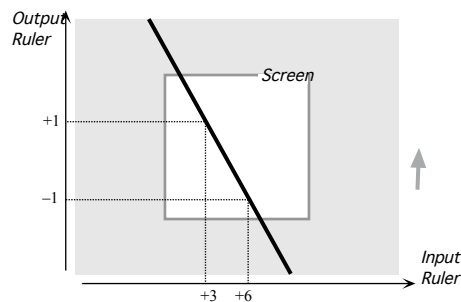
- i. We "pick" two inputs, say  $+3$  and  $+6$ ,
- ii. We compute their outputs,

$$+3 \xrightarrow{f} f(x)|_{x=+3} = -\frac{2}{3}x + 3 \Big|_{x=+3}$$

$$= +1$$

$$+6 \xrightarrow{f} f(x)|_{x=+6} = -\frac{2}{3}x + 3 \Big|_{x=+6} \\ = -1$$

iii. We plot the points and draw a straight line through the two plot points because of the THEOREM that says that the graph of an affine function is a straight line:



II-4. Given the function *CRIC* whose global input-output rule is

$$x \xrightarrow{CRIC} CRIC(x) = -3x - 12$$

find its 0-height input(s) if any.

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *output* has to be = 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. Find the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i. we *localize* near an *unspecified* input  $x_0$
  - ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, = 0, < 0 or > 0,

b. Solve the resulting equation/inequation. In the case of an *in-equation*, we always proceed in two steps:

- i. we *solve* the associated equation to get the *break-even inputs*,
- ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be  $= 0$ , we set the *equation* directly. We have

$$x \xrightarrow{f} f(x) = -4x - 6$$

and since we must have

$$f(x) = 0$$

the equation that we must solve is

$$-4x - 6 = 0$$

- b. Solving gives as only solution  $x_{0\text{-output}} = -\frac{3}{2}$

**II-5.** Given the function *FILO* whose global input-output rule is

$$x \xrightarrow{FILO} FILO(x) = +8x + 1$$

and the function *GREG* whose global input-output rule is

$$x \xrightarrow{GREG} GREG(x) = +5x - 2$$

find the inputs, if any, for which  $FILO(x) < GREG(x)$ .

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *output* of  $f$  has to be smaller than the output of  $g$ ), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i. we *localize* near an *unspecified* input  $x_0$

**ii.** we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign,  $= 0$ ,  $< 0$  or  $> 0$ ,

**b.** *Solve* the resulting equation/inequation. In the case of an *inequation*, we always proceed in two steps:

**i.** we *solve* the associated equation to get the *break-even inputs*,

**ii.** we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

**a.** We find the equation/inequation: Since we are looking for the inputs for which the *output* of  $f$  is *smaller* than the *output* of  $g$ . So, we set the *inequation* directly. We have:

$$x \xrightarrow{f} f(x) = +8x + 1$$

and

$$x \xrightarrow{g} g(x) = +5x - 2$$

and since we must have

$$f(x) < g(x)$$

the inequation that we must solve is

$$+8x + 1 < +5x - 2$$

**b.** We *solve* the inequation in two steps.

**i.** We *solve* the associated equation to get the *break-even* input(s), if any:

$$+8x + 1 = +5x - 2$$

We add  $-5x - 1$  on both sides:

$$-5x - 1 = -5x - 1$$

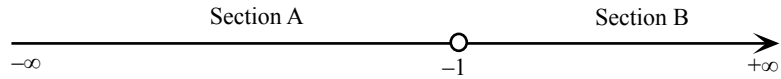
We get

$$+3x = -3$$

and therefore

$$x = -1$$

ii. We *test* an input in each one of the sections determined by the break-even point  $-1$  for the sign of the outputs:



- To test Section A, we use a number near  $-\infty$ , say  $-1000$ :

$$f(x)|_{x=-1000} = +8x + 1|_{x=-1000} = -8000 + (\dots)$$

$$g(x)|_{x=-1000} = -5x - 1|_{x=-1000} = +5000 + (\dots)$$

So, in Section A we have that  $f(x) < g(x)$  is TRUE.

- To test Section B, we use a number near  $+\infty$ , say  $+1000$ :

$$f(x)|_{x=+1000} = +8x + 1|_{x=+1000} = +8000 + (\dots)$$

$$g(x)|_{x=+1000} = -5x - 1|_{x=+1000} = -5000 + (\dots)$$

So, in Section A we have that  $f(x) < g(x)$  is FALSE.

Altogether, the solution set is:



**II-6.** Given the function *MARC* specified by the global input-output rule

$$x \xrightarrow{MARC} MARC(x) = +3x^2 + 6x - 17$$

find Slope-sign near  $\infty$ .

