Math 161 REVIEW II Discussions

[Run: 08/26/2010 at 10:10. Seed: 576. Order of Checkable Items: List.]

II-1. Given the *affine* function $HERB$ such that $HERB(-1) = +5$ and $HERB(+3) = +15$

$$\text{And} \begin{cases} \text{Height } HERB\,|_{-1} = +5 \\ \text{Height } HERB\,|_{+3} = +15 \end{cases}$$

find Slope $HERB$, that is the slope of the global graph that speficifies $HERB$.

Discussion: Since we are being asked about the slope of a *function* this is a GLOBAL question.

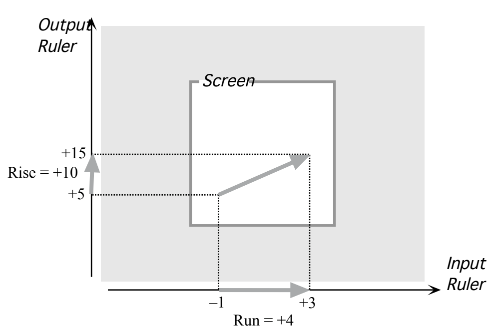
In fact, that the function f is given to be *affine* is absolutely crucial because only for *affine* functions does the graph have a *constant slope*.

a. When we go from input -1 to input $+3$, we *run* $+4$

b. When we go from the corresponding output $+5$ to the corresponding output $+15$, we *rise* $+10$.

c. The slope is therefore $\frac{+10}{+4} = +\frac{5}{2}$

Drawing a sketch helps ensuring that we count *run* and *rise* "the same way", that is from the same starting plot point to the same ending plot point.



II-2. Given that the function $DONA$ is *affine* and given the *boundary value conditions*:

$$\text{AND}\begin{cases} DONA(x)|_{x\leftarrow +4} = -1 \\ DONA(x)|_{x\leftarrow -2} = +2 \end{cases}$$

find the *global input-output rule* that specifies $DONA$

Discussion: Since we are being asked for the *global input-output rule* that specifies $DONA$, this is a *global* question which we will try to answer using the given information.

a. Since $DONA$ is given to be an *affine function*, the global input-output rule of $DONA$ is of the form:

$$x \xrightarrow{\quad DONA \quad} DONA(x) = ax + b$$

b. We use the information we have about the two inputs:

- Using the input $+4$, the global input-output rule gives

$$\begin{aligned} x|_{x\leftarrow +4} \xrightarrow{\quad DONA \quad} DONA(x)|_{x\leftarrow +4} &= ax + b|_{x\leftarrow +4} \\ &= a(+4) + b \end{aligned}$$

  that is

$$+4 \xrightarrow{\quad DONA \quad} +4a + b$$

  and since the first boundary condition was that

$$+4 \xrightarrow{\quad DONA \quad} -1$$

  because one input can have at most one output, we must have:

$$+4a + b = -1$$

- Using the input -2, the global input-output rule gives

$$\begin{aligned} x|_{x\leftarrow -2} \xrightarrow{\quad DONA \quad} DONA(x)|_{x\leftarrow -2} &= ax + b|_{x\leftarrow -2} \\ &= a(-2) + b \end{aligned}$$

  that is

$$-2 \xrightarrow{\quad DONA \quad} -2a + b$$

and since the second boundary condition was that

$$-2 \xrightarrow{DONA} = +2$$

because one input can have at most one output, we must have:

$$-2a + b = +2$$

c. We now solve the double equation problem where the unknowns are a and b:

$$\text{Both} \begin{cases} +4a + b = -1 \\ -2a + b = +2 \end{cases}$$

Given that both equations involve b with the same coefficient, namely 1, subtracting one from the other will eliminate b:

$$[+4a + b] - [-2a + b] = [-1] - [+2]$$

and since subtracting means adding the opposite

$$\begin{aligned} [+4a + b] + [+2a - b] &= [-1] + [-2] \\ +4a + b + 2a - b &= -1 - 2 \\ +6a &= -3 \\ a &= \frac{-3}{+6} \\ a &= -\frac{1}{2} \end{aligned}$$

whose solution is $-\frac{1}{2}$

To get b we can substitute $-\frac{1}{2}$ for a in either equation. We use the first one:

$$\begin{aligned} +4a + b\big|_{a \leftarrow -\frac{1}{2}} &= -1 \\ +4\left[-\frac{1}{2}\right] + b &= -1 \\ -2 + b &= -1 \\ b &= -3 \end{aligned}$$

whose solution is -3

And so the global input-output rule that specifies $DONA$ is

$$x \xrightarrow{DONA} DONA(x) = -\frac{1}{2}x - 3$$

II-3. Given the affine function $JADIH$ whose global rule is

$$x \xrightarrow{JADIH} JADIH(x) = -\frac{2}{3}x + 3$$

find its global graph.

Discussion: Since we are being asked for the global graph of a function this is a GLOBAL question.

