Sets of monotonicity for the Riemann Zeta Function

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1 Introduction

Using the methods of Bohr, Jessen, and Wintner [1, 2], it can be shown that the set

$$\mathcal{A}(T, \sigma_0) = \{t \in [T, 2T] : -\Re \frac{\zeta'}{\zeta}(\sigma + it) \geq 0 \ \forall \sigma \geq \sigma_0\}$$

has a density when $\sigma_0 > 1/2$ is fixed, in the sense that

$$\lim_{T \to \infty} \text{meas} \frac{\mathcal{A}(T, \sigma_0)}{T} \tag{1}$$

exists. In particular, such a density exists if we restrict ourselves to the smaller set

$$\mathcal{A}(T, \sigma_0, \sigma') = \{t \in [T, 2T] : -\Re \frac{\zeta'}{\zeta}(\sigma + it) \geq 0 \ \forall \sigma \in (\sigma_0, \sigma')\},$$

for $1/2 < \sigma_0 < 1 < \sigma'$.

We will choose $\sigma' > 1$ so that

$$\sum_p \frac{\log p}{p^{\sigma'}} = 29. \tag{2}$$

This choice of sigma is selected merely to overwhelm some finite sums that will appear later.

In this paper, we establish the following.

Theorem 1

$$\lim_{T \to \infty} \frac{\text{meas } \mathcal{A}(T, \sigma_0, \sigma')}{T} \gg (\sigma_0 - 1/2)^2$$

as $\sigma_0 \downarrow 1/2$ *and the limit on the left is approached uniformly for* $\sigma_0 > 1/2 + B \log\log\log\log T/\log\log\log T$ *if* B *is large enough.*

In order to do this, we argue that since the behaviour of $-\Re\frac{\zeta'}{\zeta}(\sigma+it)$ is essentially that of

$$\sum_p \frac{\log p}{p^\sigma}\cos(t\log p),$$

we are just led to investigate the σ for which this sum stays positive. This study can be further reduced to the study of the positivity of all the partial sums of

$$\sum_{p\le x} \frac{\log p}{p^{1/2}}\cos(t\log p).$$

To measure the set of all $t \in [T, 2T]$ where the partial sums of this last sum are positive, we argue that since the logarithms of the primes are linearly independent over the rationals, finding this measure is equivalent to finding the probability

$$\text{Prob}\left(\sum_{p\le y}\frac{\log p}{p^{1/2}}\mathbf{X}_p \ge 0 \text{ for all } y \le x\right), \tag{3}$$

where the $\mathbf{X}_p$ are independent, identically distributed random variables having density

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in (-1,1).$$

To justify the passage to probability theory, we use a multidimensional integral analogue of the Erdős-Turán Inequality.

The computation of the probability in (3) is calculated using the rudiments of Brownian Motion Theory.

2 Reduction to Random Variables

In this section we legitimise the passage to probability theory. We estimate the difference between the number-theoretic quantity

$$\frac{1}{T}\text{meas}\left\{t \in [T, 2T] : \sum_{p \le y} \frac{\log p}{p^\sigma} \cos(t \log p) \ \ge 0 \ \text{ for all } \ y \le x\right\}$$

and the probabilistic quantity

$$\text{Prob}\left(\sum_{p \le y} \frac{\log p}{p^\sigma} \mathbf{X}_p \ge 0 \ \text{ for all } \ y \le x\right),$$

where the $\mathbf{X}_p$ are independent, identically distributed random variables having density

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in (-1, 1).$$

There are many instances in the literature where such a passage has been necessary, [2] being probably the earliest, and the references [1], [3] and [4] providing many diverse applications of this technique. See also the work [5] for various computations of distributions.

In what follows, the letters p and q will always denote prime numbers, $\pi(x) = \sum_{p \le x} 1$ and $q = q(x)$ will denote the largest prime not exceeding x. Also, the vector $\mathbf{u}$ will have coordinates indexed by the primes, $\mathbf{u} = (u_2, u_3, u_5, \ldots, u_q)$; $\mathbf{h} = (\log 2, \log 3, \ldots, \log q)$. Observe that the vectors here live in a $\pi(x)$-dimensional space. Furthermore, we let

$$\mathbf{c}(t) = (\{t \log 2\}, \{t \log 3\}, \ldots, \{t \log q\}),$$

where $\{x\}$ denotes, as usual, the fractional part of x. Finally, we denote by $\mathcal{B}$ the collection of all boxes B contained in the unit torus with sides parallel to the axes, and $\mathbf{T} = \mathbf{R}/\mathbf{Z}$.

From Theorem 1 of [6] we deduce

Lemma 2 *Let m be a positive integer. For each $i, 1 \leq i \leq m$, let K_i be a positive integer, α_i, β_i real numbers with $\alpha_i \leq \beta_i \leq \alpha_i + 1$. Let $B = \mathsf{X}_{i=1}^m [\alpha_i, \beta_i] \subseteq \mathbf{T}^m$. Then there exist trigonometric polynomials $T_B^+(\mathbf{x})$, $T_B^-(\mathbf{x})$, such that*

$$T_B^{\pm}(\mathbf{x}) = \sum_{|k_i| \leq K_i} \widehat{T}_B^{\pm}(\mathbf{k}) e(\mathbf{k} \cdot \mathbf{x}),$$

$$T_B^-(\mathbf{x}) \leq \chi_B(\mathbf{x}) \leq T_B^+(\mathbf{x})$$

for all $\mathbf{x} \in \mathbf{T}^m$, and

$$\int_{\mathbf{T}^m} \left| T_B^{\pm}(\mathbf{x}) - \chi_B(\mathbf{x}) \right| \, d\mathbf{x} \leq 2 \left(\prod_{i=1}^m \left(\beta_i - \alpha_i + \frac{1}{K_i + 1} \right) - \prod_{i=1}^m (\beta_i - \alpha_i) \right).$$

We remark that if we write $l_i = \beta_i - \alpha_i$ and expand

$$\prod_{i=1}^{m}\left(l_i + \frac{1}{K_i+1}\right) - \prod_{i=1}^{m} l_i$$

in monomials in the l_i, then, since all the coefficients are ≥ 0, this expression increases with the l_i, and it attains its maximum when all the l_i are 1. Thus

$$\int_{\mathbf{T}^m} \left|T_B^{\pm}(\mathbf{x}) - \chi_B(\mathbf{x})\right| \, d\mathbf{x} \ \leq \ 2\left(\prod_{i=1}^{m}\left(1 + \frac{1}{K_i+1}\right) - 1\right)$$

uniformly for all boxes B.

