Difference Calculus and Generating Functions

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Contents

Preface

A work for my amusement only. Haven't gone over these in years, even though I am making them public now.

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Things to do:

- Weave functions into counting, *à la twelfold way...*

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Chapter 1

Generating Functions

1.1 Ordinary Generating Functions

Suppose we desire to change a dollar bill with pennies, nickels, dimes, quarters and half-dollars. In how many ways can we do this?

This problem is equivalent to finding the number of nonnegative solutions to the equation

$$x+5y+10z+25w+50v=100.$$

It is easy to see that this is the same as asking for the coefficient of x^{100} in the expansion of

$$\begin{aligned} C(x) &= (1+x+x^2+x^3+\ldots) \\ &\quad \cdot(1+x^5+x^{10}+x^{15}+\ldots) \\ &\quad \cdot(1+x^{10}+x^{20}+x^{30}+\ldots) \\ &\quad \cdot(1+x^{25}+x^{50}+x^{75}+\ldots) \\ &\quad \cdot(1+x^{50}+x^{100}+x^{150}+\ldots) \\ &= \frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})(1-x^{50})}. \end{aligned}$$

We call the function C the *ordinary generating function* of the change-making problem. We see that in general, the coefficient of x^n in the expansion of $C(x)$ gives us the number of nonnegative solutions to the equation

$$x+5y+10z+25w+50v=n.$$

In general, we call the function

$$G(x)=\sum_{n=0}^{\infty} a_n x^n$$

the *ordinary generating function* of the sequence $a_0, a_1, a_2, \ldots$.

1 Example Devise a pair of dice with exactly the same outcomes as ordinary dice (the sum 2 comes out once, etc.), but which are not ordinary dice.

Solution: Think of the ordinary die as the polynomial

$$x+x^2+x^3+x^4+x^5+x^6.$$

The outcomes of the ordinary pair of dice occur then as the terms of

$$(x+x^2+x^3+x^4+x^5+x^6)^2,$$

that is, the coefficients of the polynomial

$$x^2+2x^3+3x^4+4x^5+5x^6+6x^7+5x^8+4x^9+3x^{10}+2x^{11}+x^{12}$$

tell us that 2 occurs once, 3 occurs twice, etc..

We are thus looking for a_i and b_j such that

$$(x^{a_1}+\cdots+x^{a_6})(x^{b_1}+\cdots+x^{b_6}) = (x+\cdots+x^6)(x+\cdots+x^6),$$

i.e., an alternate factorisation for the dextral side of this equality. Now

$$\begin{aligned} x+\cdots+x^6 &= x\frac{1-x^6}{1-x} \\ &= x\frac{1-x^3}{1-x}(1+x^3) \\ &= x(1+x+x^2)(1+x)(1-x+x^2). \end{aligned}$$

The factors we seek must multiply to

$$x^2(1+x+x)^2(1+x)^2(1-x+x^2)^2.$$

Each factor must have an x (the condition of having *positive* entries) and each factor must have a $1+x$ and $1+x+x^2$ (the condition of having 6 faces, meaning that the value a $x=1$ is 6). The only free choice is on the distribution of the two $1-x+x^2$ factors. If we give one to each we get ordinary dice so we must try the other way. Thus we take the polynomials

$$x(1+x+x^2)(1+x) = x+2x^2+2x^3+x^4$$

and

$$x(1+x+x^2)(1+x)(1-x+x^2)^2 = x+x^3+x^4+x^5+x^6+x^8.$$

Therefore, if one die reads

$$1,2,2,3,3,4$$

and the other reads

$$1,3,4,5,6,8,$$

we meet the conditions of the problem.

2 Example Prove that the positive integers cannot be partitioned into a finite number of sets $S_1, S_2, \ldots, S_n$ each of which is an infinite arithmetic progression each with different common difference.

Solution: Assume on the contrary that

$$\mathbb{N} = S_1 \cup S_2 \cup \cdots \cup S_n,$$

with $S_k = \{a_k+d_k, a_k+2d_k, a_k+3d_k, \ldots,\}, 1 \le k \le n, d_1 > d_2 > d_3 > \cdots > d_n$. Then

$$\sum_{m\in\mathbb{N}} z^m = \sum_{m\in S_1} z^m + \sum_{m\in S_2} z^m + \cdots + \sum_{m\in S_n} z^m,$$

that is to say,

$$\frac{z}{1-z} = \frac{z^{a_1}}{1-z^{d_1}} + \frac{z^{a_2}}{1-z^{d_2}} + \cdots + \frac{z^{a_n}}{1-z^{d_n}}.$$

Letting $z \to e^{2\pi i/d_1}$, the sinistral side is finite, but the dextral side is infinite. This contradiction establishes the claim.

3 Example Partition the non-negative integers into two sets, A and B, such that every positive integer is expressible by $a+a'; a<a'; a,a' \in A$ in the same number of ways as by $b+b'; b<b'; b,b' \in B$.

Solution: Consider the generating functions

$$A(x) = \sum_{a\in A} x^a \text{ and } B(x) = \sum_{b\in B} x^b.$$

Observe that the conditions stipulate

$$A(x)+B(x) = \frac{1}{1-x}$$

and
$$A^2(x) - A(x^2) = B^2(x) - B(x^2).$$
Thus
$$(A(x) - B(x))(A(x) + B(x)) = A(x^2) - B(x^2),$$
and so
$$A(x) - B(x) = (1 - x)(A(x^2) - B(x^2)).$$
Iterating gives
$$A(x) - B(x) = \prod_{n=0}^{N-1}(1 - x^{2^n})(A(x^{2^n}) - B(x^{2^N})),$$
and letting $N \to \infty$ gives
$$A(x) - B(x) = \prod_{n=0}^{\infty}(1 - x^{2^n}).$$
This product, when multiplied out, gives $+x^k$ if k is the sum of an even number of distinct powers of 2 and $-x^k$ if k is and odd number of them. This means that
A = Set of all integers with an even number of 1 digits in its binary representation.
B = Those with an odd number of 1 digits.
So
$$A = 0, 3, 5, 6, 9, 10, 12, 15, \ldots,$$
$$B = 1, 2, 4, 7, 8, 11, 13, 14, \ldots.$$

1.2 OGFs and Linear Equations

We now present some examples of how ordinary generating functions can be used to find integral solutions to linear equations.

4 Example Find the number of integer solutions of
$$x_1 + x_2 + x_3 + x_4 = 69,$$
with $x_1 \geq 0, x_2 > 3, x_3 > 55, x_4 > 0$.

Solution: We are asking for the coefficient of x^{69} in the expansion of
$$(1 + x + x^2 + x^3 + \ldots)\cdots(x^4 + x^5 + x^6 + \ldots)$$
$$\cdot(x^{56} + x^{57} + x^{58} + \ldots)(x + x^2 + x^3 + \ldots).$$
But this expression equals
$$\frac{1}{1-x} \cdot \frac{x^4}{1-x} \cdot \frac{x^{56}}{1-x} \cdot \frac{x}{1-x} = \frac{x^{61}}{(1-x)^4}.$$
By the Generalised Binomial Theorem, the coefficient of x^8 in the expansion of $(1-x)^{-4}$ is
$$(-1)^8\binom{-4}{8} = \frac{(-4)(-5)(-6)(-7)(-8)(-9)(-10)(-11)}{8!} = 165.$$

5 Example Find the generating function for the number of integer solutions of
$$x_1 + x_2 + x_3 + x_4 + x_5 = n$$
with $0 \leq x_k \leq 4$ for all k . Then, find the number of integer solutions to $x_1 + x_2 + \cdots + x_5 = 10$ with $0 \leq x_k \leq 4$.

Solution: The generating function is easily seen to be

$$(1+x+x^2+x^3+x^4)^5=(1-x)^{-5}(1-x^5)^5.$$

For the second part of the problem, we want the coefficient of x^{10} in the expansion of the above generating function. But by the Generalised Binomial Theorem, this is

$$\binom{-5}{10}\binom{5}{0}+\binom{-5}{5}\binom{5}{1}+\binom{-5}{0}\binom{5}{2}.$$

6 Example Find the generating function for the number of integer solutions of

$$x_1+x_2+x_3+x_4=n$$

if $x_1\geq 0, x_2\geq 2, x_3\geq 8, 0\leq x_4>4$.

Solution: We want

$$(1+x+x^2+\cdots)(x^2+x^3+x^4+\cdots)(x^8+x^9+x^{10}+\cdots)(x^5+x^6+x^7+\cdots)$$

which simplifies to

$$\frac{x^{15}}{(1-x)^4}.$$

7 Example Find the generating function for the number of integer solutions of

$$x_1+x_2+x_3+x_4=n$$

with $0\leq x_1\leq x_2\leq x_3\leq x_4$.

Solution: Make the change of variables $y_1=x_1, y_2=x_2-x_1, y_3=x_3-x_2, y_4=x_4-x_3$. Then $x_1+x_2+x_3+x_4=4y_1+3y_2+2y_3+y_4$. Thus we want nonnegative solutions to the equation

$$4y_1+3y_2+2y_3+y_4=n.$$

The generating function for the number of solutions of this last equation is easily seen to be

$$(1+x^4+x^8+\cdots)(1+x^3+x^6+\cdots)(1+x^2+x^4+\cdots)(1+x+x^2+x^3+\cdots)$$

which in turn equals

$$\frac{1}{1-x^4}\cdot\frac{1}{1-x^3}\cdot\frac{1}{1-x^2}\cdot\frac{1}{1-x}.$$

8 Example Find the generating function for the number of integral solutions to

$$x_1+x_2+\cdots+x_r=n,$$

with $x_k\geq k$.

