Elementary Probability

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Contents

Preface

These notes started during the Spring of 2002. The contents are mostly discrete probability, suitable for students who have mastered only elementary algebra. No calculus is needed, except perhaps in a very few optional exercises.

I would appreciate any comments, suggestions, corrections, etc., which can be addressed to the email below.

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To the Student

These notes are provided for your benefit as an attempt to organise the salient points of the course. They are a *very terse* account of the main ideas of the course, and are to be used mostly to refer to central definitions and theorems. The number of examples is minimal, and here you will find few exercises. The *motivation* or informal ideas of looking at a certain topic, the ideas linking a topic with another, the worked-out examples, etc., are given in class. Hence these notes are not a substitute to lectures: **you must always attend to lectures**. The order of the notes may not necessarily be the order followed in the class.

There is a certain algebraic fluency that is necessary for a course at this level. These algebraic prerequisites would be difficult to codify here, as they vary depending on class response and the topic lectured. If at any stage you stumble in Algebra, seek help! I am here to help you!

Tutoring can sometimes help, but bear in mind that whoever tutors you may not be familiar with my conventions. Again, I am here to help! On the same vein, other books may help, but the approach presented here is at times unorthodox and finding alternative sources might be difficult.

Here are more recommendations:

- Read a section before class discussion, in particular, read the definitions.
- Class provides the informal discussion, and you will profit from the comments of your classmates, as well as gain confidence by providing your insights and interpretations of a topic. **Don't be absent!**
- Once the lecture of a particular topic has been given, take a fresh look at the notes of the lecture topic.
- Try to understand a single example well, rather than ill-digest multiple examples.
- Start working on the distributed homework ahead of time.
- **Ask questions during the lecture.** There are two main types of questions that you are likely to ask.
    1. *Questions of Correction: Is that a minus sign there?* If you think that, for example, I have missed out a minus sign or wrote $\boldsymbol{P}$ where it should have been $\boldsymbol{Q}$,[1] then by all means, ask. No one likes to carry an error till line XLV because the audience failed to point out an error on line I. Don't wait till the end of the class to point out an error. Do it when there is still time to correct it!
    2. *Questions of Understanding: I don't get it!* Admitting that you do not understand something is an act requiring utmost courage. But if you don't, it is likely that many others in the audience also don't. On the same vein, if you feel you can explain a point to an inquiring classmate, I will allow you time in the lecture to do so. The best way to ask a question is something like: "How did you get from the second step to the third step?" or "What does it mean to complete the square?" Asseverations like "I don't understand" do not help me answer your queries. If I consider that you are asking the same questions too many times, it may be that you need extra help, in which case we will settle what to do outside the lecture.
- Don't fall behind! The sequence of topics is closely interrelated, with one topic leading to another.

[1] My doctoral adviser used to say "I said $\boldsymbol{A}$, I wrote $\boldsymbol{B}$, I meant $\boldsymbol{C}$ and it should have been $\boldsymbol{D}$!

- The use of calculators is allowed, especially in the occasional lengthy calculations. However, when graphing, you will need to provide algebraic/analytic/geometric support of your arguments. The questions on assignments and exams will be posed in such a way that it will be of no advantage to have a graphing calculator.

- Presentation is critical. Clearly outline your ideas. When writing solutions, outline major steps and write in complete sentences. As a guide, you may try to emulate the style presented in the scant examples furnished in these notes.

1 Preliminaries

1.1 Sets

1 Definition By a *set* we will understand any well-defined collection of objects. These objects are called the *elements* of the set. A *subset* is a sub-collection of a set. We denote that the set B is a subset of A by the notation $B \subseteq A$. If a belongs to the set A, then we write $a \in A$, read "a is an element of A." If a does not belong to the set A, we write $a \notin A$, read "a is not an element of A."