The following lemma is, in a sense, a continuous analogue of the Erdős - Turán Inequality.

Lemma 3 *Let* $T > 0$. *For any positive* $K \geq \pi(x)$,

$$\sup_{B\in\mathcal{B}}\left|\frac{1}{T}\int_T^{2T} \chi_B(\mathbf{c}(t))\,dt - \operatorname{meas} B\right| \ll \frac{\pi(x)}{K} + \frac{1}{T}e^{4Kx}.$$

Proof. Consider a particular $\pi(x)$-dimensional box $B = \mathsf{X}_{i=1}^{\pi(x)}(a_i, b_i)$, $a_i \leq b_i \leq a_i + 1$. We take $K_i = K > 0$ for all i, $m = \pi(x)$ in Lemma 2. Here $\mathbf{k}$ has entries k_i with $|k_i| \leq K$. In virtue of the just cited lemma,

$$\frac{1}{T}\int_T^{2T} \chi_B(\mathbf{c}(t))\,dt - \operatorname{meas} B \ \leq \frac{1}{T}\int_T^{2T} T_B^{+}(\mathbf{c}(t))\,dt - \operatorname{meas} B.$$

Using the Fourier expansion of T_B^{+}, the above expression equals

$$\widehat{T}_B^{+}(\mathbf{0}) + \frac{1}{T}\sum_{\mathbf{k}\neq\mathbf{0}} \widehat{T}_B^{+}(\mathbf{k}) \int_T^{2T} e(t\mathbf{k}\cdot\mathbf{h})\,dt - \operatorname{meas} B.$$

The above is in turn is less than or equal to

$$2\prod_{i=1}^{\pi(x)}(b_i - a_i + \frac{1}{K+1}) - 2\prod_{i=1}^{\pi(x)}(b_i - a_i) + \frac{1}{T}\sum_{\mathbf{k}\neq\mathbf{0}} \widehat{T}_B^+(\mathbf{k}) \int_T^{2T} e(t\mathbf{k}\cdot\mathbf{h})\, dt.$$

Here, we are using the standard notation $e(z) = e^{2\pi i z}$. Since the logarithms of the prime numbers are linearly independent over the rational numbers, $\mathbf{k}\cdot\mathbf{h} \neq 0$ for $\mathbf{k} \neq \mathbf{0}$. This enables us to conclude that

$$\frac{1}{T}\int_T^{2T} \chi_B(\mathbf{c}(t))\, dt - \text{meas}\, B \ \leq 2\prod_{j=1}^{\pi(x)}(1+\frac{1}{K+1}) - 2 + \frac{2}{T}\sum_{\mathbf{k}\neq\mathbf{0}} \left|\widehat{T}_B^+(\mathbf{k})\right| \frac{1}{|\mathbf{k}\cdot\mathbf{h}|}. \quad (4)$$

Now,

$$\left|\widehat{T}_B^+(\mathbf{k})\right| \leq \left|\widehat{T}_B^+(\mathbf{k}) - \widehat{\chi}_B(\mathbf{k})\right| + |\widehat{\chi}_B(\mathbf{k})| \leq \int_{\mathbf{T}^m} |T_B^+(\mathbf{x}) - \chi_B(\mathbf{x})|\, d\mathbf{x} + |\widehat{\chi}_B(\mathbf{k})|. \quad (5)$$

We observe that $|\widehat{\chi}_B(\mathbf{k})| \leq \prod_{j=1}^{\pi(x)} \min(b_j - a_j, \frac{1}{|\pi k_j|}) \ll 1$. By the remark preceding this Lemma, the first quantity on the right-hand side of (5) is at most

$$2\left(\prod_{j=1}^{\pi(x)}(1+\frac{1}{K+1}) - 1\right).$$

Moreover, since $1 + x \leq e^x$ and $\pi(x) \leq K$, this is at most

$$2\left(\exp\left(\pi(x)/(K+1)\right) - 1\right) \ll \frac{\pi(x)}{K}.$$

Therefore $\left|\widehat{T}_B^+(\mathbf{k})\right| \ll 1 + \frac{\pi(x)}{K} \ll 1$. Upon combining this with (4) and (5), we obtain

$$\frac{1}{T}\int_T^{2T} \chi_B(\mathbf{c}(t))\, dt - \text{meas}\, B \ \ll \frac{\pi(x)}{K} + \frac{1}{T}\sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{|\mathbf{k}\cdot\mathbf{h}|}. \quad (6)$$

Also,

$$\sum_{\mathbf{k}\neq\mathbf{0}} \frac{1}{|\mathbf{k}\cdot\mathbf{h}|} \ll \sum_{0<m<n<\prod_{p\leq x} p^K} \frac{1}{\log(n/m)}.$$

To estimate this last sum, observe that if $a > 1$, then $\frac{1}{\log a} \leq 1 + \frac{1}{a-1}$. Thus

$$\sum_{0<m<n<\prod_{p\leq x} p^K} \frac{1}{\log(n/m)} \ll e^{4Kx}, \tag{7}$$

by the Chebyshev estimates.

Combining (6) and (7) we finally arrive at

$$\frac{1}{T}\int_T^{2T} \chi_B(\mathbf{c}(t))\,dt - \operatorname{meas} B \ll \frac{\pi(x)}{K} + \frac{1}{T}e^{4Kx}, \tag{8}$$

whence an upper bound is obtained. The lower bound is computed similarly.