There are two ways we can proceed:

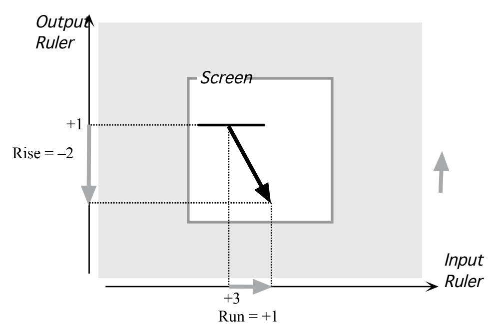
a. We can look at this as an **Initial Value Problem**, that is we can "pick" an input, say +3, localize near +3 and construct the local graph:

$$+3 + h \xrightarrow{f} f(x)|_{x=+3+h} = -\frac{2}{3}x + 3\Big|_{x=+3+h}$$

which gives after the *standard computation*:

$$= [+1]h^0 + [-2]h^1$$

and therefore the local graph where we picked the run equal to +1 so that for the slope to be −2 the rise had to be −2:



We can then extrapolate the local graph because of the THEOREM that says that the graph of an affine function is a straight line.

b. We can look at this as a **Boundary Value Problem**, that is:

i. We "pick" two inputs, say +3 and +6,

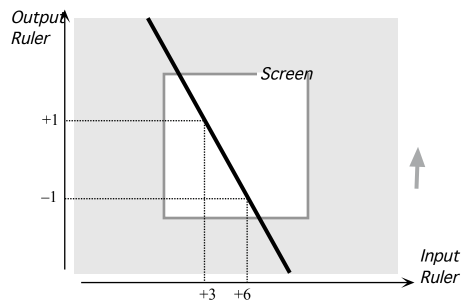
ii. We compute their outputs,

$$+3 \xrightarrow{f} f(x)|_{x=+3} = -\frac{2}{3}x + 3\Big|_{x=+3}$$

$$= +1$$

$$+6 \xrightarrow{\ f\ } f(x)|_{x=+6} = -\frac{2}{3}x + 3\Big|_{x=+6}$$
$$= -1$$

iii. We plot the points and draw a straight line through the two plot points because of the THEOREM that says that the graph of an affine function is a straight line:



II-4. Given the function $CRIC$ whose global input-output rule is

$$x \xrightarrow{\ CRIC\ } CRIC(x) = -3x - 12$$

find its 0-height input(s) if any.

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *output* has to be $= 0$), this is a GLOBAL question and the kind for which we use the

Standard approach:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we always proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be $= 0$, we set the *equation* directly. We have

$$x \xrightarrow{f} f(x) = -4x - 6$$

and since we must have

$$f(x) = 0$$

the equation that we must solve is

$$-4x - 6 = 0$$

b. Solving gives as only solution $x_{0-output} = -\frac{3}{2}$

II-5. Given the function $FILO$ whose global input-output rule is

$$x \xrightarrow{FILO} FILO(x) = +8x + 1$$

and the function $GREG$ whose global input-output rule is

$$x \xrightarrow{GREG} GREG(x) = +5x - 2$$

find the inputs, if any, for which $FILO(x) < GREG(x)$.

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *output* of f has to be smaller than the output of g), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0

ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we always proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

a. We find the equation/inequation: Since we are looking for the inputs for which the *output* of f is *smaller* than the *output* of g. So, we set the *inequation* directly. We have:

$$x \xrightarrow{\quad f \quad} f(x) = +8x + 1$$

and

$$x \xrightarrow{\quad g \quad} g(x) = +5x - 2$$

and since we must have

$$f(x) < g(x)$$

the inequation that we must solve is

$$+8x + 1 < +5x - 2$$

b. We *solve* the inequation in two steps.

i. We *solve* the associated equation to get the *break-even* input(s), if any:

$$+8x + 1 = +5x - 2$$

We add $-5x - 1$ on both sides:

$$-5x - 1 = -5x - 1$$

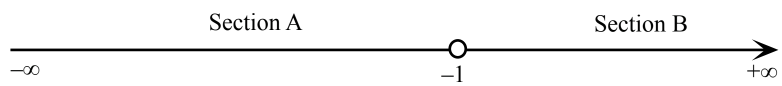
We get

$$+3x \quad = \quad -3$$

and therefore

$$x \quad = \quad -1$$

ii. We *test* an input in each one of the sections determined by the break-even point -1 for the sign of the outputs:



- To test Section A, we use a number near $-\infty$, say -1000:
  $$f(x)|_{x=-1000} = +8x+1|_{x=-1000} = -8000 + (...)$$
  $$g(x)|_{x=-1000} = -5x-1|_{x=-1000} = +5000 + (...)$$
  So, in Section A we have that $f(x) < g(x)$ is TRUE.
- To test Section B, we use a number near $+\infty$, say $+1000$:
  $$f(x)|_{x=+1000} = +8x+1|_{x=+1000} = +8000 + (...)$$
  $$g(x)|_{x=+1000} = -5x-1|_{x=+1000} = -5000 + (...)$$
  So, in Section A we have that $f(x) < g(x)$ is FALSE.

Altogether, the solution set is:

II-6. Given the function $MARC$ specified by the global input-output rule

$$x \xrightarrow{MARC} MARC(x) = +3x^2 + 6x - 17$$

find Slope-sign near ∞.

Discussion: Since we are being asked about the output of inputs *near a given input* this is a LOCAL question.