**Discussion:** Since we are being asked about the output of inputs *near a given input* this is a LOCAL question.

Since we are being asked about when  $x$  is near  $\infty$ , we localize near  $\infty$ :

$$x \text{ large} \xrightarrow{MARC} MARC(x)|_{x=\text{large}} = (+3)x^2 + (+6)x + (+1)|_{x=\text{large}}$$

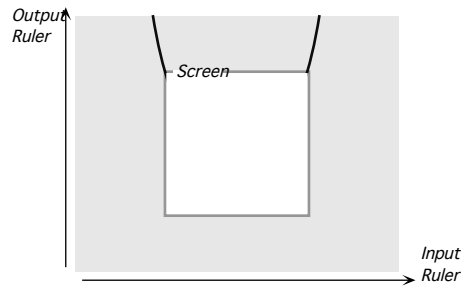
which gives after the *standard approximation*:

$$= (+3)x^2 + (\dots)$$

Since the *local* graph near  $\infty$  of the power function

$$x \longrightarrow P(x) = (+1)x^2$$

is



we have that Sign slope *MARC* near  $\infty = (\nearrow, \searrow)$ . (What is *left* of  $\infty$  is to *our* right and vice versa.)

**II-7.** Given the function *MAY* specified by the global input-output rule

$$x \xrightarrow{MAYO} MAYO(x) = -2x^2 + 4x + 6$$

find Height-sign near  $\infty$ .

**Discussion:** Since we are being asked about the output of inputs *near a given input* this is a *LOCAL* question.

Since we are being asked about when  $x$  is near  $\infty$ , we localize near  $\infty$ :

$$x \text{ large} \xrightarrow{MAYO} MAYO(x)|_{x=\text{large}} = (-2)x^2 + (+4)x + (+6)|_{x=\text{large}}$$

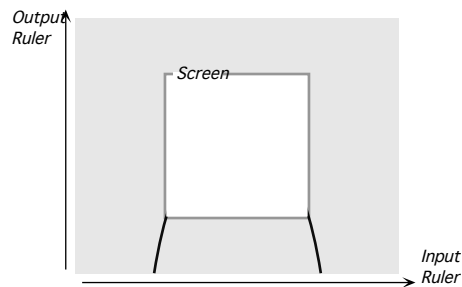
which gives after the *standard approximation*:

$$= (-2)x^2 + (\dots)$$

Since the *local* graph near  $\infty$  of the power function

$$x \longrightarrow f(x) = (-1)x^2$$

is



we have that Height-sign *MAYO* near  $\infty = (-, -)$ . (What is *left* of  $\infty$  is to *our* right and vice versa.)

**II-8.** Let the function  $f$  be specified by the global input-output rule

$$x \xrightarrow{RONI} RONI(x) = +3x^2 + 9x + 6$$

for which input(s), if any, is the output of  $f$  equal to 0?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *output* has to be = 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

**a.** *Find* the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i.** we *localize* near an *unspecified* input  $x_0$
  - ii.** we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, = 0, < 0 or > 0,

**b.** *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

- i.** we *solve* the associated equation to get the *break-even inputs*,
- ii.** we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps.

**a.** We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be = 0, we set the *inequation* directly. We have

$$x \xrightarrow{f} f(x) = +3x^2 + 9x + 6$$

and since we must have

$$f(x) = 0$$

the equation that we must solve is

$$+3x^2 + 9x + 6 = 0$$

b. To *solve* this quadratic equation, we count inputs from  $x_{0\text{-slope}} = \frac{-9}{2(+3)} = -\frac{3}{2}$  by letting  $x = -\frac{3}{2} + u$  because, then, there will be no  $u$  term:

$$-\frac{3}{2} + u \xrightarrow{f} f(x)|_{x=-\frac{3}{2}+u} = (+3)x^2 + (+9)x + (+6)|_{x=-\frac{3}{2}+u}$$

which gives after the *standard computation*:

$$= [-\frac{3}{4}] + [0]u + [+3]u^2$$

So, instead of solving the equation

$$+3x^2 + 9x + 6 = 0$$

we solve the equation

$$-\frac{3}{4} + 3u^2 = 0$$

which gives us

$$u_{0\text{-output}} = \pm\frac{1}{2}$$

and, since  $x = -\frac{3}{2} + u$ , this gives us

$$x_{0\text{-output}} = -\frac{3}{2} \pm \frac{1}{2} = -2, -1$$

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{Q} Q(x) = ax^2 + bx + c$$

the 0-output inputs depend on Discriminant  $f = b^2 - 4ac$ :