Solution: The generating function is seen to be

$$(x+x^2+\cdots)(x^3+x^4+\cdots)\cdots(x^r+x^{r+1}+\cdots)=\frac{x^{r(r+1)/2}}{(1-x)^r}.$$

1.3 Difference Calculus

We define the *forward difference operator* as

$$\Delta f(x) = f(x+1) - f(x).$$

For example:

$$\Delta x^2 = (x+1)^2 - x^2 = 2x+1.$$
$$\Delta 2^x = 2^{x+1} - 2^x = 2^x.$$

$$\begin{aligned}
\Delta \sin x &= \sin(x+1) - \sin x \\
&= \sin(x+\frac{1}{2}+\frac{1}{2}) - \sin(x+\frac{1}{2}-\frac{1}{2}) \\
&= \sin(x+\frac{1}{2})\cos(\frac{1}{2}) + \sin(\frac{1}{2})\cos(x+\frac{1}{2}) \\
&\quad -\sin(x+\frac{1}{2})\cos(\frac{1}{2}) + \sin(\frac{1}{2})\cos(x+\frac{1}{2}) \\
&= 2\sin(\frac{1}{2})\cos(x+\frac{1}{2}).
\end{aligned}$$

We define the *iterated differences* by the recursion

$$\Delta^1 = \Delta, \qquad \Delta^n = \Delta(\Delta^{n-1}) \text{ for } n > 1.$$

We also define the *forward unit push* operator as

$$Ef(x) = f(x+1).$$

For example, $Ex^2 = (x+1)^2 = x^2+2x+1$. We note in passing, that $\Delta = E - 1$. For example,

$$\begin{aligned}
\Delta x^2 &= (E-1)x^2 \\
&= Ex^2 - x^2 \\
&= (x+1)^2 - x^2.
\end{aligned}$$

If $m > 0$ is a positive integer, we define

$$x^{(m)} = x(x-1)(x-2)\cdots(x-m+1).$$

We define $x^{(0)} = 1$. Observe that

$$x^{(m+1)} = x^{(m)}(x-m) \qquad (*).$$

How must we define $x^{(m)}$ for negative integer m? Let $m = -1$ in $(*)$. We get

$$x^{(0)} = x^{(-1)}(x+1)$$

Since $x^{(0)} = 0$ we obtain $x^{(-1)} = \frac{1}{x+1}$. By recursion we see that

$$x^{(m)} = \frac{1}{(x+1)(x+2)\cdots(x+m)}$$

for negative integer m.

With $x^{(n)}$ and the operator Δ we obtain formulæ analogous to the differentiation formulæ. We can prove that $\Delta x^{(n)} = nx^{(n-1)}$. To see this, assume first that n is a positive integer. Then

$$\begin{aligned}
\Delta x^{(n)} &= (x+1)^{(n)} - x^{(n)} \\
&= (x+1)(x)(x-1)\cdots(x+1-n+1) \\
&\quad -x(x-1)\cdots(x-n+1) \\
&= x^{(n-1)}((x+1)-(x-n+1)) \\
&= x^{(n-1)}n,
\end{aligned}$$

as wanted. If n is a negative integer, the proof is similar.

The operators E and Δ are quite useful in obtaining n-th terms of sequences.
Let $u_0, u_1, u_2, u_3, \cdots$ be a sequence. We see that

$$u_1 = Eu_0$$

$$u_2 = Eu_1 = E^2u_0$$

$$u_3 = Eu_2 = E^2u_1 = E^3u_0$$

and in general, $u_k = E^k u_0$. Now, as $E = \Delta + 1$, we obtain $u_k = (1+\Delta)^k u_0$, and upon using the Binomial Theorem,

$$u_k = \sum_{j=0}^{k} \binom{k}{j} \Delta^j u_0.$$

We need thus to find the quantities $\Delta^j u_0$, $j = 0, 1, 2, \cdots$.

But on considering the following array

$$\begin{array}{ccccccc}
u_0 & & u_1 & & u_2 & & \ldots \\
 & u_1 - u_0 & & u_2 - u_1 & & u_3 - u_2 \ldots & \\
 & & u_2 - 2u_1 + u_0 & & u_3 - 2u_2 + u_1 & & \ldots
\end{array}$$

which can be written as

$$\begin{array}{ccccccccc}
u_0 & & u_1 & & u_2 & & u_3 \ldots & & \\
 & \Delta u_0 & & \Delta u_1 & & \Delta u_2 & \ldots & & \\
 & & \Delta^2 u_0 & & \Delta^2 u_1 & & \Delta^2 u_2 \ldots & & \\
 & & & \Delta^3 u_0 & & \Delta^3 u_1 \ldots & & &
\end{array}$$

Thus the sought quantities are on the first diagonal of the above array.

We now present a few examples

9 Example Find the n-th term of the sequence $4, 14, 30, 52, 80, 114, \cdots$, assuming that it grows polynomially.

Solution: We form the difference table

$$4, 14, 30, 52, 80, 114, \ldots$$

$$10, 16, 22, 28, 34, \ldots$$

$$6, 6, , 6, 6, \ldots$$

$$0, 0, 0, \ldots$$

Thus $u_0 = 4, \Delta u_0 = 10, \Delta^2 u_0 = 6, \Delta^3 u_0 = 0$ for $j \geq 3$. Now, by the Binomial Theorem,

$$\begin{aligned}
u_n = E^n u_0 &= (1+\Delta)^n u_0 \\
&= 1 \cdot u_0 + \binom{n}{1}\Delta^1 u_0 + \binom{n}{2}\Delta^2 u_0 + \binom{n}{3}\Delta^3 u_0 + \cdots \\
&= u_0 + \binom{n}{1}\Delta u_0 + \binom{n}{2}\Delta^2 u_2 \\
&= 4 + \binom{n}{1} 10 + \binom{n}{2} 6 \\
&= 4 + 10n + 3n(n-1) \\
&= 3n^2 + 7n + 4.
\end{aligned}$$

10 Example Find the n-th term of $8, 16, 0, -64, -200, -432, \ldots$, assuming that it grows polynomially.

Solution: Form the table of differences

$$8, 16, 0, -64, -200, -432, \ldots$$

$$8, -16, -64, -136, -232, \ldots$$

$$-24, -48, -72, -96, \ldots$$

$$-24, -24, -24, \ldots$$

Thus $u_0 = 8, \Delta u_0 = 8, \Delta^2 u_0 = -24, \Delta^3 u_0 = -24$ and $\Delta^j u_0 = 0$ for $j \geq 4$. Hence by the Binomial Theorem, and since $u_n = E^n u_0 = (1+\Delta)^n u_0$,

$$\begin{aligned} u_n = (1+\Delta)^n u_0 &= \binom{n}{0} u_0 + \binom{n}{1} \Delta u_0 + \binom{n}{2} \Delta^2 u_0 + \binom{n}{3} \Delta^3 u_0 + \cdots \\ &= 8 + 8n - 24\frac{n(n-1)}{2} - 24\frac{n(n-1)(n-2)}{6} \\ &= 8 + 8n - 12n(n-1) - 4n(n-1)(n-2) \end{aligned}$$

11 Example Evaluate $\frac{1}{1-\Delta} n^2$.

$$\begin{aligned} \frac{1}{1-\Delta} &= (1 + +\Delta + \Delta^2 + \Delta^3 + \cdots)n^2 \\ &= (1 + \Delta + \Delta^2 + \Delta^3 + \cdots)[n^{(2)} + n^{(1)}] \\ &= n^{(2)} + n^{(1)} + 2n^{(1)} + 1 + 2 + 0 \\ &= n^{(2)} + 3n^{(1)} + 3 \end{aligned}$$

12 Example Evaluate $\frac{1}{E^2-5E+6}\, n$.

$$\begin{aligned} \frac{1}{E^2-5E+6} n &= \frac{1}{(E-2)(E-3)} n \\ &= -(\frac{1}{E-2} - \frac{1}{E-3})n \\ &= -\frac{1}{E-2} n + \frac{1}{E-3} n \\ &= -\frac{1}{\Delta-1} n + \frac{1}{\Delta-2} n \\ &= \frac{1}{1-\Delta} n + \frac{-\frac{1}{2}}{1-\frac{\Delta}{2}} n \\ &= (1 + \Delta + \Delta^2 + \Delta^3 + \cdots)n^{(1)} \\ &\quad -(\frac{1}{2})(1 + \frac{\Delta}{2} + \frac{\Delta^2}{4} + \frac{\Delta^3}{8} + \cdots)n \\ &= n^{(1)} + 1 - \frac{1}{2} 9n^{(1)} + \frac{1}{2}) \\ &= \frac{1}{2} n^{(1)} + \frac{3}{4} \end{aligned}$$

13 Example Find

$$\frac{1}{1-\Delta} k^{(3)}.$$

Solution: Expanding $\frac{1}{1-\Delta}$ in powers of Δ,

$$\begin{aligned}\frac{1}{1-\Delta}k^{(3)} &= (1+\Delta+\Delta^2+\Delta^3+\Delta^4+\cdots)k^{(3)}\\ &= k^{(3)}+\Delta k^{(3)}+\Delta^2 k^{(3)}+\Delta^3 k^{(3)}+\Delta^4 k^{(3)}+\cdots\\ &= k^{(3)}+3k^{(2)}+6k^{(1)}+6+0+0+\cdots\\ &= k^{(3)}+3k^{(2)}+6k^{(1)}+6.\end{aligned}$$

1.4 Sum Calculus

The operator Δ is analogous to the differential operator. What is the analogue for integration? Of course, we know from Calculus that summation and integration are related, and integration is the inverse (in some way) of differentiation. We are thus going to attach the symbol $\frac{1}{\Delta}=\Delta^{-1}$ with the meaning "summation", i.e. $\Delta^{-1}=\sum$. In analogy to integration, we have

$$\Delta^{-1}x^{(n)}=\sum x^{(n)}=\frac{x^{(n+1)}}{n+1}+C \qquad n\in\mathbb{Z},\ n\neq -1,$$

where C is the summation constant. For example, $\Delta^{-1}x^{(5)}=x^{(6)}/6+C$ and $\Delta^{(-2)}x^{(5)}=\Delta^{-1}(x^{(6)}/6+C)=x^{(7)}/42+C_1x^{(1)}+C_2$.