> **Notation:** *We will normally denote sets by capital letters, say $A, B, \Omega, \mathbb{R}$, etc. Elements will be denoted by lowercase letters, say a, b, ω, r, etc. The following sets will have the special symbols below.*

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$	*denotes the set of natural numbers.*
$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$	*denotes the set of integers.*
$\mathbb{R}$	*denotes the set of real numbers.*
$\varnothing$	*denotes the empty set.*

☞ *Observe that $\varnothing \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{R}$,and that the empty set is always a subset of any set.*

2 Example There are various ways to allude to a set:

- by a verbal description, as in "the set A of all integers whose absolute value is strictly less than 2.
- by a mathematical description, as in $A = \{x \in \mathbb{Z} : |x| < 2\}$. This is read "the set of x in $\mathbb{Z}$ such that $|x|$ is strictly less than 2."
- by listing the elements of the set, as in $A = \{-1, 0, 1\}$.

Notice that the set A is the same in all three instances above.

3 Definition Given a particular situation, the *universe* or *universal set* is the set containing all the points under consideration. For any particular situation, its universe will be denoted by Ω unless otherwise noted.[1]

4 Definition An *interval* I is a subset of the real numbers with the following property: if $s \in I$ and $t \in I$, and if $s < x < t$, then $x \in I$. In other words, intervals are those subsets of real numbers with the property that every number between two elements is also contained in the set. Since there are infinitely many decimals between two different real numbers, intervals with distinct endpoints contain infinitely many members. Table 1.1 shews the various types of intervals.

Observe that we indicate that the endpoints are included by means of shading the dots at the endpoints and that the endpoints are excluded by not shading the dots at the endpoints. [2]

[1] The capital Greek letter omega.

[2] It may seem like a silly analogy, but think that in $[a; b]$ the brackets are "arms" "hugging" a and b, but in $]a; b[$ the "arms" are repulsed. "Hugging" is thus equivalent to including the endpoint, and "repulsing" is equivalent to excluding the endpoint.

Interval Notation	**Set Notation**	**Graphical Representation**
$[a;b]$	$\{x \in \mathbb{R} : a \leq x \leq b\}$	
$]a;b[$	$\{x \in \mathbb{R} : a < x < b\}$	
$[a;b[$	$\{x \in \mathbb{R} : a \leq x < b\}$	
$]a;b]$	$\{x \in \mathbb{R} : a < x \leq b\}$	
$]a;+\infty[$	$\{x \in \mathbb{R} : x > a\}$	
$[a;+\infty[$	$\{x \in \mathbb{R} : x \geq a\}$	
$]-\infty;b[$	$\{x \in \mathbb{R} : x < b\}$	
$]-\infty;b]$	$\{x \in \mathbb{R} : x \leq b\}$	
$]-\infty;+\infty[$	$\mathbb{R}$	

Table 1.1: Intervals.

5 Example Let $\Omega = \{1, 2, \ldots, 20\}$, that is, the set of integers between 1 and 20 inclusive. A subset of Ω is $E = \{2, 4, 6, \ldots, 20\}$, the set of all even integers in Ω. Another subset of Ω is $P = \{2, 3, 5, 7, 11, 13, 17, 19\}$, the set of primes in Ω. Observe that, for example, $4 \in E$ but $4 \notin P$.

6 Definition The *cardinality* of a set A, denoted by $\mathbf{card}\,(A)$ is the number of elements that it has. If the set X has infinitely many elements, we write $\mathbf{card}\,(X) = \infty$.

7 Example If $A = \{-1, 1\}$ then $\mathbf{card}\,(A) = 2$. Also, $\mathbf{card}\,(\mathbb{N}) = \infty$.

8 Example Consider the set

$$\{2, 7, 12, \ldots, 302\},$$

where the elements are in arithmetic progression. How many elements does it have? Is 286 in this set?