- If Discriminant  $f$  is negative, then  $f$  has no 0-output input
- If Discriminant  $f$  is zero, then  $f$  has one 0-output input (and  $x_{0\text{-output}} = x_{0\text{-slope}} = \frac{-b}{2a}$ )
- If Discriminant  $f$  is positive, then  $f$  has two 0-output inputs (and  $x_{0\text{-output}} = x_{0\text{-slope}} \pm \frac{\sqrt{\text{Discriminant } f}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ )

Then we compute Discriminant  $f = (+9)^2 - 4(+3)(+6) = +9$  so that  $f$  has two 0-output inputs, namely  $x_{0\text{-output}} = \frac{-9}{+6} \pm \frac{\sqrt{+9}}{+6} = \frac{-9}{+6} \pm \frac{+3}{+6} = -1, -2$

**II-9.** Let the function  $f$  be specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -3x^2 + 7x - 6$$

where, if at all, is the output of  $f$  *negative*?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *output* has to be  $< 0$ ), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

**a.** *Find* the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i.** we *localize* near an *unspecified* input  $x_0$
  - ii.** we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign,  $= 0$ ,  $< 0$  or  $> 0$ ,

**b.** *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

- i.** we *solve* the associated equation to get the *break-even inputs*,
- ii.** we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps

**a.** We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be  $< 0$ , we set the *inequation* directly. We have

$$x \xrightarrow{f} f(x) = -3x^2 + 7x - 6$$

and since we must have

$$f(x) < 0$$

the inequation that we must solve is

$$-3x^2 + 7x - 6 < 0$$

**b.** We *solve* the inequation in two steps.

**i.** We *solve* the associated equation

$$-3x^2 + 7x - 6 = 0$$

to get the *break-even inputs*:

To solve this quadratic equation, we count the inputs from  $x_{0-slope} = \frac{-7}{2(-3)} = +\frac{7}{6}$  by letting  $x = +\frac{7}{6} + u$  because, then, there will be no  $u$  term:

$$+\frac{7}{6} + u \xrightarrow{f} f(x)|_{x=+\frac{7}{6}+u} = (-3)x^2 + (+7)x + (-6)|_{x=+\frac{7}{6}+u}$$

which gives after the *standard computation*:

$$\begin{aligned} &= \left[-\frac{23}{12}\right] + [0]u + [-3]u^2 \\ &= \end{aligned}$$

So, instead of solving the equation

$$-3x^2 + 7x - 6 = 0$$

we now must solve the equation

$$-\frac{23}{12} - 3u^2 = 0$$

which has no solution

**ii.** We *test* the sections determined by the break-even inputs. Since there is no break-even input, there is only one section. So, we can any input we want.

- We can test for  $x$  near  $\infty$  and we get

$$\begin{aligned} x \text{ large} \xrightarrow{f} f(x)|_{x=\text{large}} &= -3x^2 + 7x - 6|_{x=\text{large}} \\ &= -3x^2 + (\dots) \\ &< 0 \end{aligned}$$

- We can test for  $x = 0$ , we get

$$\begin{aligned} 0 \xrightarrow{f} f(x)|_{x=0} &= -3(0)^2 + 7(0) - 6 \\ &= -6 \\ &< 0 \end{aligned}$$

- We can test for any other input but the arithmetic will be more complicated.

So, for all inputs, the output will be *negative*.

There is also a “global” way to look at this but it depends on our knowledge of quadratic functions. First, since the coefficient of  $x^2$  is negative,

for inputs near  $\infty$  the output will be negative. So, the output will be negative everywhere with a possible exception which depends on whether or not the output can be 0 which we determine by computing the discriminant. Since the discriminant is negative, there can be no 0-output and therefore the output must remain negative everywhere.

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{Q} Q(x) = ax^2 + bx + c$$

the 0-output inputs depend on Discriminant  $f = b^2 - 4ac$ :

- If Discriminant  $f$  is negative, then  $f$  has no 0-output input
- If Discriminant  $f$  is zero, then  $f$  has one 0-output input
- If Discriminant  $f$  is positive, then  $f$  has two 0-output inputs

then, by computing Discriminant  $f = (+7)^2 - 4(-3)(-6) = -23$ , we have that  $f$  has no 0-output inputs.

**II-10.** Given the function *TINA* whose global input-output rule is

$$x \xrightarrow{TINA} TINA(x) = -2x^2 + 6x + 8$$

find  $x_{0\text{-slope}}$ .