Now, what is analogous to definite integration? It must be $\sum_{k=1}^{n}a_k$. We first find the "indefinite integral" for a_k. In analogy to differential Calculus, we seek for a sequence whose first order difference is a_k. Let $\Delta y_k=a_k$. Then $y_{k+1}-y_k=a_k$. Summing from $k=1$ to $k=n$, $y_{n+1}-y_1=\sum_{k=1}^{n}a_k$, which is the quantity we want. Thus if $y_k=\Delta^{-1}a_k$, then $\sum_{k=1}^{n}a_k=y_k\Big|_1^{n+1}=y_{n+1}-y_1$.

We now observe that we can sum a series by extending the difference array upwards.

$$\begin{array}{ccccccccc} 0 & & u_0 & & u_0+u_1 & & u_0+u_1+u_2 & & u_0+u_1+u_2+u_3 \\ & u_0 & & u_1 & & u_2 & & u_3 & & u_4 \end{array}$$

We just have to find the n-th term of the sequence of partial sums.

14 Example Find a closed form for $\sum_{k=1}^{n}k^2$.

Solution: The sequence of partial sums is
0
$1^2=1$
$1^2+2^2=5$
$1^2+2^2+3^2=14$
$1^2+2^2+3^2+4^2=30$
$1^2+2^2+3^2+4^2+5^2=55$, etc.
We form the difference table for this sequence

$$0,1,5,14,30,55,\ldots,$$

$$1,4,9,16,25,\ldots,$$

$$3,5,7,9,\ldots,$$

$$2,2,2,\ldots,$$

$$0,0,\ldots.$$

Thus $u_0 = 0, \Delta^1 u_0 = 1, \Delta^2 u_0 = 3, \Delta^3 u_0 = 2, \Delta^j u_0 = 0$ for $j \geq 4$ and so,

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= u_n \\
&= (1+\Delta)^n u_0 \\
&= u_0 + \binom{n}{1}\Delta u_0 + \binom{n}{2}\Delta^2 u_0 + \binom{n}{3}\Delta^3 u_0 \\
&= n + \frac{3(n-1)}{2} + \frac{n(n-1)(n-2)}{3} \\
&= n(1 + \frac{3(n-1)}{2} + \frac{(n-1)(n-2)}{3} \\
&= n(\frac{6+9(n-1)+2(n-1)(n-2)}{6}) \\
&= \frac{n}{6}(6+9n-9+2n^2-6n+4) \\
&= \frac{n}{6}(2n^2+3n+1) \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

as it is well known.

15 Example Find a closed form for $\sum_{k=1}^{n} k(k+1)$.

Solution: Here $u_0 = 0, u_1 = 2, u_2 = 8, u_3 = 20$ and we form the difference array

$$\begin{array}{c}
0,\ 1,\ 5,\ 14,\ 30,\ 55,\ \ldots, \\
1,\ 4,\ 9,\ 16,\ 25,\ \ldots, \\
3,\ 5,\ 7,\ 9,\ \ldots, \\
2,\ 2,\ 2,\ \ldots,
\end{array}$$

and so $u_0 = 0, \Delta u_0 = 2, \Delta^2 u_0 = 4, \Delta^3 u_0 = 2$ whence

$$\begin{aligned}
\sum_{k=1}^{n} k(k+1) = u_n &= (1+\Delta)^n u_0 \\
&= 1 + \binom{n}{1} u_0 + \binom{n}{2}\Delta^2 u_0 + \binom{n}{3}\Delta^3 u_0 \\
&= 1 + 2n + 2n(n-1) + \frac{n(n-1)(n-2)}{3}.
\end{aligned}$$

The above method only works for sequences that grow polynomially, and hence their differences will ultimately be 0. For if one tries to use this to sum $\sum_{k=1}^{n} k2^k$, one obtains a false result (say)

$$\sum_{k=1}^{n} k2^k = p(n), \qquad \text{(false)}.$$

where p is a polynomial. The sinistral side is $>> n2^n$ as $n \to \infty$ but the dextral side is $<< n^{\text{degree of } p}$.

The right formula can be obtained as follows. Let

$$f(x) = \sum_{k=1}^{n} x^k = \frac{x^{n+1} - x}{x-1} = \frac{x^{n+1}}{x-1} - \frac{x}{x-1}$$

Differentiating,

$$\begin{aligned} f(x) &= \sum_{k=1}^{n} kx^{k-1} \\ &= \frac{(n+1)x^n(x-1)-x^{n+1}}{(x-1)^2} - \frac{x-1-x}{(x-1)^2} \\ &= \frac{nx^{n+1}-(n+1)x^n+1}{(x-1)^2} \end{aligned}$$

Letting $x = 2$,

$$\sum_{k=1}^{n} k2^{k-1} = n2^{n+1} - (n+1)2^n + 1$$

or

$$\sum_{k=1}^{n} k2^{k} = n2^{n+2} - (n+1)2^{n+1} + 2.$$

Or one can also argue as follows. We have

$$\begin{aligned} S &= 1\cdot 2 + 2\cdot 2^2 + 3\cdot 2^3 + \cdots + n\cdot 2^n \\ 2S &= 1\cdot 2^2 + 2\cdot 2^3 + 3\cdot 2^4 + \cdots + (n-1)2^n + n\cdot 2^{n+1} \end{aligned}$$

Upon subtraction, $S = n\cdot 2^{n+1} - 2^n - 2^{n-1} - \cdots - 2^2 - 2$. After summing the geometric series, $S = n2^{n+1} - 2^{n+1} + 2$, which is the same formula as above.

16 Example Find

$$\frac{1}{E^2 - 2E + 1} k^2.$$

Solution: Observe that $E^2 - 2E + 1 = (1+\Delta)^2 - 2(1+\Delta) + 1 = \Delta^2$. Also, $k^2 = k^{(2)} + k^{(1)}$. Thus

$$\begin{aligned} \frac{1}{E^2 - 2E + 1} k^2 &= \Delta^{-2} k^{(2)} + k^{(1)} \\ &= \Delta^{-1}\left(\frac{k^{(3)}}{3} + \frac{k^2}{2} + C_1\right) \\ &= \frac{k^{(4)}}{12} + \frac{k^{(3)}}{6} + C_1 k^{(1)} + C_2. \end{aligned}$$

17 Example Express k^3 in the form $Ak^{(3)} + Bk^{(2)} + Ck^{(1)} + D$.

Solution: We want to express k^3 as

$$k^3 = A(k)(k-1)(k-2) + B(k)(k-1) + Ck + D.$$

Letting $k = 0$, we find $D = 0$. Letting $k = 1$, we find $C = 1$. Letting $k = 2$, we find $B = 3$. Now, $A = 1$ because the degrees of both expressions is 3, and so the leading coefficients on both sides of the equality must agree. Hence

$$k^3 = k^{(3)} + 3k^{(2)} + k^{(1)}.$$

18 Example Find the sum $\sum_{k=1}^{n} k^2$.

Solution: First we express k^2 as falling factorials,

$$k^2 = Ak^{(2)} + Bk^{(1)} + C$$

which is the same as

$$k^2 = A(k)(k-1) + Bk + C.$$

Let $k = 0$.

$$0^2 = A(0)(-1) + B(0) + C = C$$

Thus $c = 0$. Let $k = 1$.

$$1^2 = A(1)(0) + B(1) + C = B$$

Thus $B = 1$. Let $k = 2$.

$$2^2 = A(2)(1) + B(2) + C = 2A + 2$$

Thus $A = 1$.

Substituting in A, B, and C, we obtain

$$k^2 = k^{(2)} + k^{(1)}.$$

$$\begin{aligned}
\sum_{k=1}^{n} k^2 &= \sum k = 1^n k^{(2)} + k^{(1)} \\
&= \left.\frac{k^{(3)}}{3} + \frac{k^{(2)}}{2}\right|_1^{n+1} \\
&= \frac{(n+1)^{(3)}}{3} + \frac{(n+1)^{(2)}}{2} - \frac{1^{(3)}}{3} - \frac{1^{(2)}}{2} \\
&= \frac{(n+1)(n)(n-1)}{3} + \frac{n(n+1)}{2} \\
&= \frac{2n(n+1)(n-1) + 3n(n+1)}{6} \\
&= \frac{[n(n+1)][2(n-1)+3]}{6} \\
&= \frac{n(n+1)(2n+1)}{6}
\end{aligned}$$

1.5 Homogeneous recurrences

If a recurrence relation has the form

$$a_0 y_{n+k} + a_1 y_{n-1+k} + a_2 y_{n-2+k} + \cdots + a_n y_k = 0,$$

with constants a_k, we call such a recurrence *linear and homogeneous*. Since by means of the push operator we can express this as

$$(a_0 E^n + a_1 E^{n-1} + \cdots + a_{n-1} E + a_n) y_k = 0,$$

we just have to determine the roots of the polynomial in E. We get several cases depending on the roots being all real and distinct, real and repeated, or complex. We will examine these different cases in the examples that follow.