Since we are being asked about when x is near ∞, we localize near ∞:

$$x \text{ large} \xrightarrow{MARC} MARC(x)|_{x=\text{ large}} = (+3)x^2 + (+6)x + (+1)\Big|_{x=\text{ large}}$$

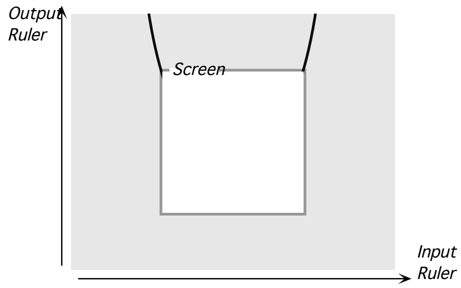
which gives after the *standard approximation*:

$$= (+3)x^2 + (...)$$

Since the *local* graph near ∞ of the power function

$$x \longrightarrow P(x) = (+1)x^2$$

is



we have that Sign slope $MARC$ near $\infty = (\diagup, \diagdown)$. (What is *left* of ∞ is to *our* right and vice versa.)

II-7. Given the function MAY specified by the global input-output rule

$$x \xrightarrow{MAYO} MAYO(x) = -2x^2 + 4x + 6$$

find Height-sign near ∞.

Discussion: Since we are being asked about the output of inputs *near a given input* this is a LOCAL question.

Since we are being asked about when x is near ∞, we localize near ∞:

$$x \text{ large} \xrightarrow{MAYO} MAYO(x)|_{x=\text{ large}} = (-2)x^2 + (+4)x + (+6)\Big|_{x=\text{ large}}$$

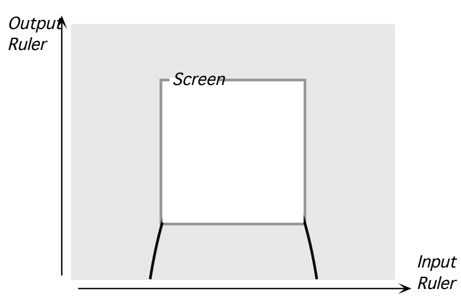
which gives after the *standard approximation*:

$$= (-2)x^2 + (...)$$

Since the *local* graph near ∞ of the power function

$$x \longrightarrow f(x) = (-1)x^2$$

is



we have that Height-sign $MAYO$ near $\infty = (-,-)$. (What is *left* of ∞ is to *our* right and vice versa.)

II-8. Let the function f be specified by the global input-output rule

$$x \xrightarrow{RONI} RONI(x) = +3x^2 + 9x + 6$$

for which input(s), if any, is the output of f equal to 0?

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *output* has to be $= 0$), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be $= 0$, we set the *inequation* directly. We have

$$x \xrightarrow{f} f(x) = +3x^2 + 9x + 6$$

and since we must have

$$f(x) = 0$$

the equation that we must solve is

$$+3x^2 + 9x + 6 = 0$$

b. To *solve* this quadratic equation, we count inputs from $x_{0-slope} = \frac{-9}{2(+3)} = -\frac{3}{2}$ by letting $x = -\frac{3}{2} + u$ because, then, there will be no u term:

$$-\frac{3}{2} + u \xrightarrow{\ f\ } f(x)|_{x=-\frac{3}{2}+u} = (+3)x^2 + (+9)x + (+6)\Big|_{x=-\frac{3}{2}+u}$$

which gives after the *standard computation*:

$$= [-\frac{3}{4}] + [0]u + [+3]u^2$$

So, instead of solving the equation

$$+3x^2 + 9x + 6 = 0$$

we solve the equation

$$-\tfrac{3}{4} + 3u^2 = 0$$

which gives us

$$u_{0-output} = \pm\tfrac{1}{2}$$

and, since $x = -\frac{3}{2} + u$, this gives us

$$x_{0-output} = -\tfrac{3}{2} \pm \tfrac{1}{2} = -2, -1$$

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{\ Q\ } Q(x) = ax^2 + bx + c$$

the 0-output inputs depend on Discriminant $f = b^2 - 4ac$:

- If Discriminant f is negative, then f has no 0-output input
- If Discriminant f is zero, then f has one 0-output input (and $x_{0-output} = x_{0-slope} = \frac{-b}{2a}$)
- If Discriminant f is positive, then f has two 0-output inputs (and $x_{0-output} = x_{0-slope} \pm \frac{\sqrt{\text{Discriminant} f}}{2a} = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$)

Then we compute Discriminant $f = (+9)^2 - 4(+3)(+6) = +9$ so that f has two 0-output inputs, namely $x_{0-output} = \frac{-9}{+6} \pm \frac{\sqrt{+9}}{+6} = \frac{-9}{+6} \pm \frac{+3}{+6} = -1, -2$

II-9. Let the function f be specified by the global input-output rule

$$x \xrightarrow{\ f\ } f(x) = -3x^2 + 7x - 6$$

where, if at all, is the output of f *negative*?