Let K be a positive integer, $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ and let B be a closed body in $[0,K]^m$ with the property

$$\text{Property } (M) : \text{if } \mathbf{x} \in B, \text{ then } X_{i=1}^m [0, x_i] \subseteq B.$$

We divide $[0,K]^m$ into K^m cells $X_{i=1}^m [k_i, k_i + 1]$ where the integers k_i satisfy $0 \leq k_i < K$. There are three kinds of cells C:

1. $C \subseteq B$; (interior)

2. $C \bigcap B = \emptyset$; (exterior)

3. $C \bigcap B \neq \emptyset, C \not\subseteq B$. (boundary)

The boundary ∂B of B is contained in the union of the cells of the third type. Let $\#(B)$ denote the number of these boundary cells, and let $f_m(K)$ denote the maximum of $\#(B)$ over all such bodies B. We will say that the $m-1$-dimensional body $X_{i=1}^{m-1}[k_i, k_i+1]X\{k_m+1\}$ is the *upper face* of the m-dimensional box $X_{i=1}^{m}[k_i, k_i+1]$ and that $X_{i=1}^{m-1}[k_i, k_i+1]X\{k_m\}$ is its *lower face*. The following will give us an upper bound for $f_m(K)$.

Lemma 4 *Let K be a positive integer and let B a closed body in $[0, K]^m$ with property (M). For every $m \geq 1$ and every positive integer K,*

$$f_m(K) \leq 2m(m+1)K^{m-1}.$$

Proof. The proof is by induction on m. Of the cells under consideration, we distinguish three types:

i. Those with an upper face lying entirely outside B;

ii. Those with a lower face lying entirely within B;

iii. All those with faces that intersect B without lying in B.

For given x, i, choose integers k_j for all $j \neq i, 1 \leq j \leq m, 0 \leq k_j < K$, and consider the set

$$\mathcal{F}(x) = \{(x_1, x_2, \ldots, x_m) \in \mathbf{R}^m : x_i = x, k_j \leq x_j \leq k_j + 1 \text{for all } j \neq i\}.$$

When x is an integer, this is the face of a cell. This set moves parallel to itself as x varies. Consider the least integer x such that $\mathcal{F}(x)$ is disjoint from B. Then $\mathcal{F}(x)$ is the upper face of a cell of type (i). Since there are m choices for i, and K^{m-1} choices for k_j, we deduce that there are at most mK^{m-1} cells of type (i). By similarly considering the greatest integer x such that $\mathcal{F}(x) \subseteq B$, we deduce that the number of cells of type (ii) is at most mK^{m-1}.

Now, fix k_i, and consider a slice through $[0, K]^m$ with $x_i = k_i$. Here $1 \leq k_i \leq K$, and we count at most $f_{m-1}(K)$ upper faces that lie partially, but not entirely, within B. By varying i and k_i, we find at most $mKf_{m-1}(K)$ such upper faces. By allowing k_i to run over $[0, K-1]$, we similarly count at most $mKf_{m-1}(K)$ such lower faces. Altogether, there are at most $2mKf_{m-1}(K)$ cell faces lying partially, but not entirely, within B. Since each cell of type (iii) has $2m$ such faces, it follows that there are at most $Kf_{m-1}(K)$ such cells. Upon assembling these estimates, we deduce that

$$f_m(K) \leq 4mK^{m-1} + Kf_{m-1}(K).$$

The result follows from the expression above and the induction hypothesis.

Let

$$\mathcal{S} = \{\mathbf{u} \in (0,1]^{\pi(x)} : \sum_{p \leq y} \frac{\log p}{p^\sigma} \cos(2\pi u_p) \ \geq 0 \text{ for all } y \leq x\}.$$

Let R be a positive integer and let $\mathcal{F}$ be the family of all $\pi(x)$-dimensional boxes

$$V = X_{i=1}^{\pi(x)} [\frac{a_i}{R}, \frac{a_i+1}{R})$$

where a_i ranges through all integers in $[0, R-1]$. We define the *in-boxes* as

$$\mathcal{S}_i = \{V \in \mathcal{F} : V \subseteq \mathcal{S}\}$$

and the *out-boxes* as

$$\mathcal{S}_o = \{V \in \mathcal{F} : V \cap \mathcal{S} \neq \emptyset\}.$$

Finally, let

$$\mathcal{D} = \bigcup_{V \in \mathcal{S}_o \setminus \mathcal{S}_i} V.$$

Corollary 5

$$\text{meas}\, \mathcal{D} \ll \pi^2(x)/R.$$

Proof. Consider the body $\mathcal{S}_R$,

$$\mathcal{S}_R = \{\mathbf{u} \in [0, R]^{\pi(x)} : \sum_{p \le y} \frac{\log p}{p^\sigma} \cos(\pi u_p/R) \ \ge 0 \text{ for all } y \le x\},$$

where R is a positive integer. With $m = \pi(x), K = R$, the body $\mathcal{S}_R$ is closed and satisfies property (M). By the preceding lemma, there are $\ll \pi^2(x) R^{\pi(x)-1}$ unit-volume boxes intersecting the boundary of $\mathcal{S}_R$. It is clear, then, that $\mathcal{D}$ consists of $\ll \pi(x)^2 R^{\pi(x)-1}$ boundary boxes, each having volume $R^{-\pi(x)}$. The result follows from this.

Theorem 6 *Let* $\mathbf{X}_p$ *be independent, identically distributed random variables having density*

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in (-1,1).$$

Then, for $x = [(\log\log T)^{1/4}]$ *and as* $T \to \infty$,

$$\left| \frac{1}{T}\text{meas}\left\{t \in [T,2T] : \sum_{p\le y} \frac{\log p}{p^{\sigma_0}} \cos(t\log p) \ \ge 0 \ \forall y \le x\right\} \right.$$
$$\left. - \text{Prob}\left(\sum_{p\le y} \frac{\log p}{p^{\sigma_0}} \mathbf{X}_p \ge 0 \ \forall y \le x \right) \right| \ll \frac{1}{(\log\log\log T)^2 (\log\log T)^{1/6}}.$$

Proof. We wish to demonstrate that

$$\left| \frac{1}{T}\int_T^{2T} \chi_{\mathcal{S}}(\mathbf{c}(t))\,dt - \text{meas}\,\mathcal{S} \right| \ll \frac{1}{(\log\log\log T)^2 (\log\log T)^{1/6}}.$$

