specify
directly
specifying-phrase
identify

Chapter 1

Relations And Functions

What is PRECALCULUS? Other than saying that it is a prequel to CALCULUS, the term is totally without meaning, mathematically or otherwise.[1] A much better question is: What is CALCULUS? And, since "to calculate" means the same as "to compute", the obvious question is: What is it one computes with in CALCULUS?

- That cannot be *numbers* since that's what we learned in ARITHMETIC.
- That cannot be *letters* since that's what we learned in ALGEBRA.

So, what is left? In order to answer that, though, we first need to look back at the ways we **specify** numbers in mathematics:

- In ARITHMETIC, *numbers* are *specified* **directly**, that is by way of a **specifying-phrase**, that is a phrase that states "clearly and definitely" what number is to be found—without actually saying what the number *is*. (See REASONABLE BASIC ALGEBRA, Chapter 3.)

  EXAMPLE 1. "The President of the USA in 1852" is a *specifying phrase* which does not say who this was.

  EXAMPLE 2. "$5 + 3$" is a *specifying phrase*, read "the result of adding three to five", which does not say what the number is.

  Then, to **identify** the number specified by a *specifying-phrase* means to find the number that meets the given *specification.*

[1] Educologists will remember that this mid seventies "invention" of theirs turned out to be just part of the publishing industry's effort to get around the used-book market.

procedure
indirectly
equation
inequation
solve

EXAMPLE 3. To identify the perpetrator of a crime means to find the person who meets a number of specifications such as: having been at the scene of the crime at the time of the crime, having a motive, etc.

EXAMPLE 4. To identify "The President of the USA in 1852" means to find that "The President of the USA in 1852" was "Millard Fillmore".

EXAMPLE 5. To *identify* "$5 + 3$" means to find that "$5 + 3$" is "8".

It is important to realize that a *specifying phrase* is not the same as the **procedure** that we *use* to identify the number specified by the specifying phrase.

EXAMPLE 6. To *identify* the *specifying-phrase* $5 + 3$ a *procedure* is to count 3 steps *forward* starting from 5:

$$5 \xrightarrow{\to 3}$$

and we then write that $5 + 3 = 8$.

- In ALGEBRA, on the other hand, *numbers* are often *specified* **indirectly**, that is by an **equation** (or an **inequation**), that is by how the number(s) should stand relative to given numbers.

EXAMPLE 7. The equation $3 + x = 5$ specifies a number, namely the number that "added to 3 gives 5".

Then, to **solve** the *equation* (or the *inequation*) means to *find* the number(s), if any, that stand in that relationship. (See REASONABLE BASIC ALGEBRA, Chapter 7.)

EXAMPLE 8. To *solve* the given *equation*

$$3 + x = 5$$

that is to find the number that "added to 3 gives 5", a *procedure* is to count 3 steps *backward* from 5:

$$5 \xrightarrow{\leftarrow 3}$$

and we then write

2 is a solution of the equation $3 + x = 5$.

And so, for the time being, we will say that CALCULUS is about many other ways to get *new* numbers from *given* numbers which correspond to all sorts of real-world processes, devices, agencies, converters, translators, etc. And so, inasmuch as the *new* numbers are "related" to the *given* numbers, we begin by investigating the idea of "relationship".

1.1 Relations

To see whether something is *changing*—or *not* changing, it is necessary to look at it in *relationship* with something else.

EXAMPLE 9. The amount of income tax changes *in relationship with* the amount of income, the amount of property tax changes *in relationship with* the amount of property, the amount of sales tax changes *in relationship with* the amount of purchases.

More precisely, in order to **observe** something that is changing—or *not* changing, we must **pair** each **observed state** that it goes through with some **reference state**, for example, if nothing else, as given by a clock and/or a calendar!

observe
pair
observed state
reference state
relation
input number
output number
point
value
input-output pair
picture
ruler, quantitative

EXAMPLE 10. We might say that someone's income tax was $\$2,753$.
But that would not be saying much since, for instance, $\$2,753$ was a lot less money in FY 2008 (FY is for Fiscal Year) than it was in, say, FY 1913—the year income tax was first established.
So, in order for $\$2,753$, the *observed state*, to make sense, we must give it along with the *reference state*, namely the Fiscal Year.

We will call **relation** whatever process, device, procedure, agency, converter, exchanger, translator, etc in terms of which the *pairing* is done and we will call **input numbers** the numbers that correspond to the *reference states* and we will call **output numbers** the numbers that correspond to the *observed states.*[2]

NOTE. These terms are taken from computer science because they are more suggestive than the more traditional terms, **point** for input and **value** for output.[3]

An *input number* together with an *output number* that it is paired with by the relation make up what we will call an **input-output pair**. (Note that we will be using parentheses to enclose input-output pairs and that this is yet *another* use of parentheses.)

EXAMPLE 11. In EXAMPLE 6, an *input-output pair* could be (FY2003, $\$2,753$).

NOTE. Eventually, the word "number" will *go without saying* and we shall just use the word "input" instead of the phrase "input number" and the word "output" instead of the phrase "output number". Occasionally, though, we will have to use the full phrases, "input number" and "output number".