19 Example Solve the following difference equation:

$$y_{k+2} - 5y_{k+1} + 6y_k = 0.$$

Solution: Using the push operator,

$$\begin{aligned}
y_{k+2} - 5y_{k+1} + 6y_k &= E^2 y_k - 5E y_k + 6y_k \\
&= (E^2 - 5E + 6) y_k \\
&= (E-3)(E-2) y_k
\end{aligned}$$

Thus $y_k = A \cdot 2^k + B \cdot 3^k$ for some constants A and B.

20 Example Solve the initial value difference equation: $y_{k+2} - 6y_{k+1} + 8y_k = 0$ if $y_0 = 3$ and $y_1 = 2$.

Solution: Using the push operator,

$$\begin{aligned} y_{k+2}-6y_{k+1}+8y_k &= E^2y_k-6Ey_k+8y_k \\ &= (E^2-6E+8)y_k \\ &= (E-4)(E-2)y_k \end{aligned}$$

Thus $y_k = A\cdot 2^k + B\cdot 4^k$ for some constants A and B. From $y_0 = 3$ and $y_1 = 2$, we plug $k = 0$ and $k = 1$ into the equation to get $3 = A\cdot 2^0 + B\cdot 2^0 = A+B$ and $2 = A\cdot 2^1 + B\cdot 4^1 = 2A+4B$. We solve these equations to get $A = 5$ and $B = -2$. Thus

$$y_k = 5\cdot 2^k - 2\cdot 4^k.$$

21 Example Solve the difference equation $y_{k+2}-2y_{k+1}+5y_k = 0$.

The equation can be written as $(E^2-2E+5)y_k = 0$. It follows that the solutions are $(1+2i)^k$ and $(1-2i)^k$. Putting these into polar form, we get

$$1+2i = \sqrt{5}(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i) = \sqrt{5}(\cos\Theta + i\sin\Theta) = \sqrt{5}e^{i\Theta}$$

$$1-2i = \sqrt{5}(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i) = \sqrt{5}(\cos\Theta - i\sin\Theta) = \sqrt{5}e^{-i\Theta}$$

where $\tan\Theta = 2$.
It follows that the two linearly independent solutions are

$$(\sqrt{5}e^{i\Theta})^k = (\sqrt{5})^k e^{ki\Theta} = 5^{\frac{k}{2}}e^{ki\Theta} = 5^{\frac{k}{2}}(\cos k\Theta + i\sin k\Theta)$$

and

$$(\sqrt{5}e^{-i\Theta})^k = (\sqrt{5})^k e^{-ki\Theta} = 5^{\frac{k}{2}}e^{-ki\Theta} = 5^{\frac{k}{2}}(\cos k\Theta + i\sin k\Theta).$$

Here we have used Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$. The general solution can be written as

$$y_k = 5^{\frac{k}{2}}(A\cos k\Theta + B\sin k\Theta).$$

22 Example Solve the difference equation $y_{k+2}-4y_{k+1}+4y_k = 0$ for $y_0 = 1$ and $y_1 = 3$.

Solution: Using the push operator, $E^2y_k - 4Ey_k + 4y_k = 0$. This can be factored as $(E-2)(E-2)y_k = 0$. The general solution is thus

$$y_k = A2^k + Bk2^k.$$

Substituting for y_0 and y_1 into the equation above, we get the following:

$$1 = A2^0 + B(0)(2^0) = A$$

and

$$3 = A2^1 + B(1)(2^1) = 2A+2B.$$

Solving the equations, we get $A = 1$ and $B = \frac{1}{2}$. Thus

$$y_k = 2^k + k2^{k-1}.$$

23 Example Solve the recurrence

$$y_{n+3}-2y_{n+2}-y_{n+1}+2y_n = 0.$$

Solution: We must solve

$$(E^3-2E^2-E+2)y_n=(E-1)(E+1)(E-2)y_n=0.$$

The general solution is

$$y_n=A+B(-1)^n+C(2)^n.$$

24 Example Solve the recurrence

$$y_{n+3}+6y_{n+2}+12y_{n+1}+8y_n=0.$$

Solution: We see that this is equivalent to $(E+2)^3y_n=0$. The solution is thus given by

$$y_n=A(-2)^n+Bn(-2)^n+Cn^2(-2)^n.$$

25 Example Solve the recurrence given by

$$(E-2)(E-3)(E-8)^{1994}y_n=0.$$

Solution: The solution is readily seen to be

$$y_n=A(2)^n+B(3)^n+C_18^n+C_2n8^n+C_3n^28^n+\cdots+C_{1994}n^{1993}8^n.$$

26 Example Solve $y_{k+4}+12y_{k+2}-64y_k=0$.

Solution: The equation is converted into $(E^4+12E^2-64)y_k=0$, which can factored as $(E^2+16)(E^2-4)y_k=0$. This can be factored again as $(E^2+16)(E-2)(E+2)y_k=0$. Factor this two more times into $(E^2-(4i)^2)(E-2)(E+2)y_k=0$, then $(E-4i)(E+4i)(E-2)(E+2)y_k=0$. The general solution for y_k is

$$y_k=A2^k+B(-2)^k+C(4i)^k+D(-4i)^k.$$

27 Example Solve the difference equation $2y_{k+1}+3y_k=0$.

Solution: This is the same as $(2E+3)y_k=(\frac{1}{2})(E+\frac{3}{2})y_k=0$. Dividing by $\frac{1}{2}$ gives $(E+\frac{3}{2})y_k=0$. The solution is $y_k=A(-\frac{3}{2})^k$.

28 Example Solve the recursion $y_{k+3}-8y_k=0$.

Solution: We have

$$\begin{aligned}
y_{k+3}-8y_k &= (E^3-8)y_k\\
&= (E-2)(E^2+2E+4)y_k\\
&= (E-2)(E^2+2E+1+3)y_k\\
&= (E-2)((E+1)^2+3)y_k\\
&= (E-2)((E+1)^2-(i\sqrt{3})^2)y_k\\
&= (E-2)(E+1-i\sqrt{3})(E+1+i\sqrt{3})y_k
\end{aligned}$$

Thus

$$y_k=A2^k+B(i-1)^k+C(-1-i)^k.$$

1.6 Inhomogeneous recurrences

We now consider the case when the difference equation is non-homogeneous.

We shall need several formal operator methods for our task. We first prove the following result.

29 Theorem (Exponential shift) Let F be a polynomial in n. Let $\phi(E)$ be a polynomial on the push operator. Then a particular solution to the equation

$$\phi(E)y_n = \alpha^n F(n)$$

is given by

$$y_{\text{particular}} = \alpha^n \frac{1}{\phi(\alpha E)} F(n).$$

Proof Let $\phi(E) = \sum_{j=0}^{m} a_j E^j$. Then

$$\begin{aligned} \phi(E)\alpha^n F(n) &= \sum_{j=0}^{m} a_j E^j \alpha^n F(n) \\ &= \sum_{j=0}^{m} a_j \alpha^{n+j} F(n+j) \\ &= \alpha^n \sum_{j=0}^{m} a_j \alpha^j E^j F(n) \\ &= \alpha^n \phi(\alpha E) F(n). \end{aligned}$$

We conclude that $\phi(E)\alpha^n F(n) = \alpha^n \phi(\alpha E)F(n)$. Thus

$$\frac{1}{\phi(E)}\alpha^n F(n) = \alpha^n \frac{1}{\phi(\alpha E)} F(n).$$

30 Corollary If $\phi(\alpha) \neq 0$, then $y_n = \frac{\alpha^n}{\phi(n)}$ is a particular solution to

$$\phi(E)y_n = \alpha^n.$$

31 Example Solve the recursion

$$y_{n+2} - 5y_{n+1} + 4y_n = 2 \cdot 3^n - 4 \cdot 7^n.$$

Solution: Using the push operator, the equation is equivalent to

$$(E^2 - 5E + 4)y_n = 2 \cdot 3^n - 4 \cdot 7^n.$$

The homogeneous solution is given by $y_n = A + B4^n$. To find a particular solution to the inhomogeneous case, we write

$$y_n = \frac{1}{E^2 - 5E + 4} 2 \cdot 3^n - \frac{1}{E^2 - 5E + 4} 4 \cdot 7^n$$

and use the Theorem of the Exponential shift

$$\begin{aligned} y_{\text{particular}} &= 3^n \tfrac{1}{3^2 - 5(3) + 4} 2 - 7^n \tfrac{1}{7^2 - 5(7) + 4} 4 \\ &= -3^n - \tfrac{2}{9} 7^n. \end{aligned}$$

The complete solution is given by the sum of the homogeneous and the particular solution

$$y_n = A + B4^n - 3^n - \frac{2}{9} 7^n.$$

32 Example (Putnam 1980) For which real numbers a does the sequence defined by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all $n \geq 0$? (Express the answer in simplest form.)

Solution: We will shew that $u_n > 0$ for all $n \geq 0$ if and only if $a \geq 3$. Let $\Delta u_n = u_{n+1} - u_n$. The difference equation takes the form $(1-\Delta)u_n = n^2$. Since n^2 is a polynomial, a particular solution is

$$u_n = (1+\Delta)^{-1}n^2 = (1+\Delta+\Delta^2+\cdots)n^2 = n^2+(2n+1)+2$$

or

$$u_n = n^2+2n+3.$$

Hence, the complete solution is $u_n = n^2+2n+3+k\cdot 2^n$, since $v_n = k\cdot 2^n$ is the solution of the associated homogeneous difference equation $v_{n+1}-2v_n = 0$. The desired solution with $u_0 = a$ is $u_n = n^2+2n+3+(a-3)2^n$. Since $\lim_{n\to\infty}[2^n/(n^2+2n+3)] = +\infty, u_n$ will be negative for large enough n if $a-3 < 0$. Conversely, if $a-3 \geq 0$, it is clear that each $u_n > 0$.