Simply observe that, as $\mathcal{S}_i \subseteq \mathcal{S} \subseteq \mathcal{S}_o$, we have

$$\frac{1}{T}\int_T^{2T} \chi_{\mathcal{S}_i}(\mathbf{c}(t))\,dt - \text{meas} \bigcup_{V\in\mathcal{S}_i} V + \text{meas} \bigcup_{V\in\mathcal{S}_i} V - \text{meas}\,\mathcal{S} \le \frac{1}{T}\int_T^{2T} \chi_{\mathcal{S}}(\mathbf{c}(t))\,dt - \text{meas}\,\mathcal{S},$$

and

$$\frac{1}{T}\int_T^{2T} \chi_{\mathcal{S}}(\mathbf{c}(t))\,dt - \text{meas}\,\mathcal{S} \le \frac{1}{T}\int_T^{2T} \chi_{\mathcal{S}_o}(\mathbf{c}(t))\,dt - \text{meas} \bigcup_{V\in\mathcal{S}_o} V + \text{meas} \bigcup_{V\in\mathcal{S}_o} V - \text{meas}\,\mathcal{S}.$$

Thus

$$\begin{aligned}\left|\frac{1}{T}\int_T^{2T}\chi_{\mathcal{S}}(\mathbf{c}(t))\,dt-\operatorname{meas}\mathcal{S}\right| &\leq \max_{j=i,o}\left|\frac{1}{T}\int_T^{2T}\chi_{\mathcal{S}_j}(\mathbf{c}(t))\,dt-\operatorname{meas}\bigcup_{V\in\mathcal{S}_j}V\right| \\ &\quad + 2\operatorname{meas}\mathcal{D} \\ &\ll R^{\pi(x)}\sup_{B\in\mathcal{B}}\left|\frac{1}{T}\int_T^{2T}\chi_B(\mathbf{c}(t))\,dt-\operatorname{meas}B\right| \\ &\quad + \operatorname{meas}\mathcal{D},\end{aligned}$$

where $\mathcal{B}$ is the collection of all boxes B contained in the unit torus with sides parallel to the axes.

By Lemma 3 and Corollary 5 the above quantity is

$$\ll R^{\pi(x)}\left(\frac{\pi(x)}{K}+\frac{e^{4Kx}}{T}\right)+\frac{\pi^2(x)}{R}.$$

Choosing $R=[(\log\log T)^{2/3}]$, $K=\left[\frac{\log T}{8x}\right]$, and $x=[(\log\log T)^{1/4}]$, we obtain the result.

3 A Probabilistic Lemma

We now estimate the probability that a random walk with shorter and shorter steps remains positive.

In 1949, Sparre Andersen proved a combinatorial identity (see [7, 8, 9]) that enables us to compute the probability

$$\mathrm{Prob}(\mathbf{W}_1 > 0, \mathbf{W}_2 > 0, \ldots, \mathbf{W}_{n-1} > 0, \mathbf{W}_n > 0),$$

where $\mathbf{W}_n = \sum_{k=1}^{n} \mathbf{Z}_k$ is the sum of symmetric, independent, identically distributed random variables $\mathbf{Z}_k$. His techniques exploited the fact that the distributions of $\sum_{k=n}^{m+n} \mathbf{Z}_k$ are identical for fixed m. They do not readily generalise. In 1961, G. Baxter [10] gave a proof utilising the all-sanctifying touch of Harmonic Analysis, exploiting the fact that identical distributions have identical characteristic functions (Fourier-Stieltjes transforms) and using the Wiener-Hopf factorisation technique ([11] p. 402, [12] pp. 581-587).

Here, we obtain an asymptotic lower bound for this probability in the case where the $\mathbf{Z}_n$ are not necessarily identically distributed. Our techniques use the fact that the random walk we are considering is a martingale. We then embed this martingale into Brownian motion by using Strassen's extension to Martingales of the Skorohod representation theorem.

For our problem, we are mainly interested in the probability

$$\mathrm{Prob}(\sum_{1 \leq n \leq y} c_n \mathbf{X}_n \geq 0 \ \forall y \ \leq x) \tag{9}$$

where the $\mathbf{X}_n$ are independent, identically distributed random variables having density function

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in (-1,1).$$

Observe that the $\mathbf{X}_n$ have mean 0 and variance 1/2. Thus, $\mathbf{X}_1 + \mathbf{X}_2 + \cdots + \mathbf{X}_n, n = 1, 2, \ldots$ forms a martingale. We note in passing that all the moments of the $\mathbf{X}_n$ exist and, in fact, the $\mathbf{X}_n$ have characteristic function

$$\frac{1}{\pi}\int_{-1}^{1} \frac{e^{2\pi i x u}}{\sqrt{1-x^2}}\, dx \;=\; J_0(2\pi u),$$

where $J_0(u) = \sum_{n=0}^{\infty}(-1)^n \frac{u^{2n}}{2^{2n}(n!)^2}$ is the 0th Bessel function.

We borrow the following result from [13] (Theorem A.1).

Lemma 7 Skorokhod's Representation *Let $\{S_n = \sum_1^n X_i, \mathcal{F}_n, n \geq 1\}$ be a zero mean, square-integrable martingale. Then there exists a probability space supporting a (standard) Brownian Motion W and a sequence of nonnegative random variables $\tau_1, \tau_2 \ldots$ with the following properties. If $T_n = \sum_1^n \tau_i, S'_n = W(T_n), X'_1 = S'_1, X'_n = S'_n - S'_{n-1}$ for $n \geq 2$, and $\mathcal{G}_n$ is the σ-field generated by $S'_1, S'_2, \ldots, S'_n$ and $W(t)$ for $0 \leq t \leq T_n$, then,*

1. $S_n, n \geq 1$ *is distributed as* $S'_n, n \geq 1$,

2. T_n *is* $\mathcal{G}_n$*-measurable,*

3. *for each real number* $r \geq 1$,

$$\mathbf{E}(\tau_n^r|\mathcal{G}_{n-1}) \leq C_r\mathbf{E}(|X_n'|^{2r}|\mathcal{G}_{n-1}) = C_r\mathbf{E}(|X_n'|^{2r}|X_1', \ldots, X_n') \quad a.s.,$$

where $C_r = 2(8/\pi^2)^{r-1}\Gamma(r+1)$, *and,*

4. $\mathbf{E}(\tau_n|\mathcal{G}_{n-1}) = \mathbf{E}(X_n'^2|\mathcal{G}_{n-1})$ *a.s.*

We remark that if the random variables above are independent, the τ_n can be chosen to be independent.