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *output* has to be < 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps

a. We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be < 0, we set the *inequation* directly. We have

$$x \xrightarrow{\quad f \quad} f(x) = -3x^2 + 7x - 6$$

and since we must have

$$f(x) < 0$$

the inequation that we must solve is

$$-3x^2 + 7x - 6 < 0$$

b. We *solve* the inequation in two steps.

i. We *solve* the associated equation

$$-3x^2 + 7x - 6 = 0$$

to get the *break-even inputs*:

To solve this quadratic equation, we count the inputs from $x_{0-slope} = \frac{-7}{2(-3)} = +\frac{7}{6}$ by letting $x = +\frac{7}{6} + u$ because, then, there will be no u term:

$$+\frac{7}{6} + u \xrightarrow{\quad f \quad} f(x)|_{x=+\frac{7}{6}+u} = (-3)x^2 + (+7)x + (-6)\Big|_{x=+\frac{7}{6}+u}$$

which gives after the *standard computation*:

$$\begin{aligned} &= [-\frac{23}{12}] + [0]u + [-3]u^2 \\ &= \end{aligned}$$

So, instead of solving the equation

$$-3x^2 + 7x - 6 = 0$$

we now must solve the equation

$$-\tfrac{23}{12} - 3u^2 = 0$$

which has no solution

ii. We *test* the sections determined by the break-even inputs. Since there is no break-even input, there is only one section. So, we can any input we want.

- We can test for x near ∞ and we get

$$\begin{aligned} x \text{ large} \xrightarrow{\quad f \quad} f(x)|_{x=\text{ large}} &= -3x^2 + 7x - 6\Big|_{x=\text{ large}} \\ &= -3x^2 + (...) \\ &< 0 \end{aligned}$$

- We can test for $x = 0$, we get

$$\begin{aligned} 0 \xrightarrow{\quad f \quad} f(x)|_{x=0} &= -3(0)^2 + 7(0) - 6 \\ &= -6 \\ &< 0 \end{aligned}$$

- We can test for any other input but the arithmetic will be more complicated.

So, for all inputs, the output will be *negative*.

There is also a “global” way to look at this but it depends on our knowledge of quadratic functions. First, since the coefficient of x^2 is negative, for inputs near ∞ the output will be negative. So, the output will be negative everywhere with a possible exception which depends on whether or not the output can be 0 which we determine by computing the discriminant. Since the discriminant is negative, there can be no 0-output and therefore the output must remain negative everywhere.

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{\ Q\ } Q(x) = ax^2 + bx + c$$

the 0-output inputs depend on Discriminant $f = b^2 - 4ac$:

- If Discriminant f is negative, then f has no 0-output input
- If Discriminant f is zero, then f has one 0-output input
- If Discriminant f is positive, then f has two 0-output inputs

then, by computing Discriminant $f = (+7)^2 - 4(-3)(-6) = -23$, we have that f has no 0-output inputs.

II-10. Given the function $TINA$ whose global input-output rule is

$$x \xrightarrow{\ TINA\ } TINA(x) = -2x^2 + 6x + 8$$

find $x_{\text{0-slope}}$.

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *slope* has to be $= 0$), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we always proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the sign of the *slope* is to be $= 0$, we find the *equation* as follows:

i. We *localize* near an unspecified input x_0:

$$x_0 + h \xrightarrow{\ f\ } f(x)|_{x=x_0+h} = (+2)x^2 + (-6)x + (+8)\Big|_{x=x_0+h}$$

which gives after the *standard computation* the coefficient of h:

$$= [\quad]h^0 + [+4x - 6]h^1 + [\quad]h^2$$

ii. We *get* the equation by setting the coefficient of h equal to 0:

$$+4x_0 - 6 = 0$$

b. We solve the equation

$$+4x_0 - 6 = 0$$

which gives us $x_{0-slope} = +\frac{3}{2}$.

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{\ Q\ } Q(x) = ax^2 + bx + c$$

the 0-slope input is $x_{0-slope} = \frac{-b}{2a}$, then we get that $x_{0-slope} = \frac{+6}{2(+2} = +\frac{3}{2}$.

(Note that the proof of the THEOREM goes exactly through the steps of the *standard approach.*)

II-11. Let f be the function specified by the global input-output rule

$$x \xrightarrow{\ f\ } f(x) = -4x^2 + 8x + 12$$

near which input(s), if any, is the output of f *decreasing*?

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *slope* has to be < 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *in-equation*, we proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the sign of the *slope* has to be < 0 we find the inequation as follows.

i. We *localize* near an unspecified input x_0:

$$x_0 + h \xrightarrow{\ f\ } f(x)\big|_{x=x_0+h} = (-4)x^2 + (+8)x + (+12)\Big|_{x=x_0+h}$$

which gives after the *standard computation* the coefficient of h:

$$= [\quad]h^0 + [-8x_0 + 8]h^1 + [\quad]h^2$$

ii. We *get* the *inequation* by setting the coefficient of h to be *negative*, that is *smaller* than 0:

$$-8x_0 + 8 < 0$$

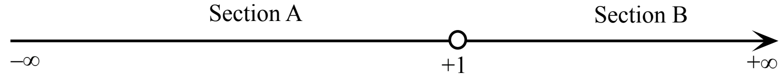
b. We *solve* the inequation in two steps.

i. We solve the *associated equation* to get the break-even inputs:

$$-8x_0 + 8 = 0$$

which gives us $x_{0-slope} = +1$.

ii. We test each of the section determined by the break-even input $x_{0-slope} = +1$:

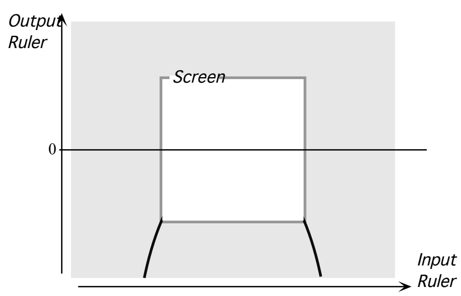


To do that, we could pick a *finite* input in each section and then check the sign of the coefficient of h for that input.