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *slope* has to be = 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

**a.** *Find* the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i.** we *localize* near an *unspecified* input  $x_0$
  - ii.** we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, = 0, < 0 or > 0,

**b.** *Solve* the resulting equation/inequation. In the case of an *in-equation*, we always proceed in two steps:

- i. we *solve* the associated equation to get the *break-even inputs*,
- ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

**a.** We *find* the equation/inequation. Since we are looking for the inputs for which the sign of the *slope* is to be  $= 0$ , we find the *equation* as follows:

- i. We *localize* near an unspecified input  $x_0$ :

$$x_0 + h \xrightarrow{f} f(x)|_{x=x_0+h} = (+2)x^2 + (-6)x + (+8)|_{x=x_0+h}$$

which gives after the *standard computation* the coefficient of  $h$ :

$$= [ \quad ]h^0 + [ +4x - 6 ]h^1 + [ \quad ]h^2$$

- ii. We *get* the equation by setting the coefficient of  $h$  equal to 0:

$$+4x_0 - 6 = 0$$

- b.** We solve the equation

$$+4x_0 - 6 = 0$$

which gives us  $x_{0-slope} = +\frac{3}{2}$ .

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{Q} Q(x) = ax^2 + bx + c$$

the 0-slope input is  $x_{0-slope} = \frac{-b}{2a}$ , then we get that  $x_{0-slope} = \frac{+6}{2(+2)} = +\frac{3}{2}$ .

(Note that the proof of the THEOREM goes exactly through the steps of the *standard approach*.)

**II-11.** Let  $f$  be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -4x^2 + 8x + 12$$

near which input(s), if any, is the output of  $f$  *decreasing*?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *slope* has to be  $< 0$ ), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

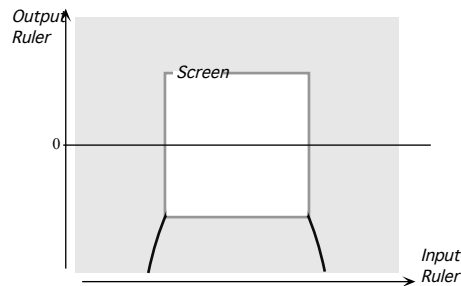


To do that, we could pick a *finite* input in each section and then check the sign of the coefficient of  $h$  for that input.

However, based on our knowledge of the power functions, it is much more efficient to test near  $\infty$  using the graph of  $f$  when  $x$  is near  $\infty$ , namely the graph of

$$\begin{aligned} x \text{ large} &\xrightarrow{f} f(x)|_{x \text{ large}} = (-4)x^2 + (+8)x + (+12)|_{x \text{ large}} \\ &= [-4]x^2 + (\dots) \end{aligned}$$

which is, essentially,



The graph shows that the slope in Section A is  $\nearrow$  and that the concavity in Section B is  $\searrow$

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{Q} Q(x) = ax^2 + bx + c$$

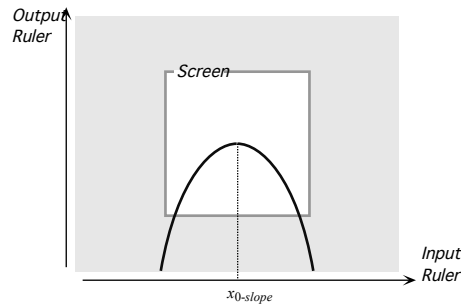
the 0-slope input is

$$x_{0-slope} = \frac{-b}{2a}$$

so that here

$$x_{0-slope} = \frac{-8}{2(-4)} = +1$$

and the THEOREM that says that when the coefficient of  $x^2$  is *negative*, the graph is like



then we can see that the *slope* will be *negative* for the inputs that are *smaller* than  $x_{0-slope} = +1$

**II-12.** Let  $f$  be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -26.06x^2 + 13.03x - 21.63$$

for which input(s), if any, is Concavity-sign  $f = (\cup, \cup)$ ?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that Concavity-sign be  $(\cup, \cup)$ ), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

**a.** Find the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i.** we *localize* near an *unspecified* input  $x_0$
  - ii.** we get the equation/inequation by setting the coefficient of the appropriate power of  $h$  to have the required sign,  $= 0, < 0$  or  $> 0$ ,

**b.** Solve the resulting equation/inequation. In the case of an *in-equation*, we proceed in two steps:

- i.** we *solve* the associated equation to get the *break-even inputs*,
- ii.** we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

**a.** We *find* the equation/inequation. Since we are looking for the inputs for which the sign of the *concavity* has to be  $> 0$  we find the inequation as follows.

**i.** We *localize* near an unspecified input  $x_0$ :

$$x_0 + h \xrightarrow{f} f(x)|_{x=x_0+h} = (-26.06x^2 + 13.03x - 21.63)|_{x=x_0+h}$$

which gives after the *standard computation* the coefficient of  $h^2$ :

$$= [ \quad ]h^0 + [ \quad ]h^1 + [-26.06]h^2$$

**ii.** We *get* the *inequation* by setting the coefficient of  $h^2$  to be *positive*, that is *larger* than 0:

$$-26.06 > 0$$

**b.** This inequation does not involve  $x_0$  and therefore is either always TRUE or always FALSE independently of  $x_0$ . Here, since  $-26.06$  is *negative*, it is always FALSE regardless of  $x_0$ .