1.2 Quantitative Rulers

Very often, we will want to **picture** the *input numbers* and the *output numbers* involved in a *relation* and we will do that with **quantitative rulers** which are essentially what goes in the real world by the name of "ruler".
NOTE. In high school, *quantitative rulers* usually go by the name of **number lines**,

[2]Educologists will wonder why we do not continue sharply to distinguish the "real world" from the "paper world" as we did in REASONABLE BASIC ALGEBRA. The answer is that, by now, we hope being able to take advantage of what Thurston calls "compression".

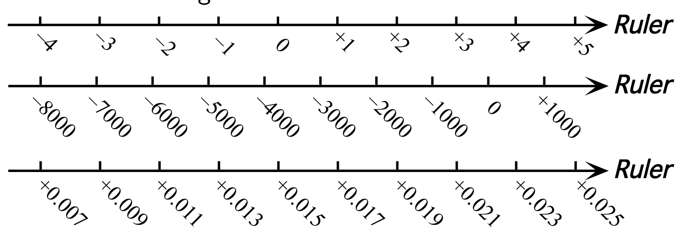
[3]Actually, never ones to shy away from the impenetrable, or perhaps just to show off, Educologists generally prefer the XIXth century terms, "independent variable" and "dependent variable".

number lines
tick-marks
labeled
in order
equally spaced
extent
curly brackets
{ }
resolution
bounded numbers
encompass

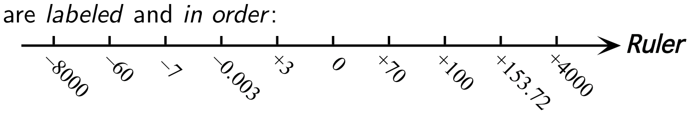
a term we will not use in this text[4].

1. More precisely, the **tick-marks** on a *quantitative ruler* must be **labeled**, **in order**, and **equally spaced**.

Example 12. The following :



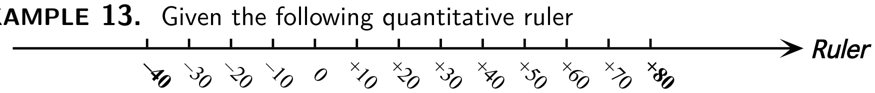
are all *quantitative* rulers but the following is *not* a quantitative ruler even though the tick-marks are *labeled* and *in order*:



Quantitative rulers are specified by two things:

- The **extent** of a given *quantitative ruler* consists of both the *smallest label* and the *largest label* which we write between **curly brackets** { }.
- The **resolution** of a given *quantitative ruler* is the *space* between the labels of two consecutive tick-marks.

Example 13. Given the following quantitative ruler

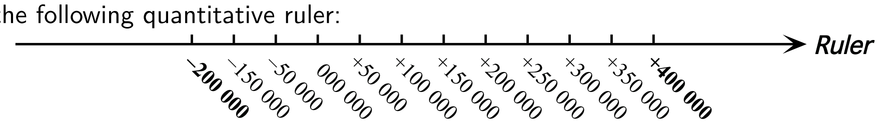


- the *extent* of the given ruler is $\{-40, +80\}$
- the *resolution* of the given ruler is 10

2. Thus, given a quantitative ruler, there are going to be two kinds of number:

- Numbers that fall *within the extent* of the quantitative ruler which will be called **bounded numbers**. It is important to realize that any given number we happen to be interested in can always be viewed as a *bounded number* since we can always draw a quantitative ruler whose extent will **encompass** the given number.

  Example 14. We can view the number $308\,195$ as a *bounded number* by using the following quantitative ruler:



[4] It would be interesting to trace the origin of this remarkably un-enlightening term. But then, it is probably due to Educologists' well known craving for the esoteric.

outlying numbers
plot
dot, solid
finite number

In fact, any bunch of numbers can be viewed as a bunch of *bounded numbers* since we need only use a quantitative ruler whose extent encompasses both the smallest number in the bunch and the largest number in the bunch.

EXAMPLE 15. The numbers $-176\,329, -53.78, +543\,830$ will be *bounded numbers* for any ruler with an extent that encompasses both $-176\,329$ and $+543\,830$ such as, for instance, $\{-200\,000, +1\,000\,000\}$

- Numbers that fall *beyond the extent* of the quantitative ruler which we will call **outlying numbers**[5]. Because there is a number of difficulties with *outlying numbers*, though, we will not deal with them right away and will return to them in **Chapter 2**.

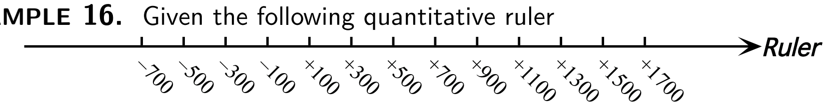
1.3 Finite Numbers And Near-Finite Numbers

As we just saw, whether a number is *bounded* or not depends only on the *extent* of the ruler and thus this is totally independent of the *resolution* of the ruler. We now investigate matters that depend on the *resolution* of the ruler, that is on how the ruler is tick-marked.