We also need the following corollary of the so-called Reflexion Principle (see [14] p. 96).

Lemma 8 *Define the* running maximum *of a Brownian Motion* $B(t)$ *as*

$$M_t = \max_{0 \leq s \leq t} B(t).$$

Put $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-u^2/2)\, du$. *Then, for any positive real number* a,

$$\text{Prob}(M_t \leq a|B(0) = 0) = 2\Phi(\frac{a}{\sqrt{t}}) - 1.$$

Theorem 9 *Let* $\mathbf{Z}_n, n = 1, 2, \ldots$ *be symmetric, independent random variables with* $\text{Prob}(\mathbf{Z}_n = 0) = 0$. *Set* $\mathbf{W}_n = \sum_{j \leq n} \mathbf{Z}_j$. *If* $\sigma_n^2 = \sum_{k=1}^{n} \mathbf{E}\mathbf{Z}_k^2$ *and if* F_1 *denotes the cumulative distribution function of* $\mathbf{Z}_1$, *then*

$$\text{Prob}(\mathbf{W}_k > 0 \ \forall \ k \leq n) \geq \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\sigma_n^2}} |x| \exp(-x^2/4\sigma_n^2)\, dF_1(x) - \frac{8}{\pi^2\sigma_n^4} \sum_{k=1}^{n} \mathbf{E}\mathbf{Z}_k^4.$$

Proof. Clearly

$$\text{Prob}(\mathbf{W}_k > 0\ \forall k,\ 1 \leq k \leq n) = \text{Prob}(\text{sgn}\mathbf{Z}_1 = 1, \mathbf{W}_k - \mathbf{Z}_1 \geq -|\mathbf{Z}_1|\ \ \forall k, 2 \leq k \leq n).$$

Since neither $\mathbf{W}_k - \mathbf{Z}_1 = \sum_{j=2}^{k} \mathbf{Z}_j$ nor $|\mathbf{Z}_1|$ depends on the sign of $\mathbf{Z}_1$, the above equals

$$\text{Prob}(\text{sgn}\mathbf{Z}_1 = 1) \cdot \text{Prob}(\mathbf{W}_k - \mathbf{Z}_1 \geq -|\mathbf{Z}_1|\ \ \forall k,\ 2 \leq k \leq n).$$

Again, since $\text{Prob}(\mathbf{Z}_n \geq a) = \text{Prob}(\mathbf{Z}_n \leq -a)$, the above quantity is equal to

$$\frac{1}{2}\text{Prob}(\mathbf{W}_k - \mathbf{Z}_1 \leq |\mathbf{Z}_1|\ \ \forall k,\ 2 \leq k \leq n).$$

By Skorokhod's Representation, we can find a series of times $T_1, T_2, \ldots$ such that $\{\mathbf{W}_n - \mathbf{W}_1,\ n \geq 2\}$ is identically distributed with $\{B(T_n),\ n \geq 1\}$. Thus, the above quantity equals

$$\frac{1}{2}\text{Prob}(B(T_k) \leq |\mathbf{Z}_1|\ \ \forall k,\ 1 \leq k \leq n-1).$$

The above quantity is at least

$$\frac{1}{2}\text{Prob}(B(t) \leq |\mathbf{Z}_1|\ \ \forall t \in [0, T_n]),$$

which in turn is at least

$$\frac{1}{2}\text{Prob}(B(t) \leq |\mathbf{Z}_1|\ \ \forall t \in [0, 2\sigma_n^2]) - \frac{1}{2}\text{Prob}(T_n > 2\sigma_n^2).$$

By Lemma 8, the above is

$$\mathbf{E}\left(\Phi\left(\frac{|\mathbf{Z}_1|}{\sqrt{2\sigma_n^2}}\right)-\frac{1}{2}\right)-\frac{1}{2}\mathrm{Prob}(T_n>2\sigma_n^2).$$

Since for non-negative x

$$\Phi(x)-\frac{1}{2}=\frac{1}{\sqrt{2\pi}}\int_0^x \exp(-u^2/2)\,du \geq \frac{x\exp(-x^2/2)}{\sqrt{2\pi}},$$

it follows that

$$\mathrm{Prob}(\mathbf{W}_k>0\ \forall\ k\leq n)\geq \int_{-\infty}^{\infty}\frac{1}{\sqrt{4\sigma_n^2}}|x|\exp(-x^2/4\sigma_n^2)\,dF_1(x)-\frac{1}{2}\mathrm{Prob}(T_n>2\sigma_n^2).$$

To estimate $\mathrm{Prob}(T_n>2\sigma_n^2)$, we observe that, by the One-sided Chebyshev Inequality,

$$\mathrm{Prob}(T_n>2\sigma_n^2)\leq\ \mathrm{var}(T_n)/\sigma_n^4.$$

Since the random variables are independent, the times τ in Skorokhod's representation can be chosen to be independent. Thus

$$\mathrm{var}(T_n)=\mathrm{var}(\sum_{k=1}^n\tau_k)\leq\sum_{k=1}^n\mathbf{E}\tau_k^2.$$

But by the inequality for the moments given in Skorokhod's Theorem, by independence, and since $\{\mathbf{W}_n-\mathbf{W}_1,\ n\geq 2\}$ is identically distributed with $\{B(T_n),\ n\geq 1\}$, $\mathbf{E}\tau_k^2\leq\frac{16}{\pi^2}\mathbf{E}\mathbf{Z}_k^4$. We thus deduce

$$\mathrm{Prob}(T_n>2\sigma_n^2)\leq\frac{16}{\sigma_n^4\pi^2}\sum_{k=1}^n\mathbf{E}\mathbf{Z}_k^4,$$

whence the lemma follows.