So, there is no input for which Concavity-sign  $f = (\cup, \cup)$

Note that if we remember the THEOREM that says that for a *quadratic* function  $QUADRATIC_{a,b,c}$

$$x \xrightarrow{QUADRATIC_{a,b,c}} QUADRATIC_{a,b,c}(x) = ax^2 + bx + c$$

the Concavity-sign is everywhere equal to  $(\text{Sign } a, \text{Sign } a)$ , then we can see that there is no input for which Concavity-sign  $f = (\cup, \cup)$

**II-13.** Let  $f$  be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +x^2 - 2x + 8$$

and let  $g$  be the function specified by the global input-output rule

$$x \xrightarrow{g} g(x) = -3x + 7$$

for how many input(s), if any, do the functions  $f$  and  $g$  return the same output?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *output* of  $f$  has to be equal to the output of  $g$ ), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

**a.** *Find* the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i.** we *localize* near an *unspecified* input  $x_0$
  - ii.** we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign,  $= 0$ ,  $< 0$  or  $> 0$ ,

**b.** *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

- i.** we *solve* the associated equation to get the *break-even inputs*,
- ii.** we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps.

**a.** We find the equation/inequation: Since we are looking for the inputs for which the *output* of  $f$  is *equal* to the *output* of  $g$  we set the *inequation* directly. We have:

$$x \xrightarrow{f} f(x) = +x^2 - 2x + 8$$

and

$$x \xrightarrow{g} g(x) = -3x + 7$$

and since we must have

$$f(x) = g(x)$$

the equation that we must solve is

$$+x^2 - 2x + 8 = -3x + 7$$

which we turn into a quadratic equation by adding  $+3x - 7$  on both sides:

$$+3x - 7 \quad + 3x - 7$$

We get

$$+x^2 + x + 1 = 0$$

b. We *solve* this quadratic equation by counting the inputs from  $x_{0\text{-slope}} = \frac{-1}{2(+1)} = -\frac{1}{2}$  by letting  $x = -\frac{1}{2} + u$  because, then, there will be no  $u$  term:

$$-\frac{1}{2} + u \xrightarrow{f} f(x)|_{x=-\frac{1}{2}+u} = +x^2 + x + 1|_{x=-\frac{1}{2}+u}$$

which gives after the *standard computation*:

$$= \left[ +\frac{3}{4} \right] + [0]u + \left[ +1 \right]u^2$$

So, instead of solving the equation

$$+x^2 + x + 1 = 0$$

we now must solve the equation

$$+\frac{3}{4} + u^2 = 0$$

which has no solution

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{Q} Q(x) = ax^2 + bx + c$$

the 0-output inputs depend on Discriminant  $f = b^2 - 4ac$ :

- If Discriminant  $f$  is negative, then  $f$  has no 0-output input
- If Discriminant  $f$  is zero, then  $f$  has one 0-output input
- If Discriminant  $f$  is positive, then  $f$  has two 0-output inputs

then, by computing Discriminant  $f = (+1)^2 - 4(+1)(+1) = -3$ , we have that  $f$  has no 0-output inputs.

**II-14.** Given the function *TITO* whose global input-output rule is

$$x \xrightarrow{TITO} TITO(x) = +3x^2 + 9x + 6$$

what is the highest output(s), if any?.

**Discussion:** Since we are being asked for “the” highest output, we must look at the output of all the inputs and so this is a GLOBAL question that requires knowledge of the general theory for quadratic functions:

a. Since the coefficient of  $x^2$  is *positive*, the graph of *TITO* is *concave up* everywhere.

**b.** In particular, the graph of *TITO* is concave up near  $x_{0\text{-slope}}$  which is therefore the input with the lowest output. (Visualize the local graph near  $x_{0\text{-slope}}$ .)

**c.** Near  $\infty$ , the output is *infinitely high*

So, *TITO* does not have a *highest* (finite) output.

**II-15.** Given the function  $f$  whose global input-output rule is

$$x \xrightarrow{f} f(x) = x^3 - 12x^2 + 45x + 10$$

what is Slope-sign  $f$  near  $+3$ ?