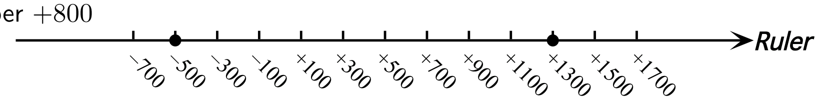
1. Given a *quantitative ruler* and given a *bounded number*, there are two possible cases

a. The given bounded number can fall *on* a tick-mark in which case we will be able to **plot** the given number by placing a **solid dot** on the tick-mark and we will say that the bounded number is a **finite number** for the given ruler.

EXAMPLE 16. Given the following quantitative ruler

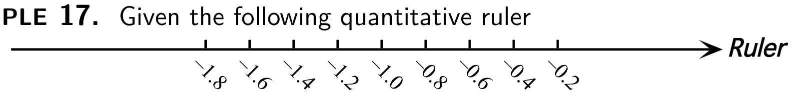


we can plot the *bounded numbers* -500 and $+1300$ but we cannot plot the bounded number $+800$



and therefore, for *this* quantitative ruler, the *bounded numbers* -500 and $+1300$ are *finite numbers* but the *bounded number* $+800$ is *not* a *finite number*.

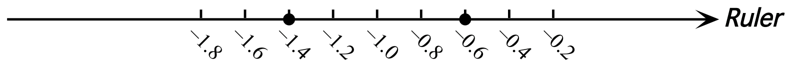
EXAMPLE 17. Given the following quantitative ruler



we can plot the *bounded numbers* -1.4 and -0.8 but we cannot plot the bounded number -1.1:

[5]Educologists will surely know why we didn't use the term "unbounded".

in general
near-finite number
center
neighborhood
halfway-mark

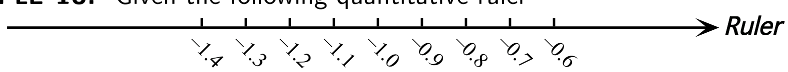


and therefore, for *this* quantitative ruler, the *bounded numbers* -1.4 and -0.8 are *finite numbers* but the *bounded number* -1.1 is *not* a *finite number*.

In order to discuss *finite numbers* **in general**, that is when the actual finite numbers that we are discussing will remain *undisclosed* for the duration, we will use the letter x with a subscript, usually $_0$ but also $_1$, $_2$ etc, so that we will use x_0 to designate a single given *finite* number but we will use x_1, x_2, etc when we have to deal with two or more given finite numbers.

b. The given bounded number can fall somewhere *between* two tick-marks and therefore cannot be *plotted*. We will call such bounded numbers **near-finite numbers** for the given ruler.

Example 18. Given the following quantitative ruler



the number

$$-1.27$$

is *near-finite* for the given quantitative ruler because it falls *between* the tick-marks -1.2 and -1.3.

In fact, this will be rather the more frequent case since there are only so many tick-marks on a *quantitative ruler* and therefore only so many *finite* number for that quantitative ruler while there can be no end numbers.

2. We saw in the previous section that any bunch of numbers we happen to be interested in can be viewed as a bunch of *bounded numbers*.

Similarly, we can view any *single* given number we happen to be interested in as a *finite number* because we can always make up a ruler such that the label of one of the tick-marks is going to be the given number.

In fact, we can view any *two* given numbers we happen to be interested in as *finite numbers* because we can always adjust the *resolution* to correspond to the difference between the two numbers.

However, given a *bunch of numbers*, it is not always possible to look at all of them as *finite numbers* because it is not always possible to find a *quantitative ruler* whose *resolution* allows us to plot *all* the given numbers, that is for which all the given numbers are labels of *tick-marks*.

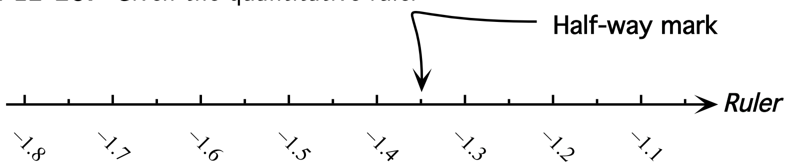
Example 19. It does not seem very feasible to come up with a quantitative ruler on which to plot the numbers $1000., 100., 10., 1., 0.1, 0.01, 0.001$ that is with a quantitative ruler with both the *extent* necessary to show 0 and $1000.$ and the *resolution* necessary to separate 0.001 from 0.01.

3. What we will do is to look at each *tick-mark* on a *quantitative ruler* as the **center** of a **neighborhood** extending between the two **halfway-marks** surrounding the tick-mark which we will mark with *parentheses*, yet

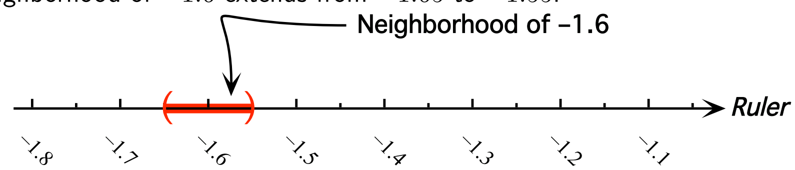
near
quantitative input ruler
quantitative output ruler
link

another use of parentheses.

EXAMPLE 20. Given the quantitative ruler

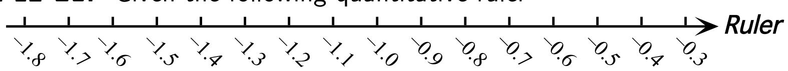


the neighborhood of -1.6 extends from -1.65 to -1.55:



Then, any given *near-finite number* will fall in the *neighborhood* of some tick-mark and, naturally, we will say that the *near-finite* number is **near** the finite number which is the label of the tick-mark.