33 Example Find a particular solution for the recursion

$$y_{n+2} - 5y_{n+1} + 4y_n = n^2 \cdot 3^n.$$

Solution: A particular solution is given by

$$y_n = \frac{1}{E^2-5E+4}\, n^2 3^n.$$

Using the exponential shift theorem,

$$\begin{array}{rcl}
y_n & = & 3^n \frac{1}{(3E)^2-5(3E)+4} n^2 \\
& = & 3^n \frac{1}{(3E-4)(3E-1)} n^2 \\
& = & 3^n \frac{-1/3}{3E-1}\, n^2 + 3^n \frac{1/3}{3E-4}\, n^2 \\
& = & 3^n \cdot \frac{-1/3}{2+3\Delta}\, n^{(2)} + n^{(1)} + 3^n \cdot \frac{1/3}{3\Delta-1}\, n^{(2)} + n^{(1)} \\
& = & \frac{-3^{n-1}}{2}(1-\frac{3}{2}\Delta+\frac{9}{4}\Delta^2+\cdots)n^{(2)}+n^{(1)} \\
& & -3^{n-1}(1+3\Delta+9\Delta^2+\cdots)n^{(2)}+n^{(1)} \\
& = & \frac{-3^{n-1}}{2}(n^{(2)}-\frac{1}{2}n^{(1)}+3) \\
& & -3^{n-1}(n^{(2)}+4n^{(1)}+21).
\end{array}$$

34 Example Find a particular solution for the recursion

$$y_n - 3y_{n-1} = 3^n.$$

Solution: The equation is $(E-3)y_{n-1} = 3^n$. A particular solution is given by

$$y_{n-1} = \frac{1}{E-3}3^n.$$

Using the exponential shift theorem,

$$y_{n-1} = 3^n \frac{1}{3E-3} 1 = 3^{n-1}\Delta^{-1}1 = 3^{n-1}(n^{(1)}+C),$$

where C is a constant. Since the homogeneous solution is of the form $A3^n$ the general solution is $y_{n-1} = A3^n + n3^{n-1}$ or $y_n = A3^n + n3^n$. (Notice how $C3^n$ and other constants $\times 3^n$ were absorbed in A.)

35 Example Solve the recursion

$$y_n - 3y_{n-1} = n+2.$$

Solution: A particular solution is given by $y_{n-1} = \frac{1}{E-3}n^{(1)} + 2 = \frac{-1}{2}\cdots\frac{1}{1-\Delta/2}n^{(1)} + 2 = -\frac{1}{2}(1+\Delta/2+\Delta^2/4+\cdots)n^{(1)} + 2 = -n/2 - 5/4$. Thus the general solution is given by

$$y_{n-1} = A3^n - n/2 - 5/4$$

or

$$y_n = A3^n - n/2 - 7/4.$$

36 Example Solve the recursion

$$y_n - 2y_{n-1} + y_{n-2} = 2^n.$$

Solution: A particular solution is given by $y_{n-2} = \frac{1}{(E-1)^2}2^n = 2^n\frac{1}{(2-1)^2}1 = 2^n$, where we have used the corollary to the exponential shift theorem. It follows that the general solution to the recursion is given by $y_{n-2} = A + Bn + 2^n$ or, what is the same, $y_n = A + Bn + 2^{n+2}$.

37 Example Solve the recursion $y_n - 2y_{n-1} + y_{n-2} = 4$.

Solution: A particular solution is given by $y_{n-2} = \frac{1}{(E-1)^2}4 = \Delta^{-2}4 = 2n^{(2)} + Cn^{(1)} + C_1 = 2n^2 + Cn + C_1$. The general solution is thus given by $y_{n-2} = A + Bn + 2n^2$.

1.7 Generalised Binomial Theorem

If x is any real number, we may define formally the symbol $\binom{x}{n}, n \in \mathbb{N}$ as

$$\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-(n-1))}{n!}.$$

Thus $\binom{x}{n}$ is a polynomial of degree n in x. We take the convention that $\binom{x}{n} = 0$ if n is not a nonnegative integer. If $n = 0$, we define $\binom{x}{0}$ as 1.

One formula which is particularly helpful is the *upper negation* formula:

$$\binom{-x}{n} = (-1)^n\binom{x+n-1}{n} \qquad n \in \mathbb{Z}, n \geq 0. \tag{1.1}$$

Its proof is easy:

$$\binom{-x}{n} = \frac{(-x)(-x-1)\cdots(-x-n+1)}{n!}.$$

Factorising the -1's, the above equals

$$(-1)^n\frac{(x)(x+1)\cdots(x+n-1)}{n!},$$

which is the same as

$$(-1)^n\binom{x+n-1}{n}.$$

38 Example Prove that

$$\sum_{k\leq m}(-1)^k\binom{a}{k} = (-1)^m\binom{a-1}{m}, \qquad m \in \mathbb{N}.$$

Solution: Using upper negation twice and the result from example (3.8) we find

$$\begin{aligned}\sum_{k\leq m}(-1)^m\binom{a}{m} &= \sum_{k\leq m}\binom{k-a-1}{k}\\ &= \binom{-a+m}{m}\\ &= (-1)^m\binom{a-1}{m},\end{aligned}$$

as wanted.

Using Taylor's Theorem, we can prove a more general version of the Binomial Theorem. For general $\alpha\in\mathbb{R}$ and $|x|<1$, the *Generalised Binomial Theorem* holds:

$$(1+x)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k}x^k.$$

Albeit we will not prove the Generalised Binomial Theorem here, we will give an example of its use.

39 Example Find the coefficient of x^{1006} in the expansion of

$$\frac{x^{1000}}{(1-5x^2)^{10}}.$$

Solution: By the Generalised Binomial Theorem

$$\frac{x^{1000}}{(1-5x^2)^{10}} = \sum_{k=0}^{\infty}\binom{-10}{k}(-5)^k x^{1000+2k}.$$

Thus we need $k=3$ and the coefficient sought is

$$(-5)^3\binom{-10}{3} = -125\frac{(-10)(-11)(-12)}{3!} = 27500.$$

1.8 Formal Power Series

We now study power series *formally*, that is, we do not worry about questions of convergence. If we have two power series

$$A(x)=\sum_{n=0}^{\infty}a_nx^n \text{ and } B(x)=\sum_{n=0}^{\infty}b_nx^n,$$

their sum is given by

$$A(x)+B(x)=\sum_{n=0}^{\infty}(a_n+b_n)x^n.$$

Their product $A(x)B(x)$ can be computed using the *Abel-Cauchy convolution formula*:

$$A(x)B(x)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_kb_{n-k}\right)x^n.$$

Some power series occur so often that the student will do well in memorising them. The most common are

$$\frac{1}{1-x} = 1+x+x^2+x^3+\cdots \tag{1.2}$$

$$\frac{1}{1+x} = 1-x+x^2-x^3+\cdots \tag{1.3}$$

$$\log\frac{1}{1-x} = x+\frac{x^2}{2}+\frac{x^3}{3}\frac{x^4}{4}+\cdots \tag{1.4}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots \tag{1.5}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \tag{1.6}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{1.7}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{1.8}$$

40 Example Obtain 1.5 from 1.3.

Solution: Integrating term by term

$$\int_0^y \frac{1}{1+x}\,dx = \int_0^y (1 - x + x^2 - x^3 + x^4 - \cdots)\,dx,$$

whence

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{5} - \cdots$$

41 Example Obtain 1.8 from 1.7.

Solution: Differentiating term by term

$$\frac{d}{dx}\sin x = \frac{d}{dx}\,(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots)$$

we obtain

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

as wanted.

42 Example Find the power series expansion of $\arctan x$.

Solution: We have

$$\frac{1}{1+y^2} = 1 - y^2 + y^4 - y^6 + y^8 - \cdots$$

Integrating term by term

$$\int_0^x \frac{dy}{1+y^2} = \int_0^x (1 - y^2 + y^4 - y^6 + y^8 - \cdots)\,dx,$$

which is to say

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

43 Example In the expansion of

$$(1 + x + x^2 + x^3)^4,$$

1. Find the coefficient of x^6.
2. Find the coefficient of x^{24}.

Solution: (1) As $1-x^4=(1-x)(1+x+x^2+x^3)$, we have

$$(1+x+x^2+x^3)^4=(1-x^4)^4(1-x)^{-4}.$$

From this, the exponent 6 can be obtained in two ways: x^0x^6 and x^4x^2. Using the Binomial Theorem (and generalised Binomial Theorem) we see that the coefficient sought is

$$\binom{4}{0}\binom{-4}{6}-\binom{4}{1}\binom{-4}{2}.$$

(2) The exponent 24 can be obtained from $(1-x^4)^4(1-x)^{-4}$ in five ways: from x^0x^{24}, x^4x^{20}, x^8x^{16}, $x^{12}x^{12}$, and from $x^{16}x^8$. Using the Abel-Cauchy convolution identity, the numerical value of this coefficient is

$$\binom{4}{0}\binom{-4}{24}-\binom{4}{1}\binom{-4}{20}+\binom{4}{2}\binom{-4}{16}-\binom{4}{3}\binom{-4}{12}+\binom{4}{4}\binom{-4}{8}.$$

44 Example Prove that for integer $n\geq 1$,

$$\binom{2n}{n}=\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}.$$

Solution: The strategy is to split $\binom{n}{k}\binom{n}{n-k}$ into $\binom{n}{k}$ and $\binom{n}{n-k}$. By the Binomial Theorem, $\binom{2n}{n}$ is the coefficient of n is the expansion of $(1+x)^{2n}$. As $(1+x)^{2n}=(1+x)^n(1+x)^n$, using the Abel-Cauchy convolution, the coefficient of x^n on the dextral side is $\sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$, and so the identity is established.