►**Solution:** *Observe that*

$$2 = 2 + 5 \cdot 0, \quad 7 = 2 + 5 \cdot 1, \quad 12 = 2 + 5 \cdot 2, \quad \ldots, \quad 302 = 2 + 5 \cdot 60,$$

and hence, there are $60 + 1 = 61$ *elements, where we add the* 1 *because our count started at* 0. *Notice that every element has the form* $2 + 5k$. *If* $286 = 2 + 5k$ *then* $k = \dfrac{284}{5}$, *which is not an integer, and hence* 286 *is not in this set.*

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One of our main preoccupations will be to obtain the cardinality of a set. If the set is finite, then, in theory, we could list all of its elements and count them. But a quick realisation shews that this is not so easy if the number of elements is large. For example, if the set has a million elements, say, we would be quite discouraged to write all of its elements down. (I usually get tired after writing ten elements!) Most of the next chapter will be spent on counting finite but large sets. If the set is infinite we, of course, could not list all of the elements down. Infinite sets are trickier for another reason. The infinite sets that we will see in this course can be classified into one of two types: *countably infinite* and *uncountably infinite*.

Roughly speaking a countably infinite set is one where we can list all its elements, that is, that it has as many elements as the natural numbers. For example, the set of even numbers and the set positive multiples of 3 are countably infinite, as evinced by the following array

$$\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\ 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 & \ldots \\ 0 & 3 & 6 & 9 & 12 & 15 & 18 & 21 & \ldots \end{array}$$

We have now arrived at what is called *Galileo's Paradox:* a proper subset (in this case, the even numbers or the multiples of 3) has as many elements as its parent set (in this case the natural numbers). That this is impossible to do for finite sets is somewhat obvious—but still, requires proof, which we will not include here—and hence "Galileo's Paradox" is a defining feature of infinite sets. It can be proved—but we will not do it here—that the integers $\mathbb{Z}$, and the rational numbers $\mathbb{Q}$ are also countably infinite. We denote this smallest infinity of the natural numbers by the symbol $\aleph_0$.

Uncountably infinite sets are somewhat larger or denser than countably infinite sets. That is, their "type of infinity" is larger than the "type of infinity" of the natural numbers. This is somewhat difficult to prove, and it was only in the XIX-th century, thanks to the work of George Cantor, that these concepts were discovered. We content ourselves with mentioning here that any non-degenerate interval, for example $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$ is uncountably infinite, and that the set of real numbers $\mathbb{R}$ is uncountably infinite. Hence, in a sense, there are as many numbers between 0 and 1 as there are real numbers! We denote this larger infinity of the real numbers by the symbol $\mathfrak{c}$.

9 Definition The set of all subsets of a set A is the *power set* of A, denoted by 2^A. In symbols

$$2^A = \{X : X \subseteq A\}.$$[3]

10 Example Find all the subsets of $\{a, b, c\}$.

▶**Solution:** *They are*

$S_1 = \varnothing$	$S_4 = \{c\}$	$S_7 = \{c, a\}$
$S_2 = \{a\}$	$S_5 = \{a, b\}$	
$S_3 = \{b\}$	$S_6 = \{b, c\}$	$S_8 = \{a, b, c\}$

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11 Example Find all the subsets of $\{a, b, c, d\}$.

▶**Solution:** *The idea is the following. We use the result of example 10. Now, a subset of $\{a, b, c, d\}$ either contains d or it does not. This means that $\{a, b, c, d\}$ will have $2 \times 8 = 16$ subsets. Since the subsets of $\{a, b, c\}$ do not contain d, we simply list all the subsets of $\{a, b, c\}$ and then to each one of them we add d. This gives*

$S_1 = \varnothing$	$S_6 = \{b, c\}$	$S_{12} = \{c, d\}$
$S_2 = \{a\}$	$S_7 = \{c, a\}$	$S_{13} = \{a, b, d\}$
$S_3 = \{b\}$	$S_8 = \{a, b, c\}$	$S_{14} = \{b, c, d\}$
$S_4 = \{c\}$	$S_9 = \{d\}$	$S_{15} = \{c, a, d\}$
$S_5 = \{a, b\}$	$S_{10} = \{a, d\}$	$S_{16} = \{a, b, c, d\}$
	$S_{11} = \{b, d\}$	

[3]This is read "the collection of X such that X is a subset of A.