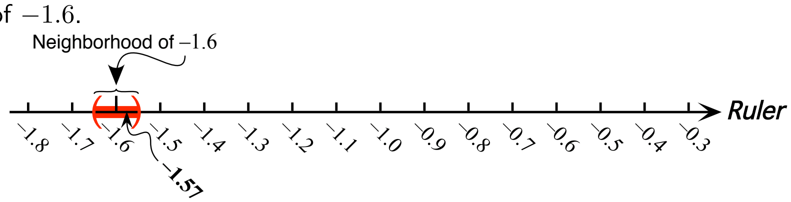
EXAMPLE 21. Given the following quantitative ruler



since the number

$$-1.57$$

falls between -1.6 and -1.5 and -1.6 is the closest tick-mark, -1.57 is in a *neighborhood* of -1.6.



Or, we can just say that -1.57 is *near* -1.6

1.4 Quantitative Screens

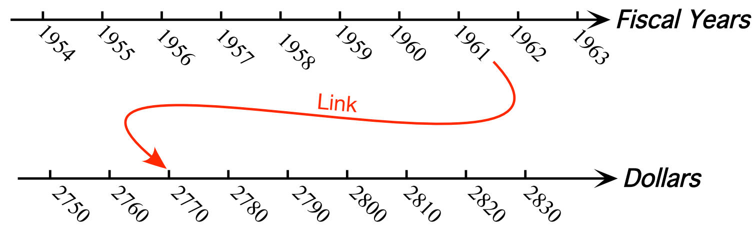
Given a *relation*, we will often want to *picture* the *input-output pairs* involved in the relation.

1. The simplest way, of course, is to use:

- a **quantitative input ruler**, that is a quantitative ruler to *plot* input numbers
- a **quantitative output ruler**, that is a quantitative ruler to *plot* output numbers
- **links** to *pair* the input and the output into an input-output pair.

quantitative screen
screen
outlying-space

EXAMPLE 22. Given the *input-output pair* (FY1961, \$2, 750), we could picture it as follows:

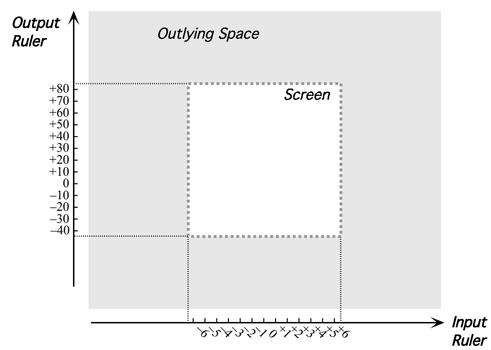


Rather obviously, though, this approach is not going to work very well when the relation involves a lot of input-output pairs because the links will very quickly turn into something looking like a pile of overcooked spaghetti.

2. So, for a better way to picture the input-output pairs of a relation, we will use **quantitative screens** which will consist of:

- A **screen**, that is a rectangular area in which we will *link* the input and the output of an input-output pair,
- An *quantitative input ruler* placed under the *screen* with the *extent* of the input ruler corresponding to the *width* of the screen,
- An *quantitative output ruler* placed left of the *screen* with the *extent* of the output ruler corresponding to the *height* of the screen.
- Some **outlying-space** beyond the *screen*.

EXAMPLE 23.



Before going any further, we need to make a few remarks about the way *quantitative screens* look.

The idea is that, in the real world, we are always working in a *bounded space*, be it on a workbench, in an office, on a construction site, in a city, etc. When we want to picture relations, we do that on a piece of paper, a blackboard, a computer monitor, etc. So, no matter what, we always work in a *bounded space*.

Of course, we know that we can always get a larger piece of paper, a larger blackboard or a larger monitor. But no matter how large the piece of paper, the blackboard or the monitor, once we have decided on the piece of paper, the blackboard or the monitor, that is what we have to work with: a *bounded space*.

re-scale
input point
input level line
output point
output level line
plot point

In a *quantitative screen*, the *screen* will represent the *bounded space* to which we have access at the time we are working and the *outlying space* will serve to remind us that there is more space beyond the screen that might turn out to be important and which we can always access by using a larger screen or by **re-scaling** the rulers, that is by using *larger extents*—and therefore *smaller resolutions*. However, there is a lot more to this than meets the eye and we will have to return to this very issue after we have dealt with some more immediate concerns.