**Discussion:** Since we are being asked about the output of inputs *near a given input* this is a LOCAL question.

Since we are being asked about when  $x$  is near a *finite* input, we localize near that input and since we are looking for the *slope*, we look for the coefficient of  $h$  in the output:

$$+3 + h \xrightarrow{f} f(x)|_{x=+3+h} = (+1)x^3 + (-12)x^2 + (+45)x + (+10)|_{x=+3+h}$$

which gives after the *standard computation* the coefficient of  $h$ :

$$= [ \quad ]h^0 + [0]h^1 + [ \quad ]h^2 + [ \quad ]h^3$$

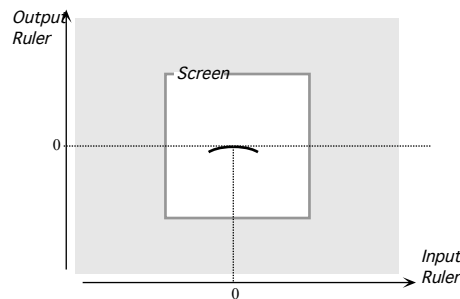
So, as it happens, here  $+3$  is a 0-slope input and the slope near  $+3$  will have to come from the coefficient of  $h^2$ :

$$\begin{aligned} &= [ \quad ]h^0 + [0]h^1 + [(+1) \cdot (3 \cdot +3) + (-12) \cdot 1]h^2 + [ \quad ]h^3 \\ &= [ \quad ]h^0 + [0]h^1 + [-3]h^2 + [ \quad ]h^3 \end{aligned}$$

Since the *local* graph of the power function

$$h \xrightarrow{f} f(x)|_{x=+3+h} = (-3)h^2$$

is



we have that Sign slope  $f$  near  $+3 = (\nearrow, \searrow)$  (What is left of a *finite* point is to *our* left etc.)

**II-16.** Given the function  $f$  whose global input-output rule is

$$x \xrightarrow{f} f(x) = -2x^3 + 12x^2 + 8x - 7$$

what is Concavity-sign  $f$  near  $+3$ ?

**Discussion:** Since we are being asked about the output of inputs *near a given input* this is a LOCAL problem.

Since we are being asked about the *concavity*, we look for the coefficient of  $h^2$  in the output:

$$\begin{aligned} +3 + h &\xrightarrow{f} f(x)|_{x=+3+h} = (-2)x^3 + (+12)x^2 + (+8)x + (-7) \Big|_{x=+3+h} \\ &= (-2) [+3 + h]^3 + (+12) [+3 + h]^2 + (+8) [+3 + h] + (-7) \\ &= (-2) [+27 + 27h + 9h^2 + \dots] + (+12) [+9 + 6h + h^2] + (+8) [+3 + h] + (-7) \end{aligned}$$

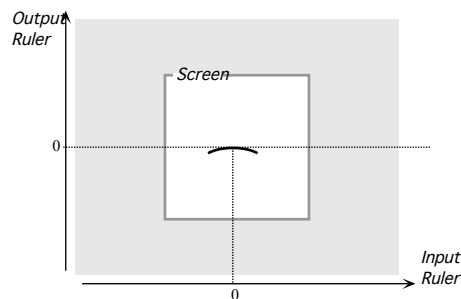
which gives after the *standard computation* the coefficient of  $h^2$ :

$$= [ \quad ]h^0 + [ \quad ]h^1 + [-6]h^2 + [ \quad ]h^3$$

Since the *local* graph of the power function

$$h \xrightarrow{f} f(x)|_{x=+3+h} = (-6)h^2$$

is



we have that Sign concavity  $f$  near  $+3 = (\cap, \cap)$

**II-17.** Given the function  $f$  whose global input-output rule is

$$x \xrightarrow{f} f(x) = x(x - 2)^2$$

where is the output equal to 0?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *output* has to be = 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

**a.** *Find* the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i.** we *localize* near an *unspecified* input  $x_0$
  - ii.** we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, = 0, < 0 or > 0,

**b.** *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

- i.** we *solve* the associated equation to get the *break-even inputs*,
- ii.** we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps.

**a.** We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be = 0, we set the *equation* directly. We have

$$x \xrightarrow{f} f(x) = x(x - 2)^2$$

and since we must have

$$f(x) = 0$$

the equation that we must solve is

$$+x(x - 2)^2 = 0$$

b. We *solve* the equation. This is a *cubic* equation and, normally, we don't know how to solve a *cubic* equation. Here, though, the cubic is *factored* and so solving amounts to the same as solving:

$$\text{OR } \begin{cases} x = 0 \\ (x - 2)^2 = 0 \end{cases}$$

i. The first equation,

$$x = 0$$

gives us  $x_{0\text{-output}} = 0$

ii. The second equation,

$$(x - 2)^2 = 0$$

gives us  $x_{0\text{-output}} = +2$

**II-18.** Given the function  $f$  whose global input-output rule is

$$x \xrightarrow{f} f(x) = -(x - 1)(x - 2)(x - 3)$$

where is the output of  $f$  *negative*?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *output* has to be  $< 0$ ), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i. we *localize* near an *unspecified* input  $x_0$
  - ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign,  $= 0$ ,  $< 0$  or  $> 0$ ,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