Corollary 10 *Let* $\mathbf{X}_p$ *be independent random variables having density function*

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}}, \qquad x \in (-1,1),$$

and let $x_0 > 0$*. Then*

$$\text{Prob}\left(\sum_{x_0<p\le y} \frac{\log p}{p^{1/2}}\mathbf{X}_p > 0 \ \forall y \ x_0 < y \le x\right) \gg 1/\log x,$$

as $x \to \infty$.

Proof. This immediately follows from the above theorem, since $\sum_{p\le x} \frac{\log^2 p}{p} \sim \frac{1}{2}\log^2 x$ and all moments of the $\mathbf{X}_p$ are uniformly bounded.

4 Proof of Theorem 1

We start by quoting the following result from [15].

Lemma 11

$$\begin{aligned} \frac{\zeta'}{\zeta}(s) &= -\sum_{n<x^2} \frac{\Lambda(n)}{n^s}\omega_x(n) + \frac{x^{2(1-s)} - x^{1-s}}{(1-s)^2\log x} \\ & \quad + \frac{1}{\log x}\sum_{q=1}^{\infty} \frac{x^{-2q-s} - x^{-2(2q+s)}}{(2q+s)^2} + \frac{1}{\log x}\sum_{\rho} \frac{x^{\rho-s} - x^{2(\rho-s)}}{(\rho-s)^2}, \end{aligned}$$

where $\omega_x(n) = 1\ (1 \le n \le x),\ \frac{\log(x^2/n)}{\log x},\ (x \le n \le x^2)$.

We shall need the following zero-density result (see [15]).

Lemma 12 *Let* $N(\sigma, T)$ *denote the number of zeroes* $\beta + i\gamma$ *of* $\zeta(\sigma + it)$ *with* $\beta \geq \sigma, |\gamma| \leq T$. *Then, for* $1/2 \leq \sigma \leq 1$,

$$N(\sigma, T) \ll T^{1-(\sigma-1/2)/4} \log T.$$

Lemma 13 *If* $x_0 = \sqrt{\log \log x}$,

$$\text{Prob}\left(\sum_{x_0 < p \leq x} \frac{\log p}{p^{\sigma}} \mathbf{X}_p \geq 0 \text{ for all } \sigma > 1/2 \right) \gg 1/\log x.$$

Proof. By Riemann-Stieltjes integration

$$\begin{aligned} \sum_{p \leq x} \frac{\log p}{p^{\sigma}} \mathbf{X}_p &= \int_{1-}^{x} u^{1/2-\sigma} \, d\left(\sum_{p \leq u} \frac{\log p}{p^{1/2}} \mathbf{X}_p \right) \\ &= x^{1/2-\sigma} \sum_{p \leq x} \frac{\log p}{p^{1/2}} \mathbf{X}_p + (\sigma - 1/2) \int_{1}^{x} u^{-1/2-\sigma} \sum_{p \leq u} \frac{\log p}{p^{1/2}} \mathbf{X}_p \, du. \end{aligned}$$

The result now follows upon appealing to Corollary 10.

Lemma 14

$$\text{Prob}\left(\sum_{p > x_0} \frac{\log p}{p^{\sigma}} \mathbf{X}_p \geq 0 \text{ for all } \sigma > \sigma_0 \right) \gg \frac{\sigma_0 - 1/2}{-\log(\sigma_0 - 1/2)}.$$

as $\sigma_0 \downarrow 1/2, \sigma_0 \geq 1/2 + (A \log \log x)/\log x$ *for some positive constant* A.

Proof. Write

$$\sum_{p > x_0} \frac{\log p}{p^{\sigma}} \mathbf{X}_p = \sum_{x_0 < p \leq x} \frac{\log p}{p^{\sigma}} \mathbf{X}_p + \sum_{r=1}^{\infty} \sum_{2^{r-1} x < p \leq 2^r x} \frac{\log p}{p^{\sigma}} \mathbf{X}_p. \tag{10}$$

For some large and positive constant c_1, we have, via Kolmogorov's Inequality,

$$\text{Prob}\left(\max_{2^{r-1}x<y\leq 2^r x}\left|\sum_{2^{r-1}x<p\leq y}\mathbf{X}_p\right|\leq c_1 r^2\sqrt{2^r x}\right)\geq 1-\frac{c_2}{r^5}. \tag{11}$$

Therefore, via independence,

$$\text{Prob}\left(\max_{2^{r-1}x<y\leq 2^r x}\left|\sum_{2^{r-1}x<p\leq y}\mathbf{X}_p\right|\leq c_1 r^2\sqrt{2^r x}\text{ for all } r=1,2,\dots\right)\geq\prod_{r=1}^{\infty}(1-\frac{c_2}{r^5}). \tag{12}$$

The infinite product on the right-hand side of (12) is some positive constant c_3, thanks to the convergence of $\sum_{r=1}^{\infty}\frac{1}{r^5}$.

If the event in (11) does hold, then

$$\sum_{2^{r-1}x<p\leq 2^r x}\frac{\log p}{p^\sigma}\mathbf{X}_p\ll r^2\sqrt{2^r x}((\log 2^r x)/(2^r x)^\sigma), \tag{13}$$

upon summing by parts. Summing over all $r\geq 1$,

$$\sum_{r=1}^{\infty} r^2\sqrt{2^r x}\left((\log 2^r x)/(2^r x)^\sigma\right)\ll x^{1/2-\sigma}\left((\sigma-1/2)^{-4}+(\sigma-1/2)^{-3}\log x\right),$$

for $\sigma>1/2$ sufficiently close to $1/2$. Thus

$$\sum_{r=1}^{\infty}\sum_{2^{r-1}x<p\leq 2^r x}\frac{\log p}{p^\sigma}\mathbf{X}_p\ll x^{1/2-\sigma}\left((\sigma-1/2)^{-4}+(\sigma-1/2)^{-3}\log x\right), \tag{14}$$

provided the event on (12) holds. Calling the quantity on the right-hand side of (14) $A(x,\sigma)$, we see that $A(x,\sigma_0)\ll(\log x)/(\log\log x)^3$ uniformly for $\sigma_0\geq 1/2+A\log\log x/\log x$ for large enough A. Now, since $\sum_{x_0<p\leq x}\log^2 p/p^{2\sigma}\sim$