45 Example Find a closed form for

$$\sum_{k\leq n}k\binom{n}{k}^2.$$

Solution: The strategy is to split $k\binom{n}{k}^2=k\binom{n}{k}\binom{n}{n-k}$ into $k\binom{n}{k}$ and $\binom{n}{n-k}$. Now, $k\binom{n}{k}$ occurs in the derivative of $(1+x)^n$, as

$$nx(1+x)^{n-1}=\sum_{k\leq n}k\binom{n}{k}x^k.$$

The term $\binom{n}{n-k}$ occurs, of course, in the binomial expansion

$$(1+x)^n=\sum_{k\leq n}\binom{n}{n-k}x^{n-k}.$$

If we multiply these two sums together using the Abel-Cauchy convolution formula, the coefficient of x^n in the expansion of

$$nx(1+x)^{n-1}(1+x)^n$$

is

$$\sum_{k\leq n}k\binom{n}{k}\binom{n}{n-k}.$$

But this coefficient, is the same as the coefficient of x^n in the expansion of

$$nx(1+x)^{2n-1},$$

that is, $n\binom{2n-1}{n-1}$. Therefore

$$\sum_{k\leq n}k\binom{n}{k}^2=n\binom{2n-1}{n-1}.$$

Although we said that we were going to consider power series formally, we mention in passing the important *Abel's Limit Theorem.*

Abel's Limit Theorem: Let $r > 0$, and suppose that $\sum_{n\geq 0} a_n r^n$ converges. Then $\sum_{n\geq 0} a_n x^n$ converges absolutely for $|x| < r$, and

$$\lim_{x\to r^-} \sum_{n\geq 0} a_n x^n = \sum_{n\geq 0} a_n r^n.$$

Abel's Limit Theorem can also be extended to cover the case when one is in a region of the complex plane.

46 Example Find the exact numerical value of

$$\sum_{n\geq 1} \frac{(-1)^{n-1}}{n}.$$

Solution: This alternating series converges by Leibniz's Test. Consider more generally

$$f(x) = \sum_{n\geq 1} \frac{(-1)^{n-1} x^n}{n} = \log(1+x),$$

by (3.4.4). We see that $f(1) = \log 2$. Thus

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2,$$

by Abel's Limit Theorem.

47 Example Prove that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

Solution: The given alternating series converges by Leibniz's Test. The result follows immediately from example (3.28) by letting $x \to 1$.

48 Example Find the exact numerical value of

$$\sum_{n\geq 0} \frac{(n+1)^2}{n!}.$$

Solution: Let

$$f(x) = xe^x = \sum_{n\geq 0} \frac{x^{n+1}}{n!}.$$

Then

$$f'(x) = xe^x + e^x = \sum_{n\geq 0} \frac{(n+1)x^n}{n!}.$$

Multiplying by x,

$$xf'(x) = x^2 e^x + xe^x = \sum_{n\geq 0} \frac{(n+1)x^{n+1}}{n!}.$$

Differentiating this last equality,

$$xf''(x) + f'(x) = 2xe^x + x^2 e^x + xe^x + e^x = \sum_{n\geq 0} \frac{(n+1)^2 x^n}{n!}.$$

Letting $x \to 1$, we obtain

$$\sum_{n\geq 0} \frac{(n+1)^2}{n!} = 2e + e + e + e = 5e.$$

1.9 Series Multisection

In what follows we will need *DeMoivre's Theorem*: if n is a natural number, then

$$(\cos x + i\sin x)^n = \cos xn + i\sin xn.$$

Consider the polynomial

$$1 - x^n = (1-x)(1+x+x^2+\cdots+x^{n-1}).$$

Its n roots $\varepsilon_k, k = 0,1,2,\ldots,n-1$ are the nth roots of unity

$$\varepsilon_k = \cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n} = e^{2\pi ik/n},\ k = 0,1,\ldots,n-1.$$

For example, the roots of $x^3-1=0$ are $\cos\frac{2\pi\cdot 0}{3}+i\sin\frac{2\pi\cdot 0}{3}=1, \cos\frac{2\pi\cdot 1}{3}+i\sin\frac{2\pi\cdot 1}{3}=-1/2+i\sqrt{3}/2$, and $\cos\frac{2\pi\cdot 2}{3}+i\sin\frac{2\pi\cdot 2}{3}=-1/2-i\sqrt{3}/2$. Suppose that $\varepsilon^n = 1$ but $\varepsilon \neq 1$. Then

$$1+\varepsilon+\varepsilon^2+\varepsilon^3+\cdots+\varepsilon^{n-1} = \frac{\varepsilon^n-1}{\varepsilon-1} = 0.$$

Hence

$$1+x+x^2+\cdots+x^{n-1} = \begin{cases} 0 & x = \cos\frac{2\pi k}{n}+i\sin\frac{2\pi k}{n} 1\le k\le n-1 \\ n & x=1. \end{cases}$$

The above identity is quite useful for "multisecting" a power series. For example, suppose we wanted to find the sum of every third term of $\sum_{k=0}^{27}\binom{27}{k}$, starting with the first one, that is, $S=\sum_{k=0}^{9}\binom{27}{3k}$. Then we use the fact that for $\varepsilon_1 = -1/2+i\sqrt{3}/2$ and $\varepsilon_2 = -1/2-i\sqrt{3}/2$ we have

$$\varepsilon_k^3 = 1, \text{ and } 1+\varepsilon_k+\varepsilon_k^2 = 0,\ k=1,2.$$

Thus

$$(*) \qquad \varepsilon_k^s+\varepsilon_k^{s+1}+\varepsilon_k^{s+2} = 0,\ k=1,2,\ s\in\mathbb{Z}.$$

From this

$$\begin{aligned}
(1+1)^{27} &= \binom{27}{0}+\binom{27}{1}+\binom{27}{2}+\binom{27}{4}+\cdots+\binom{27}{26}+\binom{27}{27} \\
(1+\varepsilon_1)^{27} &= \binom{27}{0}+\binom{27}{1}\varepsilon_1+\binom{27}{2}\varepsilon_1^2+\binom{27}{3}\varepsilon_1^3+\cdots+\binom{27}{27}\varepsilon_1^{27} \\
(1+\varepsilon_2)^{27} &= \binom{27}{0}+\binom{27}{1}\varepsilon_2+\binom{27}{2}\varepsilon_2^2+\binom{27}{3}\varepsilon_2^3+\cdots+\binom{27}{27}\varepsilon_2^{27}
\end{aligned}$$

Summing column-wise and noticing that because of $(*)$ only the terms $0,3,6,\ldots,27$ survive,

$$2^{27}+(1+\varepsilon_1)^{27}+(1+\varepsilon_2)^{27} = 3\binom{27}{0}+3\binom{27}{3}+3\binom{27}{6}+\cdots+3\binom{27}{27}.$$

By DeMoivre's Theorem, $(1-1/2+i\sqrt{3}/2)^{27} = \cos 9\pi + i\sin 9\pi = -1$ and $(1-1/2-i\sqrt{3}/2)^{27} = \cos 45\pi + i\sin 45\pi = -1$. Thus

$$\binom{27}{0}+\binom{27}{3}+\binom{27}{6}+\cdots+\binom{27}{27} = \frac{1}{3}(2^{27}-2).$$

The above procedure can be generalised as follows. Suppose that

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

If $\omega = e^{2\pi i/q}, q\in\mathbb{N}, q>1$, then $\omega^q = 1$ and $1+\omega+\omega^2+\omega^3+\cdots+\omega^{q-1}=0$. Then in view of

$$\frac{1}{q}\sum_{1\le b\le q}\omega^{kb} = \begin{cases} 1 & \text{if } q \text{ divides } k, \\ 0 & \text{else,} \end{cases}$$

we have

$$\sum_{\substack{n=0 \\ n\equiv a \bmod q}}^{\infty} c_n x^n = \frac{1}{q}\sum_{b=1}^{q} \omega^{-ab} f(\omega^b x). \tag{1.9}$$

We may use complex numbers to select certain sums of coefficients of polynomials. The following problems use the fact that if k is an integer

$$i^k + i^{k+1} + i^{k+2} + i^{k+3} = i^k(1+i+i^2+i^3) = 0 \tag{1.10}$$

49 Example Find the sum of all the coefficients once the following product is expanded and like terms are collected:

$$(1-x^2+x^4)^{109}(2-6x+5x^9)^{1996}.$$

Solution: Put

$$p(x) = (1-x^2+x^4)^{109}(2-6x+5x^9)^{1996}.$$

Observe that $p(x)$ is a polynomial of degree $4\cdot 109+9\cdot 1996 = 18400$. Thus $p(x)$ has the form

$$p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{18400}x^{18400}.$$

The sum of all the coefficients of $p(x)$ is

$$p(1) = a_0 + a_1 + a_2 + \cdots + a_{18400},$$

which is also $p(1) = (1-1^2+1^4)^{109}(2-6+5)^{1996} = 1$. The desired sum is thus 1.

50 Example Let

$$(1+x^4+x^8)^{100} = a_0 + a_1x + a_2x^2 + \cdots + a_{800}x^{800}.$$

Find:

❶ $a_0 + a_1 + a_2 + a_3 + \cdots + a_{800}$.

❷ $a_0 + a_2 + a_4 + a_6 + \cdots + a_{800}$.

❸ $a_1 + a_3 + a_5 + a_7 + \cdots + a_{799}$.