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Reasoning inductively, as in the last two examples, we obtain the following theorem.

12 Theorem If $\text{card}\,(A) = n < \infty$, then $\text{card}\,\left(2^A\right) = 2^n$.

Proof: *We use induction. Clearly a set A with $n = 1$ elements has $2^1 = 2$ subsets: $\varnothing$ and A itself. Assume every set with $n-1$ elements has 2^{n-1} subsets. Let B be a set with n elements. If $x \in B$ then $B \setminus \{x\}$ is a set with $n-1$ elements and so by the induction hypothesis it has 2^{n-1} subsets. For each subset $S \subseteq B \setminus \{x\}$ we form the new subset $S \cup \{x\}$. This is a subset of B. There are 2^{n-1} such new subsets, and so B has a total of $2^{n-1} + 2^{n-1} = 2^n$ subsets. A different proof will be given in Theorem 55.*❑

Homework

Problem 1.1.1 *Given the set $A = \{a, b\}$, find 2^A and* $\text{card}\,(2^A)$.

Problem 1.1.2 *Let A be the set of all 3-element subsets of $\{1, 2, 3, 4\}$. List all the elements of A and find* $\text{card}\,(A)$.

Problem 1.1.3 *List all the elements of the set*

$$A = \{x \in \mathbb{Z} : x^2 < 6\},$$

that is, the set of all integers whose squares are strictly less than 6. Is the set A the same as the set

$$B = \{t \in \mathbb{Z} : t^2 < 9\}?$$

Problem 1.1.4 *How many subsets does the set $\varnothing$ have? How many subsets does a set with 10 elements have?*

Problem 1.1.5 *Is there a difference between the sets $\varnothing$ and $\{\varnothing\}$?*

Problem 1.1.6 *Consider the set*

$$\{1, 7, 13, \ldots, 397\},$$

where the elements are in arithmetic progression. How many elements does it have? Is 295 in this set? What is the sum of the elements of this set?

1.2 Sample Spaces and Events

13 Definition A situation whose results depend on chance will be called an *experiment*.

14 Example Some experiments in our probability context are

❶ rolling a die,
❷ flipping a coin,
❸ choosing a card from a deck,
❹ selecting a domino piece.
❺ spinning a roulette.
❻ forming a committee from a given group of people.
❼ waiting for a bus.

15 Definition A set $\Omega \neq \varnothing$ is called a *sample space* or *outcome space*. The elements of the sample space are called *outcomes*. A subset $A \subseteq \Omega$ is called an *event*. In particular, $\varnothing \subseteq \Omega$ is called the *null* or *impossible* event.

16 Example If the experiment is flipping a fair coin and recording whether heads H or tails T is obtained, then the sample space is

$$\Omega = \{\text{heads}, \text{tails}\} \tag{1.1}$$

17 Example If the experiment is rolling a fair die once and observing how many dots are displayed, then the sample space is the set