However, based on our knowledge of the power functions, it is much more efficient to test near ∞ using the graph of f when x is near ∞, namely the graph of

$$x \text{ large} \xrightarrow{\ f\ } f(x)|_{x \text{ large}} = (-4)x^2 + (+8)x + (+12)\Big|_{x \text{ large}}$$
$$= \big[-4\big]x^2 + (...)$$

which is, essentially,



The graph shows that the slope in Section A is ⁄ and that the concavity in Section B is ＼

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{\ Q\ } Q(x) = ax^2 + bx + c$$

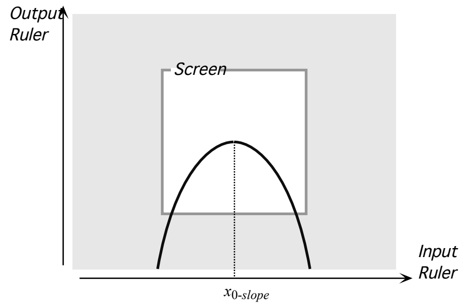
the 0-slope input is

$$x_{0-slope} = \tfrac{-b}{2a}$$

so that here

$$x_{0-slope} = \tfrac{-8}{2(-4} = +1$$

and the THEOREM that says that when the coefficient of x^2 is *negative*, the graph is like



then we can see that the *slope* will be *negative* for the inputs that are *smaller* than $x_{0-slope} = +1$

II-12. Let f be the function specified by the global input-output rule

$$x \xrightarrow{f} f(x) = -26.06x^2 + 13.03x - 21.63$$

for which input(s), if any, is Concavity-sign $f = (\cup, \cup)$?

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that Concavity-sign be $(\cup, \cup)$), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power of h to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the sign of the *concavity* has to be > 0 we find the inequation as follows.

i. We *localize* near an unspecified input x_0:

$$x_0 + h \xrightarrow{\quad f \quad} f(x)\big|_{x=x_0+h} = (-26.06x^2 + 13.03x - 21.63)\Big|_{x=x_0+h}$$

which gives after the *standard computation* the coefficient of h^2:

$$= [\quad]h^0 + [\quad]h^1 + [-26.06]h^2$$

ii. We *get* the *inequation* by setting the coefficient of h^2 to be *positive*, that is *larger* than 0:

$$-26.06 > 0$$

b. This inequation does not involve x_0 and therefore is either always TRUE or always FALSE independently of x_0. Here, since -26.06 is *negative*, it is always FALSE regardless of x_0.

So, there is no input for which Concavity-sign $f = (\cup, \cup)$

Note that if we remember the THEOREM that says that for a *quadratic* function $QUADRATIC_{a,b,c}$

$$x \xrightarrow{\quad QUADRATIC_{a,b,c} \quad} QUADRATIC_{a,b,c}(x) = ax^2 + bx + c$$

the Concaviity-sign is everywhere equal to $(\text{Sign } a, \text{Sign } a)$, then we can *see* that there is no input for which Concavity-sign $f = (\cup, \cup)$

II-13. Let f be the function specified by the global input-output rule

$$x \xrightarrow{\quad f \quad} f(x) = +x^2 - 2x + 8$$

and let g be the function specified by the global input-output rule

$$x \xrightarrow{\quad g \quad} g(x) = -3x + 7$$

for how many input(s), if any, do the functions f and g return the same output?

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *output* of f has to be equal to the output of g), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps.

a. We find the equation/inequation: Since we are looking for the inputs for which the *output* of f is *equal* to the *output* of g we set the *inequation* directly. We have:

$$x \xrightarrow{\quad f \quad} f(x) = +x^2 - 2x + 8$$

and

$$x \xrightarrow{\quad g \quad} g(x) = -3x + 7$$

and since we must have

$$f(x) = g(x)$$

the equation that we must solve is

$$+x^2 - 2x + 8 = -3x + 7$$

which we turn into a quadratic equation by adding $+3x-7$ on both sides:

$$+3x - 7 \qquad +3x - 7$$

We get

$$+x^2 + x + 1 = 0$$

b. We *solve* this quadratic equation by counting the inputs from $x_{0-slope} = \frac{-1}{2(+1)} = -\frac{1}{2}$ by letting $x = -\frac{1}{2} + u$ because, then, there will be no u term:

$$-\frac{1}{2} + u \xrightarrow{\ f\ } f(x)\big|_{x=-\frac{1}{2}+u} = +x^2 + x + 1\Big|_{x=-\frac{1}{2}+u}$$

which gives after the *standard computation*:

$$= [+\frac{3}{4}] + [0]u + [+1]u^2$$

So, instead of solving the equation

$$+x^2 + x + 1 = 0$$

we now must solve the equation

$$+\tfrac{3}{4} + u^2 = 0$$

which has no solution

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{\ Q\ } Q(x) = ax^2 + bx + c$$

the 0-output inputs depend on Discriminant $f = b^2 - 4ac$:

- If Discriminant f is negative, then f has no 0-output input
- If Discriminant f is zero, then f has one 0-output input
- If Discriminant f is positive, then f has two 0-output inputs

then, by computing Discriminant $f = (+1)^2 - 4(+1)(+1) = -3$, we have that f has no 0-output inputs.