- i. we *solve* the associated equation to get the *break-even inputs*,
- ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps

a. We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be  $< 0$ , we set the *inequation* directly. We have

$$x \xrightarrow{f} f(x) = -(x-1)(x-2)(x-3)$$

and since we must have

$$f(x) < 0$$

the inequation that we must solve is

$$-(x-1)(x-2)(x-3) < 0$$

b. We *solve* the inequation in two steps.

i. We *solve* the associated equation

$$-(x-1)(x-2)(x-3) = 0$$

to get the *break-even inputs*: This is a *cubic* equation and, normally, we don't know how to solve a *cubic* equation. Here, though, the cubic is *factored* and so solving amounts to the same as solving:

$$\text{OR } \begin{cases} x-1=0 \\ x-2=0 \\ x-3=0 \end{cases}$$

- The first equation,

$$x-1=0$$

gives us the break-even input  $x_{0\text{-output}} = +1$

- The second equation,

$$x-2=0$$

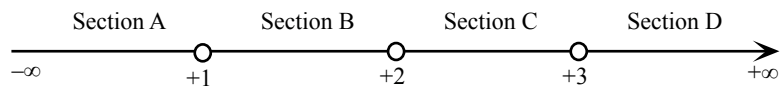
gives us the break-even input  $x_{0\text{-output}} = +2$

- The third equation,

$$x-3=0$$

gives us the break-even input  $x_{0\text{-output}} = +3$

ii. We test each of the four sections determined by the three break-even inputs  $x_{0\text{-slope}} = +1$ ,  $x_{0\text{-slope}} = +2$ ,  $x_{0\text{-slope}} = +3$ , :

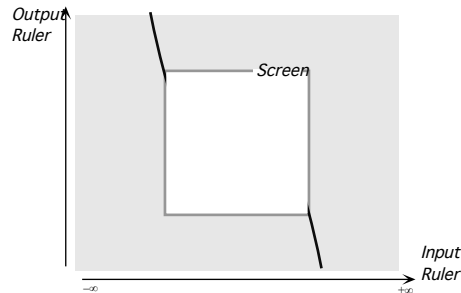


To do that, we could pick a *finite* input in each section and then check the sign of the coefficient of  $h^0$  (that is, in fact, the sign of the output) for that input.

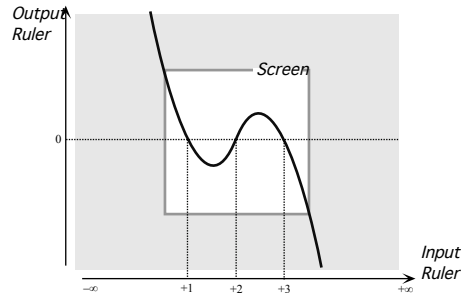
However, based on our knowledge of the cubic functions, it is much more efficient to test near  $\infty$  using the graph of  $f$  when  $x$  is near  $\infty$ , namely the graph of

$$\begin{aligned} x \text{ large} &\xrightarrow{f} f(x)|_{x \text{ large}} = -(x-1)(x-2)(x-3)|_{x \text{ large}} \\ &= -(x+(\dots))(x+(\dots))(x+(\dots)) \\ &= -x^3 + (\dots) \end{aligned}$$

which is, essentially,



and then to use the global graph of  $f$  which must be:



The global graph shows that the output in Section A is +, the output in Section B is -, the output in Section C is +, the output in Section D is -.

**II-19.** Let  $f$  be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +2x^3 + x^2 + 10x + 7$$

where is the slope of  $f$  equal to 0?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *slope* has to be = 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

**a.** *Find* the equation/inequation that the inputs must be a solution of in order for the “something” to be the case.

- When the “something” involves *outputs*, we set the equation/inequation directly.
- When the “something” involves the *slope* or the *concavity*,
  - i. we *localize* near an *unspecified* input  $x_0$
  - ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, = 0, < 0 or > 0,

**b.** *Solve* the resulting equation/inequation. In the case of an *inequation*, we always proceed in two steps:

- i. we *solve* the associated equation to get the *break-even inputs*,
- ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

**a.** We *find* the equation/inequation. Since we are looking for the inputs for which the sign of the *slope* is to be = 0, we find the *equation* as follows:

- i. We *localize* near an unspecified input  $x_0$ :

$$x_0 + h \xrightarrow{f} f(x)|_{x=x_0+h} = (+2)x^3 + (+1)x^2 + (+10)x + (+7)|_{x=x_0+h}$$

which gives after the *standard computation* the coefficient of  $h$ :

$$= 2[x_0 + h]^3 + [x_0 + h]^2 + 10[x_0 + h] + 7$$

using the *addition formulas* for cube and for square

$$\begin{aligned} &= 2 \left[ x_0^3 + 3x_0^2h + [\dots] \right] + \left[ x_0^2 + 2x_0h + [\dots] \right] + 10[x_0 + h] + 7 \\ &= [ \quad ]h^0 + [ +6x_0^2 + 2x_0 + 10 ]h^1 + [ \quad ]h^2 + [ \quad ]h^3 \end{aligned}$$

- ii. We *get* the equation by setting the coefficient of  $h$  equal to 0:

$$+6x_0^2 + 2x_0 + 10 = 0$$

**b.** We *solve* this quadratic equation by counting the inputs from  $x_{0\text{-slope}} = \frac{-2}{2(+6)} = -\frac{1}{6}$  by letting  $x = -\frac{1}{6} + u$  because, then, there will be no  $u$  term:

$$-\frac{1}{6} + u \xrightarrow{f} f(x)|_{x=-\frac{1}{6}+u} = +6x^2 + 2x + 10|_{x=-\frac{1}{6}+u}$$

which gives after the *standard computation*:

$$= \left[ + \frac{59}{6} \right] + [0]u + [+6]u^2$$

So, instead of solving the equation

$$+6x^2 + 2x + 10 = 0$$

we now must solve the equation

$$+ \frac{59}{6} + 6u^2 = 0$$

which has no solution

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{Q} Q(x) = ax^2 + bx + c$$

the 0-output inputs depend on Discriminant  $f = b^2 - 4ac$ :

- If Discriminant  $f$  is negative, then  $f$  has no 0-output input
- If Discriminant  $f$  is zero, then  $f$  has one 0-output input
- If Discriminant  $f$  is positive, then  $f$  has two 0-output inputs

then, by computing Discriminant  $f = (+2)^2 - 4(+6)(+10) = -236$ , we have that the equation  $+6x^2 + 2x + 10 = 0$  has no solution.

**II-20.** Given the function  $f$  specified by the global input-output rule

$$x \xrightarrow{f} f(x) = +x^3 - 9x^2 + 15x - 11$$

where is  $f$  *concave up*?

**Discussion:** Since we are being asked for all the input(s), if any, for which “something” is required to be the case (here it is that the *concavity* has to be  $> 0$ ), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

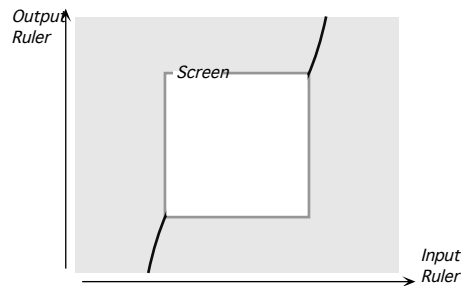


To do that, we could pick a *finite* input in each section and then check the sign of the coefficient of  $h$  for that input.

However, based on our knowledge of the power functions, it is much more efficient to test near  $\infty$  using the graph of  $f$  when  $x$  is near  $\infty$ , namely the graph of

$$\begin{aligned} x \text{ large} &\xrightarrow{f} f(x)|_{x \text{ large}} = (+1)x^3 + (-9)x^2 + (+15)x + (-11)|_{x \text{ large}} \\ &= [+1]x^3 + (\dots) \end{aligned}$$

which is, essentially,



The graph shows that the concavity in Section A is  $\cap$  and that the concavity in Section B is  $\cup$

Note that if we remember the THEOREM that says that, for a *cubic* function

$$x \xrightarrow{Q} Q(x) = ax^3 + bx^2 + cx + d$$

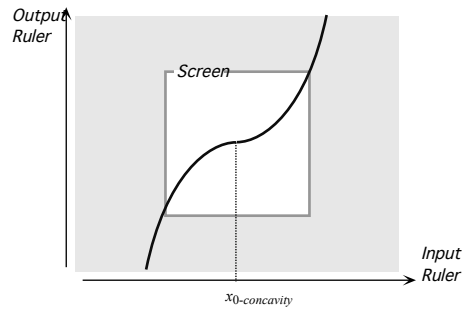
the 0-concavity input is

$$x_{0\text{-concavity}} = \frac{-b}{3a}$$

so that here

$$x_{0\text{-concavity}} = \frac{+9}{3(+1)} = +3$$

and the THEOREM that says that when the coefficient of  $x^3$  is *positive*, the graph is like



then we can see that the *concavity* will be *cup* for the inputs that are larger than  $x_{0\text{-concavity}} = +3$