3. We now turn to the *procedure* for *linking* on a quantitative screen the input and the output of an *input-output pair*:

i. We *plot* the input number, that is we mark the **input point**, that is the tick-mark on the *input ruler* whose label is the *input number*,

ii. We draw the **input level line**, that is the *vertical* line through the *input point*,

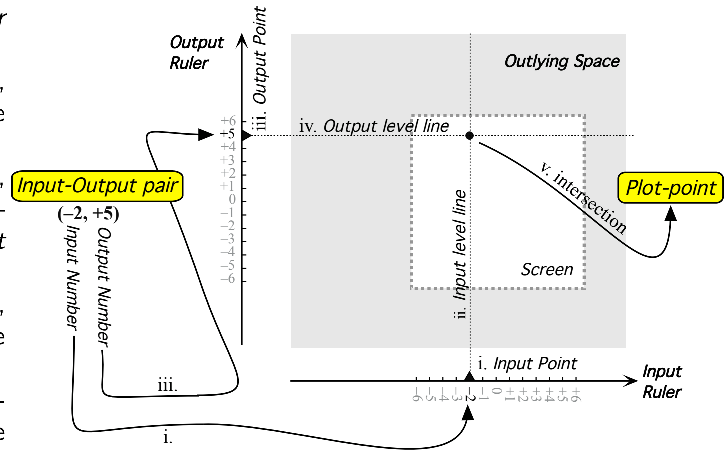
iii. We *plot* the output number, that is we mark the **output point**, that is the tick-mark on the *output ruler* whose label is the *output number*,

iv. We draw the **output level line**, that is the *horizontal* line through the *output point*,

v. Then, the **plot point** that pictures the *input-output pair* on the *screen* is at the intersection of the *input level line* and the *output level level line.*

EXAMPLE 24. In order to picture the *input-output pair* $(-2, +5)$,

i. We mark the *input point*, that is the tick-mark on the *input ruler* whose label is the *input number* -2
ii. We draw the *input level line*, that is the *vertical* line through the *input point*
iii. We mark the *output point*, that is the tick-mark on the *output ruler* whose label is the *output number* $+5$
iv. We draw the *output level line*, that is the *horizontal* level line through the *output point*,
v. Then, the *plot point* that pictures the *pair* $(-2, +5)$ on the *screen* is at the intersection of the input level line and the output level level line.



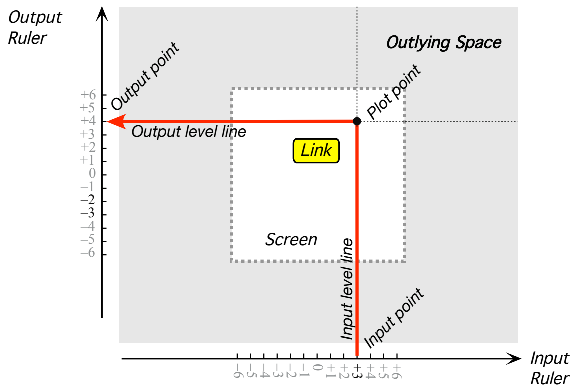
NOTE. The fact that we use “side rulers” instead of “crosshairs” in the middle of the screen does not conform with the usual practice in textbooks—but it does conform with the usual practice just about anywhere else. One reason *we* will use “side rulers” as opposed to “crosshairs” is that *rulers* and “0-level lines” are entirely separate entities which we want to keep separate and so we do not want to draw the rulers on top of level lines[6]. Another reason, as we will soon see, will be our heavy use of “qualitative rulers” in which, as we will see, the rulers *cannot* go across the screen.

Beyond that, though, the above arrangement is rather *arbitrary* and, in particular, there is no *reason* why the input ruler should have to go from left to right and the output ruler from bottom to top and why the rulers should be at a 90 degree angle, etc. In this text, though, this will be our standard arrangement.

4. In this way of picturing input-output pairs, the *link* consists of part of the input level line and part of the output level line.

EXAMPLE 25. Given the input-output pair $(+3, +4)$ the *link* consists of the part of the $+3$-input level line from the input point to the plot point and the part of the $+4$-output level line from the plot point to the output point:

[6]This is another one of these cases where Educologists blissfully ignore a profound difference, here that between “identitical” and “identified”. In the present case, they identify the “number line” where x lives with the level line where the pair $(x, 0)$ lives. Just to simplify matters one supposes. But for whom?



good picture
recover

However, just the *plot-point* is a **good picture** of an *input-output pair* because, once we have drawn the plot-point, we can erase the input point and the output point as well as the two level lines and we are still able to **recover** the *input-output pair* of which this plot-point is the picture. All we have to do is to go backwards through the above steps:

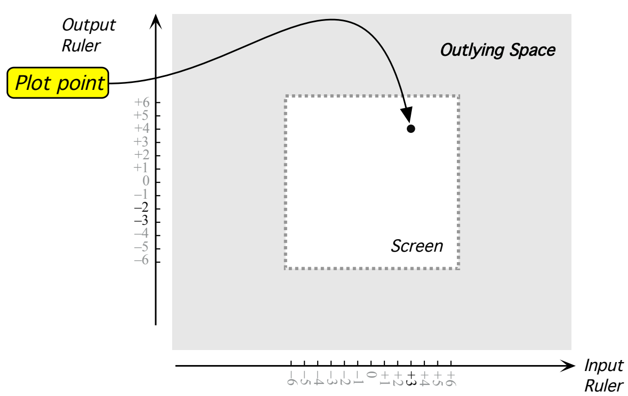
i. We draw the *input level line* (vertical) through the given plot point,

ii. The input point (that represent the *input number*) is where the input level line intersects the input ruler,

iii. We draw the *output level line* (horizontal) through the given plot point,

iv. The output point (that represent the *output number* is where the output level line intersects the output ruler.