❹ $a_0 + a_4 + a_8 + a_{12} + \cdots + a_{800}$.

❺ $a_1 + a_5 + a_9 + a_{13} + \cdots + a_{797}$.

Solution: Put

$$p(x) = (1+x^4+x^8)^{100} = a_0 + a_1x + a_2x^2 + \cdots + a_{800}x^{800}.$$

Then

❶
$$a_0 + a_1 + a_2 + a_3 + \cdots + a_{800} = p(1) = 3^{100}.$$

❷
$$a_0 + a_2 + a_4 + a_6 + \cdots + a_{800} = \frac{p(1)+p(-1)}{2} = 3^{100}.$$

❸
$$a_1 + a_3 + a_5 + a_7 + \cdots + a_{799} = \frac{p(1)-p(-1)}{2} = 0.$$

❹
$$a_0 + a_4 + a_8 + a_{12} + \cdots + a_{800} = \frac{p(1)+p(-1)+p(i)+p(-i)}{4} = 2\cdot 3^{100}.$$

❺

$$a_1 + a_5 + a_9 + a_{13} + \cdots + a_{797} = \frac{p(1) - p(-1) - ip(i) + ip(-i)}{4} = 0.$$

51 Example Let

$$(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n} x^{2n}.$$

Find formulæ for

1. $\sum_{k=0}^{2n} a_k$
2. $\sum_{0 \le k \le n/2} a_{2k}$
3. $\sum_{1 \le k \le n/2} a_{2k-1}$
4. $a_0 + a_4 + a_8 + \cdots$
5. $a_1 + a_5 + a_9 + \cdots$

Solution: Let $f(x) = (1 + x + x^2)^n$.
(1) Clearly $a_0 + a_1 + a_2 + a_3 + a_4 + \cdots = f(1) = 3^n$.
(2) We have

$$\begin{array}{lcl} f(1) & = & a_0 + a_1 + a_2 + a_3 + \cdots \\ f(-1) & = & a_0 - a_1 + a_2 - a_3 + \cdots \end{array}$$

Summing these two rows,

$$f(1) + f(-1) = 2a_0 + 2a_2 + 2a_4 + \cdots,$$

whence

$$a_0 + a_2 + a_4 + \cdots = \frac{1}{2}(f(1) + f(-1)) = \frac{1}{2}(3^n + 1).$$

(3) We see that

$$f(1) - f(-1) = 2a_1 + 2a_3 + 2a_5 + \cdots$$

Therefore

$$a_1 + a_3 + a_5 + \cdots = \frac{1}{2}(f(1) - f(-1)) = \frac{1}{2}(3^n - 1).$$

(4) Since we want the sum of every fourth term, we consider the fourth roots of unity, that is, the complex numbers with $x^4 = 1$. These are $\pm 1, \pm i$. Now consider the equalities

$$\begin{array}{lcl} f(1) & = & a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots \\ f(-1) & = & a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - a_9 + \cdots \\ f(i) & = & a_0 + ia_1 - a_2 - ia_3 + a_4 + ia_5 - a_6 - ia_7 + a_8 + ia_9 + \cdots \\ f(-i) & = & a_0 - ia_1 - a_2 + ia_3 + a_4 - ia_5 - a_6 + ia_7 + a_8 - ia_9 + \cdots \end{array}$$

Summing these four rows,

$$f(1) + f(-1) + f(i) + f(-i) = 4a_0 + 4a_4 + 4a_8 + \cdots,$$

whence

$$a_0 + a_4 + a_8 + \cdots = \frac{1}{4}(f(1) + f(-1) + f(i) + f(-i)) = \frac{1}{4}(3^n + 1 + i^n + (-i)^n).$$

(5) Consider the equalities

$$\begin{array}{lcl} f(1) & = & a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + \cdots \\ -f(-1) & = & -a_0 + a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7 - a_8 + \cdots \\ -if(i) & = & -ia_0 + a_1 + ia_2 - a_3 - ia_4 + a_5 + ia_6 - a_7 - ia_8 + \cdots \\ if(-i) & = & ia_0 + a_1 - ia_2 - a_3 + ia_4 + a_5 - ia_6 - a_7 + ia_8 + \cdots \end{array}$$

Adding

$$f(1)-f(-1)-if(i)+if(i)=4a_1+4a_5+4a_9+\cdots,$$

whence

$$a_1+a_5+a_9+\cdots=\frac{1}{4}\left(3^n-1-i^{n+1}-(-i)^{n+1}\right).$$

52 Example Find the exact numerical value of

$$\sum_{k=0}^{665}\binom{1995}{3k}.$$

Solution: Since we want every third term starting with the zeroth one, we consider the cube roots of unity, that is, $\omega^3=1$. These are $\omega=-1/2-\sqrt{3}/2, \omega^2=-1/2+\sqrt{3}/2$ and $\omega^3=1$. If $\omega\neq 1$, then $1+\omega+\omega^2=0$. If $\omega=1, 1+\omega+\omega^2=3$. Thus if k is not a multiple of 3, $1^k+\omega^k+\omega^{2k}=0$, and if k is a multiple of 3, then $1^k+\omega^k+\omega^{2k}=3$. By the Binomial Theorem we then have

$$\begin{aligned}(1+1)^{1995}+(1+\omega)^{1995} & \\ +(1+\omega^2)^{1995} &= \sum_{k\leq 1995}(1^k+\omega^k+\omega^{2k})\binom{1995}{k}\\ &= \sum_{k\leq 665}3\binom{1995}{3k}.\end{aligned}$$

But $(1+\omega)^{1995}=(-\omega^2)^{1995}=-1$, and $(1+\omega^2)^{1995}=(-w)^{1995}=-1$. Hence

$$\sum_{k\leq 665}\binom{1995}{3k}=\frac{1}{3}(2^{1995}-2).$$

1.10 Miscellaneous examples

53 Example Shew that the series obtained from the harmonic series by deleting all the terms that contain at least one 0 converges.

Solution: Let $\mathscr{S}$ be the set of integers that do not have any 0 in their decimal representation. Take any $n\in\mathscr{S}$ with k digits. Then $n\geq 10^{k-1}$ and there are 9^k possible n. Therefore, the series satisfies

$$\begin{aligned}\sum_{n\in\mathscr{S}}\frac{1}{n} &= \sum_{k=1}^{\infty}\sum_{\substack{10^{k-1}\leq n<10^k\\ n\in\mathscr{S}}}\frac{1}{n}\\ &\leq \sum_{k=1}^{\infty}\frac{9^k}{10^{k-1}}\\ &= 90.\end{aligned}$$

Hence the sum is majorised by a converging geometric series and its sum is at most 90.

54 Example Let $\alpha(n)$ denote the number of zeroes that n has, for example $\alpha(660006)=3$. For $N\in\mathbb{N}$, evaluate

$$L=\lim_{N\to\infty}\frac{\ln S(N)}{\ln N}$$

where

$$S(N)=\sum_{n=1}^{N}666^{\alpha(n)}.$$

Solution: Suppose n has k digits, i.e. $10^{k-1} \leq n < 10^k$. Let us count how many k-digit numbers have exactly $j, 0 \leq j \leq k-1$ digits. We can choose the first digit from the left in 9 distinct ways (it cannot be 0). Of the remaining $k-1$ slots, we can choose j of them to contain the j 0's in $\binom{k-1}{j}$ ways. The remaining $k-1-j$ can be filled in 9^{k-1-j} ways (they cannot be 0). Thus, there are $9^{k-j}\binom{k-1}{j}$ k-digit numbers having exactly j 0's. Therefore

$$\begin{aligned}
\sum_{n=1}^{N} 666^{\alpha(n)} &= \sum_{1\leq k\leq \log_{10} N} \sum_{10^{k-1}\leq n<10^k} 666^{\alpha(n)} \\
&= \sum_{1\leq k\leq \log_{10} N} \sum_{0\leq j\leq k-1} \sum_{\substack{10^{k-1}\leq n<10^k \\ \alpha(n)=j}} 666^{\alpha(n)} \\
&= \sum_{1\leq k\leq \log_{10} N} \sum_{0\leq j\leq k-1} 9^{k-j}\binom{k-1}{j} 666^j \\
&= \sum_{1\leq k\leq \log_{10} N} 9(666+9)^{k-1} \\
&= 9\frac{675^{[\log_{10} N]+1}-1}{674}.
\end{aligned}$$

So

$$\ln S(N) \sim [\log_{10} N] \ln 675,$$

whence

$$L = \log_{10} 675.$$

Homework

55 Problem Find the ordinary generating functions for the following sequences.

1. $a_n = 1, n = 0, 1, 2, \ldots$
2. $a_n = n, n = 0, 1, 2, \ldots$
3. $a_n = n^2, n = 0, 1, 2, \ldots$
4. $a_n = 1/n!, n = 0, 1, 2, \ldots$
5. $a_n = 1/(2n)!$ if $n \geq 0$ is even and $a_n = 0$ if $n \geq 1$ is odd.
6. $a_n = 1/(3n)!$ if n is a multiple of 3 and $a_n = 0$ otherwise.

56 Problem Find the generating function of the sequence of the central binomial coefficients

$$\binom{2n}{n}, \qquad n \geq 0.$$

57 Problem Let n be a positive integer. What is the ordinary generating function for the binomial coefficients

$$\binom{n}{k}, \quad 0 \leq k \leq n?$$

58 Problem A sequence $a_0 = 1, a_1, a_2, \ldots$ satisfies

$$\sum_{k\leq n} a_k a_{n-k} = 1$$

for every $n \geq 1$. Find its generating function.