II-14. Given the function $TITO$ whose global input-output rule is

$$x \xrightarrow{\ TITO\ } TITO(x) = +3x^2 + 9x + 6$$

what is the highest output(s), if any?.

Discussion: Since we are being asked for "the" highest output, we must look at the output of all the inputs and so this is a GLOBAL question that requires knowledge of the general theory for quadratic functions:

a. Since the coefficient of x^2 is *positive*, the graph of $TITO$ is *concave up* everywhere.

b. In particular, the graph of $TITO$ is concave up near $x_{0-slope}$ which is therefore the input with the lowest output. (Visualize the local graph near $x_{0-slope}$.)

c. Near ∞, the output is *infinitely high*

So, $TITO$ does not have a *highest* (finite) output.

II-15. Given the function f whose global input-output rule is

$$x \xrightarrow{f} f(x) = x^3 - 12x^2 + 45x + 10$$

what is Slope-sign f near $+3$?

Discussion: Since we are being asked about the output of inputs *near a given input* this is a LOCAL question.

Since we are being asked about when x is near a *finite* input, we localize near that input and since we are looking for the *slope*, we look for the coefficient of h in the output:

$$+3 + h \xrightarrow{f} f(x)|_{x=+3+h} = (+1)x^3 + (-12)x^2 + (+45)x + (+10)\Big|_{x=+3+h}$$

which gives after the *standard computation* the coefficient of h:

$$= [\quad]h^0 + [0]h^1 + [\quad]h^2 + [\quad]h^3$$

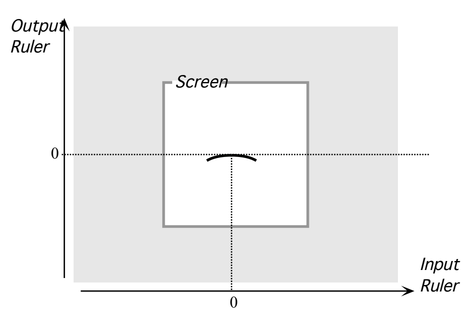
So, as it happens, here $+3$ is a 0-slope input and the slope near $+3$ will have to come from the coefficient of h^2:

$$\begin{aligned} &= [\quad]h^0 + [0]h^1 + [(+1)\cdot(3\cdot +3) + (-12)\cdot 1]h^2 + [\quad]h^3 \\ &= [\quad]h^0 + [0]h^1 + [-3]h^2 + [\quad]h^3 \end{aligned}$$

Since the *local* graph of the power function

$$h \xrightarrow{f} f(x)|_{x=+3+h} = (-3)h^2$$

is



we have that Sign slope f near $+3 = (\diagup, \diagdown)$ (What is left of a *finite* point is to *our* left etc.)

II-16. Given the function f whose global input-output rule is

$$x \xrightarrow{\ f\ } f(x) = -2x^3 + 12x^2 + 8x - 7$$

what is Concavity-sign f near $+3$?

Discussion: Since we are being asked about the output of inputs *near a given input* this is a LOCAL problem.

Since we are being asked about the *concavity*, we look for the coefficient of h^2 in the output:

$$\begin{aligned} +3+h \xrightarrow{\quad f \quad} f(x)|_{x=+3+h} &= (-2)x^3 + (+12)x^2 + (+8)x + (-7)\Big|_{x=+3+h} \\ &= (-2)\,[+3+h]^3 + (+12)\,[+3+h]^2 + (+8))\,[+3+h] + (-7) \\ &= (-2)\left[+27 + 27h + 9h^2 + [...]\right] + (+12)\left[+9 + 6h + h^2\right] + (+8)\,[+3+h] + (-7) \end{aligned}$$

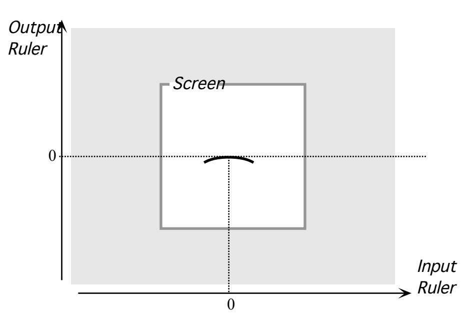
which gives after the *standard computation* the coefficient of h^2:

$$= [\quad]h^0 + [\quad]h^1 + [-6]h^2 + [\quad]h^3$$

Since the *local* graph of the power function

$$h \xrightarrow{\quad f \quad} f(x)|_{x=+3+h} = (-6)h^2$$

is



we have that Sign concavity f near $+3 = (\cap, \cap)$

II-17. Given the function f whose global input-output rule is

$$x \xrightarrow{f} f(x) = x(x-2)^2$$

where is the output equal to 0?

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *output* has to be $= 0$), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be $= 0$, we set the *equation* directly. We have

$$x \xrightarrow{\quad f \quad} f(x) = x(x-2^2$$

and since we must have

$$f(x) = 0$$

the equation that we must solve is

$$+x(x-2^2 = 0$$

b. We *solve* the equation. This is a *cubic* equation and, normally, we don't know how to solve a *cubic* equation. Here, though, the cubic is *factored* and so solving amounts to the same as solving:

$$\text{OR}\begin{cases} x = 0 \\ (x-2)^2 = 0 \end{cases}$$

i. The first equation,

$$x = 0$$

gives us $x_{0-output} = 0$

ii. The second equation,

$$(x-2)^2 = 0$$

gives us $x_{0-output} = +2$

II-18. Given the function f whose global input-output rule is

$$x \xrightarrow{f} f(x) = -(x-1)(x-2)(x-3)$$

where is the output of f *negative*?