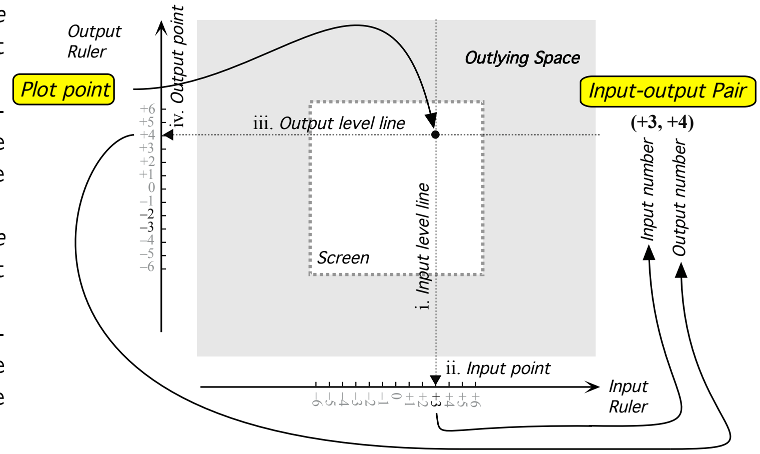
EXAMPLE 26. Given, the following *plot-point*



we can *recover* the input-output pair as follows:

loss of information
function
return

i. We draw the *input level line* (vertical) through the given plot point,
ii. The input point (that represents the *input number*) is where the input level line intersects the input ruler,
iii. We draw the *output level line* (horizontal) through the given plot point,
iv. The output point (that represents the *output number*) is where the output level line intersects the output ruler.



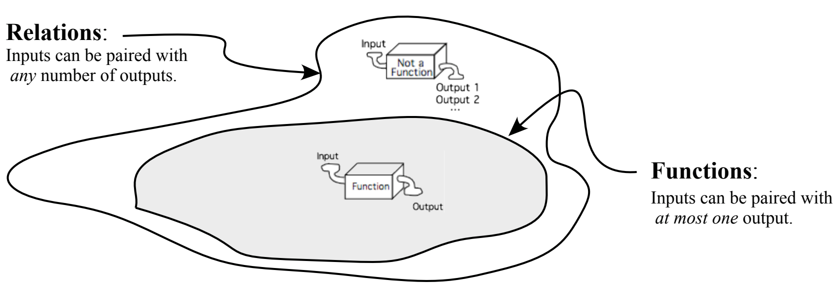
In other words, a *plot point* does not involve any **loss of information** compared to the *input-output pair* that it pictures.

1.5 Functions

Relations can get to be surprisingly complicated and so, from now on, we will only investigate **functions**, that is relations that meet the requirement that:

No input shall be paired with more than one output.

In other words, given *any* input, a *function* may either **return** *one* output or *no* output at all[7] but *never more than one* output.



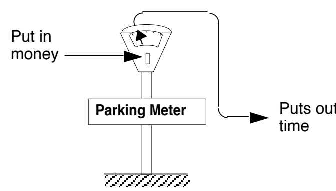
[7]Educologists will rightfully object that functions should not be allowed to return *no* output and therefore that we should introduce the notion of **domain**. But while, of course, pontificating about domains can be gratifying to the instructor, it also complicates the students' mathematical life quite unnecessarily since, at this point, the loophole thereby opened is quite unlikely to occur to beginners and thus to puzzle them.

However, that there is nothing to prevent a function from pairing *many* inputs with a *same, single output*. In other words, a function may return the *same output* for *different inputs*.

input-output table
function, tabular

In terms of what we are observing, this means that once a reference state has been decided upon, *no more than one* state can be observed—but a same state can be observed from any number of reference states.

EXAMPLE 27. A *parking meter*



is a *function* because, whatever the amount of money we input, we can be sure that anyone who inputs the same amount of money will get the same parking times as we did. For instance, given an input, say 1 **Quarter**, the parking meter outputs a definite amount of parking time, say 20 **Minutes**.

However, since there usually is a maximum parking time, any amounts of money we input above the maximum will return the same amount of parking time namely the maximum parking time.

EXAMPLE 28. A *slot machine* is *not a function* because, given an input, say 1 **Quarter**, a slot machine can output just about *any* number of **Quarters**.

EXAMPLE 29. We definitely want the relation between taxable income and income tax to be a *function* because, presumably, we don't want two persons with the same taxable income to pay different amounts of income tax!

1.6 Functions Specified By A Global I-O Rule

Functions can be *specified* in several ways:

1. In some sciences, such as PSYCHOLOGY, SOCIOLOGY, BUSINESS, ACCOUNTING, etc functions are usually *specified* by **input-output tables**. We will call such function **tabular functions**.

EXAMPLE 30. A business may be described by its profits/losses over the years, that is by a *tabular function* specified by the following *input-output table*:

specified indirectly
functional equation
specified directly
input-output rule, global
unspecified input
place holder
specific inputs

Fiscal Year	Profit/Loss
2001	+5,924
2002	-2,351
2003	+6,753
2004	+5,636
2005	-3,753
2006	+8,482

Picturing tabular functions is usually not difficult:

- We choose the extent of the input ruler so that it *encompasses* the largest and the smallest of the inputs,
- We choose the extent of the output ruler so that it *encompasses* the largest and the smallest of the outputs,
- The only difficulty may be to find a *resolution* for which as many numbers as possible are *finite*.