59 Problem Let k be a fixed positive integer. Prove that

$$\sum_{\substack{a_1+a_2+\cdots+a_k=n\\ a_j\in\mathbb{N}}} a_1a_2\cdots a_k = \frac{n(n^2-1^2)\cdots(n^2-(k-1)^2)}{(2k-1)!}$$

(Hint: Prove that the ordinary generating function of the sinistral side is $\frac{x^k}{(1-x)^{2k}}$.)

60 Problem Find the generating function for the number of selections of r distinct integers from $\{1,2,\ldots,n\}$ such that $|x-y|>2$ for all x,y selected.

61 Problem Find the ordinary generating function for the number of ways of putting k identical balls in n different boxes such that, no box contains more than one ball, and no two empty boxes are adjacent.

62 Problem Find the generating function for the number of ways of choosing r distinct integers from $\{1,2,\ldots,n\}$ no two of them consecutive.

63 Problem Find the ordinary generating function for the number of ways a roll of n distinct dice will produce a sum of r.

64 Problem Use induction to prove the *Factorial Binomial Theorem*

$$(x+y)^{(n)} = \sum_{k=0}^{n}\binom{n}{k}x^{(k)}y^{(n-k)},\ n\in\mathbb{N}.$$

65 Problem Find the value of

$$\frac{1}{3+\Delta}\,k^{(2)}.$$

(Hint: $\frac{1}{3+\Delta}=\frac{1/3}{1+\Delta/3}$.)

66 Problem Prove that

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

using finite differences.

67 Problem Prove that

$$\sum_{k=1}^{n}k^3=\frac{n^2(n+1)^2}{4},$$

and thus

$$(1+2+3+\cdots+n)^2 = 1^3+2^3+\cdots+n^3.$$

68 Problem Assuming that $\sum_{k=0}^{\infty}x^k=\frac{1}{1-x}$ for $|x|<1$, find the exact numerical value of

$$\sum_{n=1}^{\infty}\frac{n}{2^n},$$

$$\sum_{n=2}^{\infty}\frac{n(n-1)}{3^n},$$

and

$$\sum_{n=1}^{\infty}\frac{1}{(n+1)3^n}.$$

69 Problem Prove that
$$\sum_{k=1}^{n} k(k+1)(k+2) = \frac{(n+3)(n+2)(n+1)(n)}{4}.$$

70 Problem Find a closed form for
$$\sum_{k \leq n} \frac{1}{(k+1)(k+2)(k+3)(k+4)}$$

71 Problem (AIME 1994) The function f has the property that, for each real number
$$f(x) + f(x-1) = x^2.$$
If $f(19) = 94$, what is the remainder when $f(94)$ is divided by 1000?

72 Problem Find the solution to the recursion
$$a_n = na_{n-1}, \ n > 1, \ a_1 = 343.$$

73 Problem (Putnam 1969) The terms of a sequence T_n satisfy
$$T_n T_{n+1} = n \quad (n = 1, 2, 3, \cdots)$$
and
$$\lim_{n\to\infty} \frac{T_n}{T_{n+1}} = 1.$$
Shew that $\pi T_1^2 = 2$.

74 Problem (AHSME 1992) The increasing sequence of positive integers $a_1, a_2, a_3, \cdots$ has the property that $a_{n+2} = a_n + a_{n+1}$ for all $n \geq 1$. If $a_7 = 120$, the a_8 is what number?

75 Problem (AHSME 1992) For a finite sequence
$$A = (a_1, a_2, a_3, \cdots, a_n)$$
of numbers, the *Cesàro sum* of A is defined to be
$$\frac{S_1 + S_2 + S_3 + \cdots + S_n}{n},$$
where $S_k = a_1 + a_2 + a_3 + \cdots + a_k (1 \leq k \leq n)$. If the Cesàro sum of the 99-term sequence $(a_1, a_2, a_3, \cdots, a_{99})$ is 1000, what is the Cesàro sum of the 100-term sequence $(1, a_1, a_2, a_3, \cdots, a_{99})$?

76 Problem (AHSME 1993) Consider the non-decreasing sequence of positive integers
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \cdots$$
in which the n-th positive integer appears n times. The remainder when the 1993-th term is divided by 5 is what number?

77 Problem (AHSME 1993) Let $a_1, a_2, a_3, \cdots a_k$ be a finite arithmetic sequence with $a_4 + a_7 + a_{10} = 17$ and $a_4 + a_5 + a_6 + \cdots + a_{14} = 77$. If $a_k = 13$, then $k =$?

78 Problem (AHSME 1994) In the sequence
$$\cdots, a, b, c, d, 0, 1, 1, 2, 3, 5, 8, \cdots$$
each term is the sum of the two terms to its left. Find a.

79 Problem (AHSME 1994) Find the sum of the arithmetic series

$$20+20\frac{1}{5}+20\frac{2}{5}+\cdots+40.$$

80 Problem (Putnam 1948) Let a_n be a decreasing sequence of positive numbers with limit 0 such that $b_n = a_n - 2a_{n+1} + a_{n+2} \geq 0$ for all n. Prove that

$$\sum_{n=1}^{\infty} nb_n = a_1.$$

81 Problem (Putnam 1952) Let $a_j (j = 1, 2, \cdots, n)$ be arbitrary numbers. Prove that

$$a_1 + \sum_{i=2}^{n} a_i \prod_{j=1}^{i-1} (1-a_j) = 1 - \prod_{j=1}^{n} (1-a_j).$$

82 Problem Prove that

$$\sum_{k\geq 0} \binom{n+k}{2k}\binom{2k}{k}(-1)^k = \frac{\binom{0}{n}}{n+1}$$

for nonnegative integer n.

83 Problem Prove that

$$\sum_{k\geq 0} \binom{n+k}{n+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} = \binom{n-1}{m-1},$$

for natural numbers n and m.

84 Problem Prove that

$$(1-4x)^{-1/2} = \sum_{n\geq 0} \binom{2n}{n} x^n.$$

85 Problem Simplify

$$\sum_{n\geq 0} \binom{2n-1}{n} x^n.$$

86 Problem Find the coefficient of x^9 in the expansion of

$$\frac{x}{1+x^2+x^4}.$$

87 Problem Prove that

$$\frac{1}{(1-x)^{k+1}} = \sum_{n\geq 0} \binom{n+k}{n} x^n.$$

(Hint: Use the Generalised Binomial Theorem).

88 Problem Prove that

$$\arcsin x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \cdots$$

(Hint: Use the Generalised Binomial Theorem to expand $\frac{1}{\sqrt{1-x^2}}$)

89 Problem Find a closed form for

$$\sum_{k\geq 0} kx^k.$$

90 Problem Find the exact numerical value of

$$\sum_{k\geq 0} \frac{k}{2^k}.$$

91 Problem Find a closed form for

$$\sum_{k\geq 0} k^2x^k.$$

92 Problem Find the exact numerical value of

$$\sum_{k\geq 0} \frac{k^2}{2^k}.$$

93 Problem Prove that

$$\sum_{k\leq n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n.$$

94 Problem Prove that

$$1-\frac{1}{4}+\frac{1}{7}-\frac{1}{10}+\cdots = \frac{1}{3}\left(\log 2+\frac{\pi}{\sqrt{3}}\right).$$

(Hint: Expand $(1+x^3)^{-1})$ into a power series. Integrate $(1+x^3)^{-1}$ using partial fractions. Use Abel's Limit Theorem.)

95 Problem Prove that

$$\sum_{k\leq n} \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{n(n+1)}{2}.$$

96 Problem Let $f(x) = 1+x+x^2+\cdots+x^n$. Find a closed formula for

$$\sum_{k\leq n} k^2$$

by considering

$$\lim_{x\to 1^-} \frac{d}{dx}(x\frac{d}{dx}f(x)).$$

97 Problem Prove that

$$\sum_{k\leq n} \frac{2^{k+1}}{k+1}\binom{n}{k} = \frac{3^{n+1}-1}{n+1}.$$

98 Problem Prove that

$$\sum_{k\leq n} \binom{n}{k}\binom{n}{r+k} = \frac{(2n)!}{(n-r)!(n+r)!}.$$

99 Problem Prove that

$$\sum_{k\leq n} (2k+1)\binom{n}{k} = 2^n(n+1).$$

100 Problem Prove that

$$\sum_{n\geq 1}^{\infty}\frac{\sin n}{n}=\frac{\pi-1}{2}.$$

(Hint: Expand $\log(1-e^i)$ and consider real and imaginary parts. Use Abel's Limit Theorem.)

101 Problem (AHSME 1989) Find

$$\sum_{k=0}^{49}(-1)^k\binom{99}{2k}.$$

102 Problem Let

$$(1+x^2+x^4)^{100}=a_0+a_1x+\cdots+a_{400}x^{400}.$$

Find

$$a_0+a_3+a_6+\cdots+a_{399}.$$

103 Problem Find the exact numerical value of

$$\sum_{k=0}^{664}\binom{1995}{3k+1}.$$

104 Problem Find a closed form for

$$\sum_{n\geq 0}\frac{x^{3n}}{(3n)!}$$

105 Problem Find a closed form for

$$\sum_{n\geq 0}\frac{x^{3n+1}}{(3n+1)!}.$$

106 Problem For a set with one hundred elements, how many subsets are there whose cardinality is a multiple of 3?

107 Problem Prove that

$$\sum_{1\leq k\leq n/2}(-1)^{k+1}\binom{n}{2k-1}=2^{n/2}\sin\frac{n\pi}{4}$$

and

$$\sum_{0\leq k\leq n/2}(-1)^{k+1}\binom{n}{2k}=2^{n/2}\cos\frac{n\pi}{4}.$$

Answers

71 561

74 194

75 991

76 3

77 18

78 -3

79 3030