$\frac{1}{2}\log^2 x$ as $\sigma \downarrow 1/2, x \to \infty$, by the Central Limit Theorem and the Berry-Esseen Inequality,

$$\text{Prob}\left(\left|\sum_{x_0<p\le x}\frac{\log p}{p^\sigma}\mathbf{X}_p\right| \le c_4 A(x,\sigma_0)\right) \ll \int_{-c_4/\log x(\log\log x)^3}^{c_4/\log x(\log\log x)^3} e^{-u^2/2}du, \quad (15)$$

which is in turn

$$\ll \frac{1}{\log x(\log\log x)^3}.$$

for a positive constant c_4 chosen appropriately. The first quantity on the right-hand side of (10) will be positive for all $\sigma > \sigma_0$ with probability $\gg \frac{\sigma_0-1/2}{-\log(\sigma_0-1/2)}$ in view of Lemma 13. Combining this with (15), we obtain the result.

We are now in position to prove our main result.

Proof of Theorem 1. By Lemma 11, if $s = \sigma + it$,

$$\begin{aligned}
-\Re\frac{\zeta'}{\zeta}(s) &= \sum_{p\le x^2}\frac{\log p}{p^\sigma}w_x(p)\cos(t\log p) + \sum_{p^2\le x^2}\frac{\log p}{p^{2\sigma}}w_x(p^2)\cos(t\log p^2) \\
&\quad + \frac{1}{\log x}\Re\sum_{q=1}^{\infty}\frac{x^{-2q-s}-x^{-2(2q+s)}}{(2q+s)^2} \\
&\quad - \Re\frac{x^{2(1-s)}-x^{1-s}}{(1-s)^2\log x} + \sum_{n=3}^{\infty}\sum_{p^n\le x^2}\frac{\log p}{p^{n\sigma}}w_x(p^n)\cos(t\log p^n) - \\
&\quad - \Re\frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s}-x^{2(\rho-s)}}{(\rho-s)^2}. \qquad (16)
\end{aligned}$$

Our strategy is the following. We decompose $\sum_{p\le x^2}\frac{\log p}{p^\sigma}w_x(p)\cos(t\log p)$

as

$$\sum_{p\le x_0}\frac{\log p}{p^\sigma}w_x(p)\cos(t\log p)+\sum_{x_0<p\le x^2}\frac{\log p}{p^\sigma}w_x(p)\cos(t\log p). \tag{17}$$

We force the first term above to be positive, at the expense of some small probability, and we use this term to overwhelm the effect of every other term in the sum (16). We then calculate the measure of the set of $t \in [T, 2T]$ such that the second term on the right-hand side above be positive for all $\sigma > \sigma_0$.

To determine the proportion of $t \in [T, 2T]$ such that the quantity

$$\sum_{p\le x^2}\frac{\log p}{p^\sigma}w_x(p)\cos(t\log p)+\sum_{p^2\le x^2}\frac{\log p}{p^{2\sigma}}w_x(p^2)\cos(t\log p^2)$$

be positive for $\sigma > \sigma_0$, it is enough to determine the probability that

$$\sum_{p\le x^2}\frac{\log p}{p^\sigma}w_x(p)\mathbf{X}_p+\sum_{p^2\le x^2}\frac{\log p}{p^{2\sigma}}w_x(p^2)(2\mathbf{X}_p^2-1)$$

be positive for $\sigma > \sigma_0$. Since the weights $\omega_x(n)$ are decreasing, it is enough to compute the probability that

$$\sum_{x_0<p\le x^2}\frac{\log p}{p^\sigma}\mathbf{X}_p$$

be positive for $\sigma > \sigma_0$.

We note that $\mathrm{Prob}(\mathbf{X}_p > 3/4 \text{ for all } p \le x_0) = (\int_{3/4}^1 (1-x^2)^{-1/2}\,dx)^{\pi(x_0)} \gg (\sigma_0 - 1/2)^{1/2}$ if $\sigma > \sigma_0 \ge 1/2 + A\log\log x/\log x, x_0 = (\log\log x)^{1/2}$. Thus

$$\sum_{p\le x_0}\frac{\log p}{p^\sigma}w_x(p)\mathbf{X}_p \gg x_0^{1-\sigma_0}/(1-\sigma_0) \gg (\log\log x)^{1/5}.$$

If $\sigma > 3/4$, then $\sum_{p^2 \le x^2} \frac{\log p}{p^{2\sigma}} w_x(p^2)(2\mathbf{X}_p^2 - 1) \ll 1$. Consider the event

$$\left| \sum_{p^2 \le x^2} \frac{\log p}{p^{2\sigma}} w_x(p^2)(2\mathbf{X}_p^2 - 1) \right| > (\log\log x)^{1/5},$$

for some $\sigma \in [1/2, 3/4]$. Let $\sigma_k = 1/2 + \frac{k}{\log^2 x}$, $0 \le k \le \frac{3\log^2 x}{4}$. Then, the event described above is contained in the event

$$\bigcup_k \left| \sum_{p^2 \le x^2} \frac{\log p}{p^{2\sigma_k}} w_x(p^2)(2\mathbf{X}_p^2 - 1) \right| > (\log\log x)^{1/6},$$

by the Mean Value Theorem. Choose y as large as possible so that $\sum_{p \le y} \frac{\log p}{p} \le \frac{1}{2}(\log\log x)^{1/6}$. Then the preceding union of events is contained in

$$\bigcup_k \left(\left| \sum_{y \le p \le x} \frac{\log p}{p^{2\sigma_k}} w_x(p^2)(2\mathbf{X}_p^2 - 1) \right| \right) > (\log\log x)^{1/6}.$$

Thus, for a fixed k,

$$\text{Prob}\left(\left| \sum_{y \le p \le x} \frac{\log p}{p^{2\sigma_k}} w_x(p^2)(2\mathbf{X}_p^2 - 1) \right| > (\log\log x)^{1/6} \right) \le \exp\left(\frac{-a_3(\log\log x)^{1/3}}{\sum_{p \le y} \frac{\log p}{p}} \right).$$