And, that is about all there is to say about *tabular functions*.

2. In other sciences, such as PHYSICS, ELECTRONICS, ... and MATHEMATICS, many *functions* are **specified indirectly**, that is by a **functional equation**[8] which is the equivalent for functions of the *equations* we use to specify numbers *indirectly* in ALGEBRA.

But, just like ALGEBRA is more difficult than ARITHMETIC because it is more difficult to solve equations than it is to identify specifying phrases, solving *functional equations* can be quite difficult and we will leave this for the next volume in this series: REASONABLE TRANSCENDENTAL FUNCTIONS.

3. What we will do *here* will be the equivalent of using *specifying phrases* in ARITHMETIC, that is we will investigate only functions **specified directly** by a **global input-output rule** constructed as follows:

i. We must already *have* or, if not, *create* a name for the function. (*By default*, that is in the absence of any other name, the letter f is often used.)

ii. We use a letter, usually x, which we will call **unspecified input**, as **place holder** standing for **specific inputs**, that is we will be allowed at any time to replace the *unspecified input* x by any *specific input* we want.

iii. Then, $f(x)$, to be read as "f of x", yet *another* use of parentheses, will stand for the *output* returned by the function f for the *unspecified input* x.

iv. Finally, we must give a *specifying phrase* to specify $f(x)$ in terms of x

EXAMPLE 31. Let $JILL$ be the function that doubles the input and adds 5 to the result to give the output. Then the function $JILL$ is specified by the *global input-output rule*

[8] Educologists will agree that this is the most natural way to characterize transcendental functions.

$$x \xrightarrow{JILL} JILL(x) = 2x + 5$$

replace vertical bar $\leftarrow$

where $2x + 5$ is the *formula* that specifies $JILL(x)$ in terms of x

4. Then, given a function specified by a global input-ouput rule, the procedure to *identify* the output for a *given* input is as follows:

i. We *indicate* by which *given* input we want to **replace** the *unspecified* input x. To do that:

a. We draw, to the right of the *unspecified input* a **vertical bar** extending a bit *below* the line, which we read as "where"

b. We write to the bottom right of the *vertical bar*:

- the *unspecified input* x

followed by

- the symbol $\leftarrow$, to be read as "is to be replaced by",

followed by

- the *specific input* that we want to replace x with.

ii. We *carry out* the replacement of the unspecified input by the given specific input.

iii. We *identify* the resulting *specifying phrase*.

EXAMPLE 32. Given the function $JACK$ specified by the global input-output rule

$$x \xrightarrow{JACK} JACK(x) = -4x + 2$$

and given the input -3, we get the output as follows.

a. We indicate that x is to be replaced by -3

$$x|_{x\leftarrow -3} \xrightarrow{JACK} JACK(x)|_{x\leftarrow -3} = -4x + 2|_{x\leftarrow -3}$$

b. We *carry out* the replacement:

$$= (-4) \cdot (-3) + 2$$

c. We *compute* the output:

$$= +12 + 2$$
$$= +14$$

d. We can then write

$$-3 \xrightarrow{JACK} +14$$

or

$$JACK(-3) = +14$$

or

quantitative bounded graph

$$(-3, +14) \text{ is an input-output pair for the function } JACK$$

.

NOTE. Most introductory textbooks ignore the "input side" of the global input-output rule. Ignoring the "input side", though, usually loses the idea of *input-output rule* in that we do not know anymore which numbers were part of the global input-output rule and which number was the input. So, we will not follow this practice.

EXAMPLE 33. Given the function$JACK$ specified by the global input-output rule

$$x \xrightarrow{\quad JACK \quad} JACK(x) = -4x - 3$$

and given the input -3, so-called *introductory* textbooks would "simply" write[9]:

$$\begin{aligned} JACK(-3) &= (-4) \cdot (-3) - 3 \\ &= +12 - 3 \\ &= +9 \end{aligned}$$

1.7 Functions Specified By A Curve

Any *curve* we draw on a *quantitative screen* specifies a *relation* because each point on the curve can be looked upon as being a *plot point* from which we can recover the *input-output pair* as we did in **Section 1.4** above and, when we take a number of points on the curve, we can even get an *input-output table*.

1. If it happens that *no* input-level line intersects the curve in more than one point, then the *relation* meets the requirement that

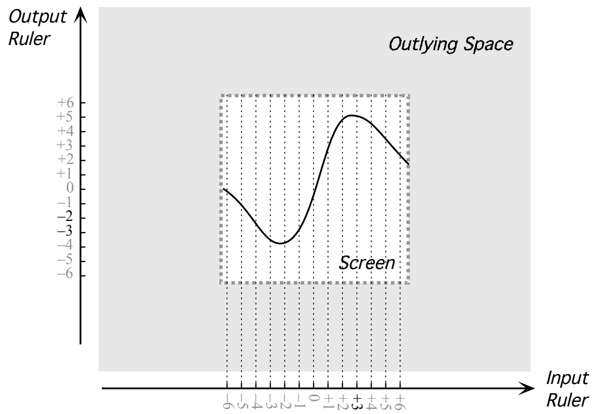
No input shall be paired with more than one output.

and the *curve* thus specifies a *function*. The curve is then called the **quantitative bounded graph** of the function.