$$\Omega = \{⚀, ⚁, ⚂, ⚃, ⚄, ⚅\}.$$

The event E of observing an even number of dots is

$$E = \{⚁, ⚃, ⚅\}$$

and the event O of observing an odd number of dots is

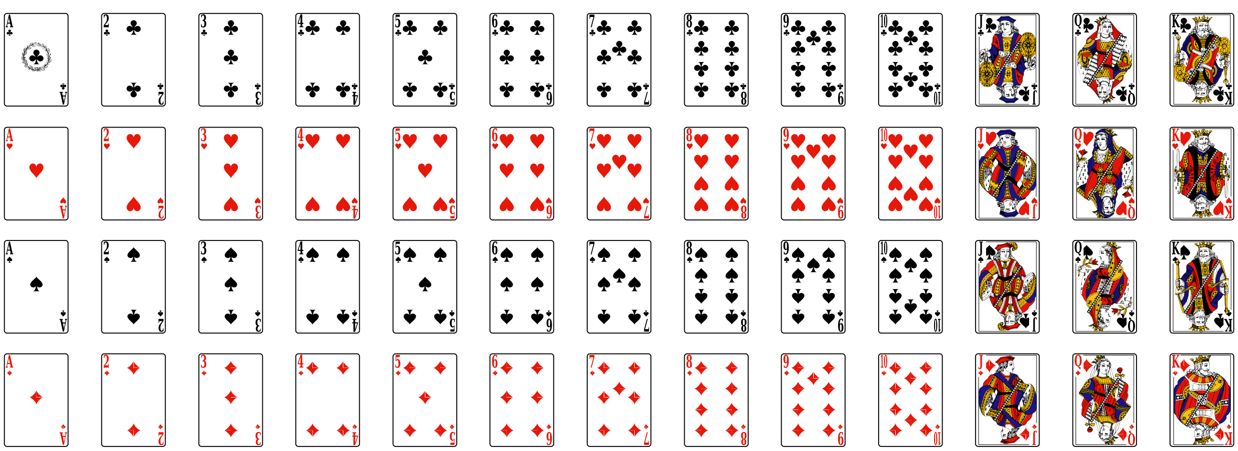
$$O = \{⚀, ⚂, ⚄\}.$$

The event P of observing a prime number score is

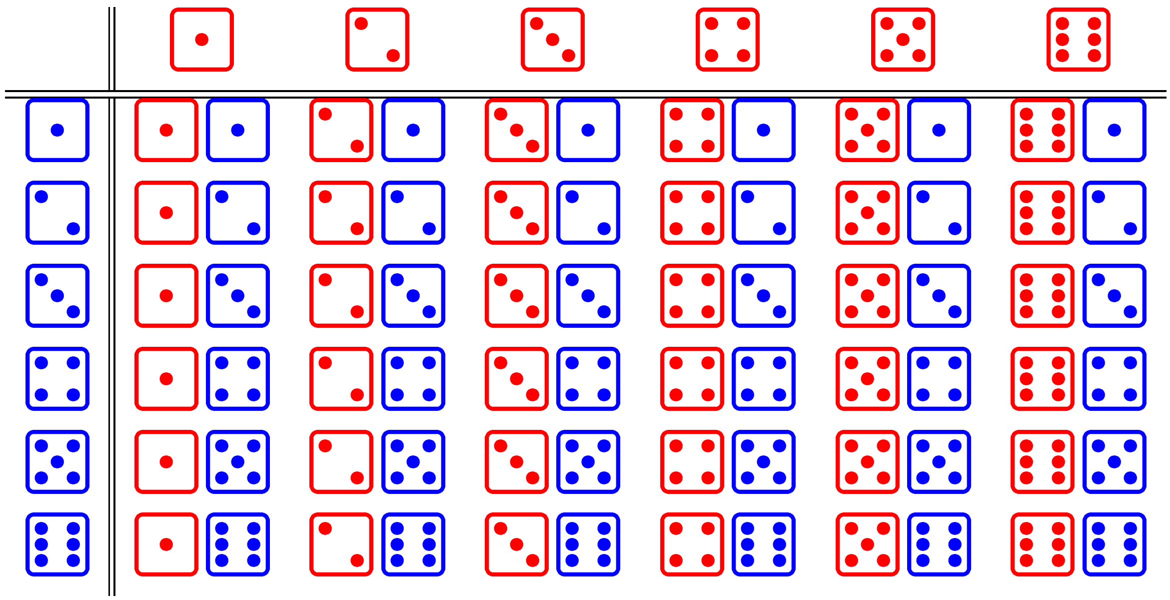
$$P = \{⚁, ⚂, ⚄\}.$$

18 Example If the experiment consists of measuring the time until the bus comes, then the sample space is $[0; +\infty[$, that is the time could be any positive real number. If we allow for the possibility that the bus will never shew up (say, we are in a dungeon, where there is no bus service), then a more precise sample space would be $[0; +\infty[\cup\{+\infty\}$.