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *output* has to be < 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We follow the above steps

a. We *find* the equation/inequation. Since we are looking for the inputs for which the *output* is to be < 0, we set the *inequation* directly. We have

$$x \xrightarrow{\ f\ } f(x) = -(x-1)(x-2)(x-3)$$

and since we must have

$$f(x) < 0$$

the inequation that we must solve is

$$-(x-1)(x-2)(x-3) < 0$$

b. We *solve* the inequation in two steps.

i. We *solve* the associated equation

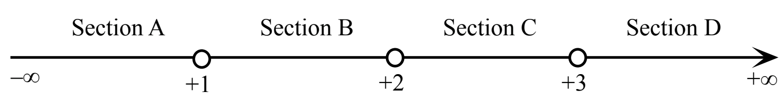
$$-(x-1)(x-2)(x-3) = 0$$

to get the *break-even inputs*: This is a *cubic* equation and, normally, we don't know how to solve a *cubic* equation. Here, though, the cubic is *factored* and so solving amounts to the same as solving:

$$\text{OR}\begin{cases} x-1=0 \\ x-2=0 \\ x-3=0 \end{cases}$$

- The first equation,
$$x-1=0$$
gives us the break-even input $x_{0-output} = +1$
- The second equation,
$$x-2=0$$
gives us the break-even input $x_{0-output} = +2$
- The third equation,
$$x-3=0$$
gives us the break-even input $x_{0-output} = +3$

ii. We test each of the four sections determined by the three break-even inputs $x_{0-slope} = +1$, $x_{0-slope} = +2$, $x_{0-slope} = +3$, :

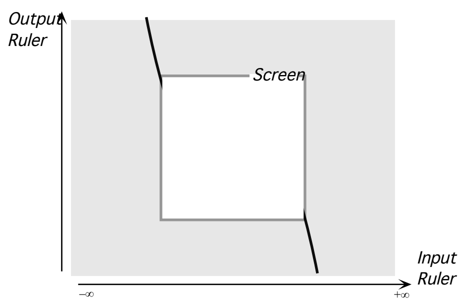


To do that, we could pick a *finite* input in each section and then check the sign of the coefficient of h^0 (that is, in fact, the sign of the output) for that input.

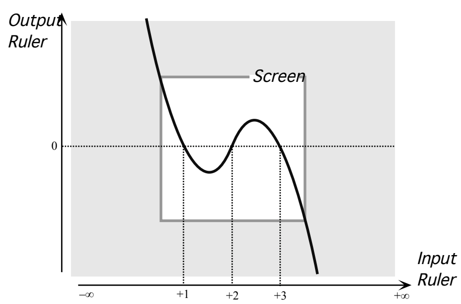
However, based on our knowledge of the cubic functions, it is much more efficient to test near ∞ using the graph of f when x is near ∞, namely the graph of

$$\begin{aligned} x \text{ large} \xrightarrow{\ f\ } f(x)|_{x \text{ large}} &= -(x-1)(x-2)(x-3)|_{x \text{ large}} \\ &= -\big(x+(...)\big)\big(x+(...)\big)\big(x+(...)\big) \\ &= -x^3+(...) \end{aligned}$$

which is, essentially,



and then to use the global graph of f which must be:



The global graph shows that the output in Section A is +, the output in Section B is −, the output in Section C is +, the output in Section D is −.

II-19. Let f be the function specified by the global input-output rule

$$x \xrightarrow{\ f\ } f(x) = +2x^3 + x^2 + 10x + 7$$

where is the slope of f equal to 0?.

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *slope* has to be $= 0$), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we always proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the sign of the *slope* is to be $= 0$, we find the *equation* as follows:

i. We *localize* near an unspecified input x_0:

$$x_0 + h \xrightarrow{\quad f \quad} f(x)|_{x=x_0+h} = (+2)x^3 + (+1)x^2 + (+10)x + (+7)\Big|_{x=x_0+h}$$

which gives after the *standard computation* the coefficient of h:

$$= 2[x_0 + h]^3 + [x_0 + h]^2 + 10[x_0 + h] + 7$$

using the *addition formulas* for cube and for square

$$= 2\left[x_0^3 + 3x_0^2 h + [...]\right] + \left[x_0^2 + 2x_0 h + [...]\right] + 10\,[x_0 + h] + 7$$
$$= [\quad]h^0 + [+6x_0^2 + 2x_0 + 10]h^1 + [\quad]h^2 + [\quad]h^3$$

ii. We *get* the equation by setting the coefficient of h equal to 0:

$$+6x_0^2 + 2x_0 + 10 = 0$$

b. We *solve* this quadratic equation by counting the inputs from $x_{0-slope} = \frac{-2}{2(+6)} = -\frac{1}{6}$ by letting $x = -\frac{1}{6} + u$ because, then, there will be no u term:

$$-\frac{1}{6} + u \xrightarrow{\ f\ } f(x)\big|_{x=-\frac{1}{6}+u} = +6x^2 + 2x + 10\Big|_{x=-\frac{1}{6}+u}$$

which gives after the *standard computation*:

$$= [+\frac{59}{6}] + [0]u + [+6]u^2$$

So, instead of solving the equation

$$+6x^2 + 2x + 10 = 0$$

we now must solve the equation

$$+\tfrac{59}{6} + 6u^2 = 0$$

which has no solution

Note that if we remember the THEOREM that says that, for a *quadratic* function

$$x \xrightarrow{\ Q\ } Q(x) = ax^2 + bx + c$$

the 0-output inputs depend on Discriminant $f = b^2 - 4ac$:

- If Discriminant f is negative, then f has no 0-output input
- If Discriminant f is zero, then f has one 0-output input
- If Discriminant f is positive, then f has two 0-output inputs

then, by computing Discriminant $f = (+2)^2 - 4(+6)(+10) = -236$, we have that the equation $+6x^2 + 2x + 10 = 0$ has no solution.

II-20. Given the function f specified by the global input-output rule

$$x \xrightarrow{\ f\ } f(x) = +x^3 - 9x^2 + 15x - 11$$

where is f *concave up*?

Discussion: Since we are being asked for all the input(s), if any, for which "something" is required to be the case (here it is that the *concavity* has to be > 0), this is a GLOBAL question and the kind for which we use the

STANDARD APPROACH:

a. *Find* the equation/inequation that the inputs must be a solution of in order for the "something" to be the case.

- When the "something" involves *outputs*, we set the equation/inequation directly.
- When the "something" involves the *slope* or the *concavity*,
  i. we *localize* near an *unspecified* input x_0
  ii. we get the equation/inequation by setting the coefficient of the appropriate power to have the required sign, $= 0$, < 0 or > 0,

b. *Solve* the resulting equation/inequation. In the case of an *inequation*, we proceed in two steps:

i. we *solve* the associated equation to get the *break-even inputs*,

ii. we *test* the sections determined by the break-even inputs.

EXECUTION: We will follow the above steps.

a. We *find* the equation/inequation. Since we are looking for the inputs for which the sign of the *concavity* has to be > 0 we find the inequation as follows.

i. We *localize* near an unspecified input x_0:

$$x_0 + h \xrightarrow{\ f\ } f(x)\big|_{x=x_0+h} = (+1)x^2 + (-9)x^2 + (+15)x + (-11)\Big|_{x=x_0+h}$$

which gives after the *standard computation* the coefficient of h^2:

$$= [\quad]h^0 + [\quad]h^1 + [+6x_0 - 18]h^2 + [\quad]h^3$$

ii. We *get* the *inequation* by setting the coefficient of h^2 to be *positive*, that is *larger* than 0:

$$+6x_0 - 18 > 0$$

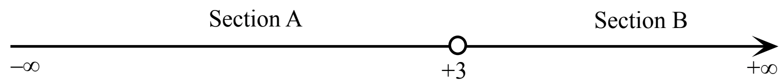
b. We *solve* the inequation in two steps.

i. We solve the *associated equation* to get the break-even inputs:

$$+6x_0 - 18 = 0$$

which gives us $x_{0-concavity} = +3$.

ii. We test each of the section determined by the break-even input $x_{0-concavity} = +3$:

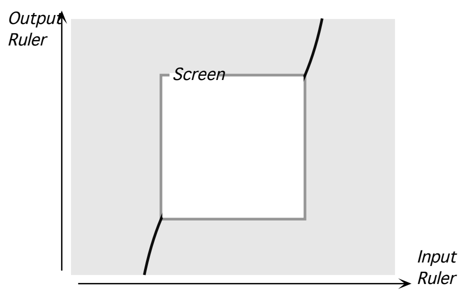


To do that, we could pick a *finite* input in each section and then check the sign of the coefficient of h for that input.

However, based on our knowledge of the power functions, it is much more efficient to test near ∞ using the graph of f when x is near ∞, namely the graph of

$$x \text{ large} \xrightarrow{\quad f \quad} f(x)|_{x \text{ large}} = (+1)x^3 + (-9)x^2 + (+15)x + (-11)\Big|_{x \text{ large}}$$
$$= \big[+1\big]x^3 + (...)$$

which is, essentially,



The graph shows that the concavity in Section A is $\cap$ and that the concavity in Section B is $\cup$

Note that if we remember the THEOREM that says that, for a *cubic* function

$$x \xrightarrow{\quad Q \quad} Q(x) = ax^3 + bx^2 + cx + d$$

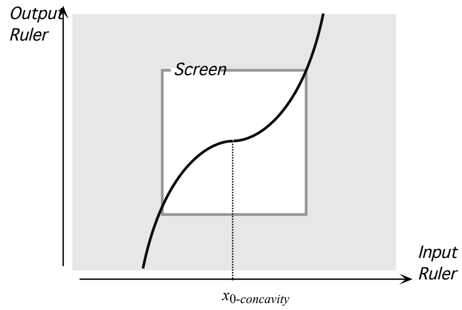
the 0-concavity input is

$$x_{0-concavity} = \frac{-b}{3a}$$

so that here

$$x_{0-concavity} = \frac{+9}{3(+1} = +3$$

and the THEOREM that says that when the coefficient of x^3 is *positive*, the graph is like



then we can see that the *concavity* will be *cup* for the inputs that are *larger* than $x_{0-concavity} = +3$