But by our choice of y, this is

$$\le \exp\left(\frac{-a_3 y (\log\log x)^{1/3}}{\log y} \right) \le \exp\left(-\exp\left(a_4 (\log\log x)^{1/6} \right) \right) \ll \frac{1}{\log^{10} x}.$$

Summing over k,

$$\text{Prob}\left(\bigcup_k \left| \sum_{y \le p \le x} \frac{\log p}{p^{2\sigma_k}} w_x(p^2)(2\mathbf{X}_p^2 - 1) \right| > (\log\log x)^{1/6} \right) \ll \frac{1}{\log^8 x}.$$

Thus

$$\left|\sum_{p^2\le x^2}\frac{\log p}{p^{2\sigma}}w_x(p^2)(2\mathbf{X}_p^2-1)\right|\le\frac{1}{2}\sum_{p\le x_0}\frac{\log p}{p^{\sigma}}w_x(p)\mathbf{X}_p,$$

except for a set of measure $\ll \frac{1}{\log^8 x}$. We then deduce that the probability that

$$\sum_{p\le x^2}\frac{\log p}{p^{\sigma}}w_x(p)\mathbf{X}_p+\sum_{p^2\le x^2}\frac{\log p}{p^{2\sigma}}w_x(p^2)(2\mathbf{X}_p^2-1)$$

be positive for all $\sigma>\sigma_0$ is $\gg(\sigma_0-1/2)^2$ in view of the preceding Lemma.

We observe that since $p^{7/5}<p^{3/2}-p^{1/2}$ for $p>5$,

$$\begin{aligned}\sum_p\frac{\log p}{p^{3/2}-p^{1/2}} &< \sum_{p\le 5}\frac{\log p}{p^{3/2}-p^{1/2}}+\sum_{n=7}^{\infty}\frac{\log n}{n^{7/5}}\\ &< \sum_{p\le 5}\frac{\log p}{p^{3/2}-p^{1/2}}+\int_1^{\infty}u^{-7/5}\log u\,du\\ &< 1+\frac{25}{4}\Gamma(2)=7.25.\end{aligned}$$

We choose σ' so that

$$\sum_p\frac{\log p}{p^{\sigma'}}=29.$$

We set $x=(\log\log T)^{1/4}$ and let A be the positive constant from Lemma 14. To all zeroes ρ in the rectangle

$$1/2+A\log\log\log\log T/\log\log\log T\ \le\ \sigma\ \le\sigma',\qquad T\ \le t\ \le 2T,$$

make a circle centred at the zero with radius $\log^2 T$, and then delete these circles from the rectangle. Also, make a circle of radius of $\log T$ around $s=1$ and delete it.

We now show that, independent of $\sum_{x_0<p\le x^2}\frac{\log p}{p^\sigma}\mathbf{X}_p$ being positive for all $\sigma>\sigma_0$, the absolute values of the last four terms in (17) are no larger than $\sum_{p\le x_0}\frac{\log p}{p^\sigma}\mathbf{X}_p$ for all $\sigma>\sigma_0$, on a set of positive measure.

First observe that

$$\sum_{|t-\gamma|\ge \log^2 T}\frac{1}{(t-\gamma)^2}\le \sum_{n\ge \log^2 T}\frac{1}{n^2}\sum_{n\le|t-\gamma|<n+1}1.$$

This last sum is

$$O\left(\sum_{n\ge\log^2 T}\frac{\log(t+n+1)}{n^2}\right),$$

which is in turn

$$O\left(\int_{(\log^2 T)/2}^{\infty}\frac{\log T+\log u}{u^2}\,du\right)=O\left(\frac{1}{(\log T)^{1/2}}\right).$$

Therefore, outside the neighbourhoods of the zeroes,

$$\left|\frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s}-x^{2(\rho-s)}}{(\rho-s)^2}\right|\ll\frac{x^{1/2-A\log\log\log\log T/(\log\log\log T)}}{\log x}\sum_{|t-\gamma|\ge\log^2 T}\frac{1}{(t-\gamma)^2}$$

For T sufficiently large, this can be made

$$<\frac{1}{6}\sum_{p\le x_0}\frac{\log p}{p^\sigma}.$$

This will hold true for all $\sigma\in[\sigma_0,\sigma')$, except for a set of $t\in[T,2T]$ of proportion $T^{-A\log\log\log\log T/4(\log\log\log T)}(\log T)^3$ in view of Lemma 12.

To treat $\sum_{n=3}^{\infty}\sum_{p^n\le x^2}\frac{\log p}{p^{n\sigma}}w_x(p^n)\cos(t\log p^n)$, we note that

$$\begin{aligned}\left|\sum_{n=3}^{\infty}\sum_{p^n\le x^2}\frac{\log p}{p^{n\sigma}}w_x(p^n)\cos(t\log p^n)\right| &< \sum_{p}\frac{\log p}{p^{3/2}-p^{1/2}}\\ &< \frac{1}{2}\sum_{p\le x_0}\frac{\log p}{p^{\sigma}},\end{aligned}$$

on a set of positive measure, for all $\sigma\in(\sigma_0,\sigma']$.

To treat $\frac{1}{\log x}\Re\sum_{q=1}^{\infty}\frac{x^{-2q-s}-x^{-2(2q+s)}}{(2q+s)^2}$ we observe that since $\sigma > 1/2$ the series converges absolutely, and being multiplied by $1/\log x$, we will have eventually

$$\frac{1}{\log x}\left|\Re\sum_{q=1}^{\infty}\frac{x^{-2q-s}-x^{-2(2q+s)}}{(2q+s)^2}\right| < \frac{1}{6}\sum_{p\le x_0}\frac{\log p}{p^{\sigma}},$$

for all $\sigma\in(\sigma_0,\sigma']$ on a set of positive measure.

Finally, if T is chosen large enough, $\left|\frac{x^{2(1-s)}-x^{1-s}}{(1-s)^2\log x}\right| < \frac{1}{6}\sum_{p\le x_0}\frac{\log p}{p^{\sigma}}$ for all $\sigma\in(\sigma_0,\sigma']$, except for a set of measure $(\log T)/T$.

Upon gathering all of the above, we achieve the result.

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