NOTE. The reason we need to use the long phrase "quantitative bounded graph" as opposed to just the single word "graph" is because we will use many different kinds of graphs which it will be good to distinguish.

EXAMPLE 34. Given the following curve

[9]Educologists often "simplify" matters even further with "To compute $y = -4x - 3$ for -3, just replace x by -3: $y = -4(-3) - 3 = +12 - 3 = +9$". It is difficult to imagine a more efficient way to ensure the complete eradication of the underlying *function*.



none of input-level lines intersects the curve in more than one point so that the curve is the *bounded graph* of a *function*.

Strictly speaking of course, whether or not the curve specifies a function depends on what extents we used for the input ruler and for the output ruler and it could well be that with larger extents the relation specified by the curve would not meet the requirement any more.

The reason that this is not going to bother us is that we will not investigate functions *specified by* a *given* curve but, given a function specified by an *global input-output rule*, we will want to find a *curve* that specifies the *same* function. We will discuss this at some length in the next section.

2. When a *function* is given by a *curve*, and given an input, we get the output for that input with the following *procedure*:

i. We mark the *input point*, that is the tick-mark on the *input ruler* whose label is the *input number*,

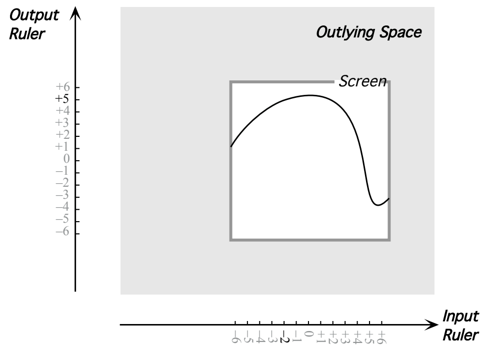
ii. We draw the *input level line*, that is the *vertical* line through the *input point*,

iii. We mark the *plot point*, that the point at the intersection of the input level line with the curve,

iv. We draw the *output level line*, that is the *horizontal* line through the *plot point*,

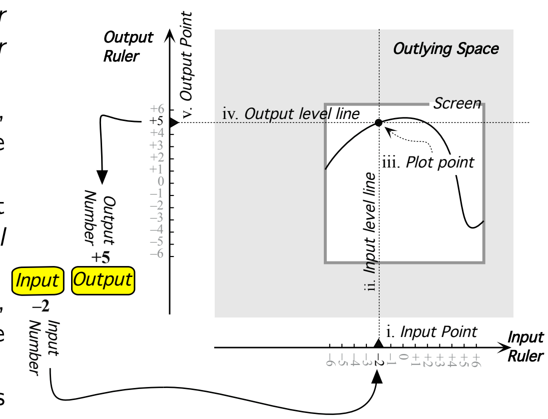
v. Then, the *output point* that pictures the *output number* is at the intersection of the output level line and the output ruler.

EXAMPLE 35. Given the function specified by the curve



and given the input -2, find the output.

i. We mark the *input point*, that is the tick-mark on the *input ruler* whose label is the *input number* -2
ii. We draw the *input level line*, that is the *vertical* line through the *input point*
iii. We mark the *plot point*, that is the point where the *input level line* intersects the *curve*
iv. We draw the *output level line*, that is the *horizontal* level line through the *plot point*,
v. Then, the *output point* is where the output level line intersects the output ruler.



3. However, the reader may have realized that things may not go as smoothly as we just made them out.

- Given a bunch of input numbers, there might not be an input ruler for which all the given input numbers are *finite numbers*. For our purposes, though, it will be enough to look at *one* input number at a time and we have seen that we can always arrange the input ruler for the given input number to be a *finite number*.
- While *we* can pick the input numbers, we have no control over the outputs as it is the *function* that will determine the outputs and we may not be able to find an output ruler on which to plot all these output numbers because output numbers could come out wildly different.

EXAMPLE 36. Given the function $MERLE$ specified by the input-output table

Inputs	Outputs
+3	1000.
+2	100.
+1	10.
0	1.
-1	0.1
-2	0.01
-3	0.001

there is not output ruler with both the *extent* necessary to show 0 and $1000.$ and the *resolution* necessary to show 0.001.

- Even if we look at just one input number, the *output point* may not be a *tick-mark* on the output ruler in which case we will not be able to say what the output number *is*. As it happens, though, this will not be much of a problem for us either because, for our purposes, it will be enough for us to say *near* which *finite number* the output number is.
- Then there there might be the issue of how we are to reconcile the *computational approximation* with the *resolution* of the ruler.

EXAMPLE 37. Given the function $RECIPROCAL$ specified by the global input-output rule

$$x \xrightarrow{RECIPROCAL} RECIPROCAL(x) = \frac{1}{x}$$

just to plot the input-ouput pair for the input 3, should we use an output ruler with:

– a *resolution* of 0.1 so as to be able to plot 0.3?
– a *resolution* of 0.01 so as to be able to plot 0.33?
– a *resolution* of 0.001 so as to be able to plot 0.333?
– etc

However, here again, this will not be an issue for us.