19 Example An experiment consists of drawing one card from a standard (52-card) deck and recording the card. The sample space is the set of 52 cards



20 Example If the experiment consists of tossing two (distinguishable) dice (say one red, one blue), then the sample space consists of the 36 ordered pairs:



Here we record first the number on the red die and then the number on the blue die in the ordered pair (R, B). The event S of obtaining a sum of 7 is the set of ordered pairs

$$S = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}.$$

Homework

Problem 1.2.1 *An experiment consists of flipping a fair coin twice and recording each flip. Determine its sample space.*

Problem 1.2.2 *In the experiment of tossing two distinguishable dice in example 20, determine the event X of getting a product of 6, the event T of getting a sum smaller than 5, and the event U of getting a product which is a multiple of 7.*

Problem 1.2.3 *An urn has two blue and three red marbles. Three marbles are drawn one by one—without replacement—and their colour noted. Define a sample space for this experiment.*

Problem 1.2.4 *A small bookshelf has room for four books: two different Spanish books, an Italian book and a German book. Define a sample space for the number of ways of arranging the books in a row in this bookshelf. Also, describe the event E that the Spanish novels remain together.*

Problem 1.2.5 *A purse has two quarters, three nickels, one dime and four pennies. Two coins are drawn one by one, at random and without replacement. Define a sample space for the following experiments:*

1. *Drawing 26¢,*
2. *Drawing 29¢,*
3. *Drawing at least 10¢ but at most 24¢.*

1.3 Combining Events

21 Definition The *union* of two events A and B is the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Observe that this "*or*" is inclusive, that is, it allows the possibility of x being in A, or B, or possibly both A and B.

The *intersection* of two events A and B, is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The difference of events A *set-minus* B, is

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Figures 1.1 through 1.3 represent these concepts pictorially, through the use of *Venn Diagrams*.

22 Definition Two events A and B are *disjoint* or *mutually exclusive* if $A \cap B = \varnothing$.

23 Definition Let $A \subseteq \Omega$. The *complement* of A with respect to Ω is $A^c = \{\omega \in \Omega : \omega \notin A\} = \Omega \setminus A$. This is sometimes written as $\complement_\Omega A$.

Observe that A^c is all that which is outside A. The complement A^c represents the event that A does not occur. We represent A^c pictorially as in figure 1.4.

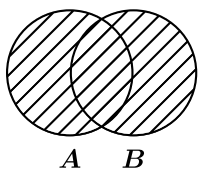


Figure 1.1: $A \cup B$

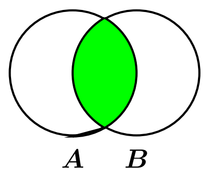


Figure 1.2: $A \cap B$

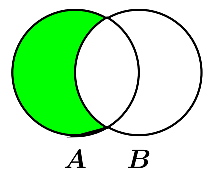


Figure 1.3: $A \setminus B$

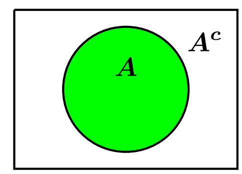


Figure 1.4: A^c

24 Example Let $\Omega = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ be the universal set of the decimal digits and let $A = \{0, 2, 4, 6, 8\} \subseteq \Omega$ be the set of even digits. Then $A^c = \{1, 3, 5, 7, 9\}$ is the set of odd digits.

Observe that

$$(A^c) \cap A = \varnothing. \tag{1.2}$$

The following equalities are known as the *De Morgan Laws*, and their truth can easily be illustrated via Venn Diagrams.

$$(A \cup B)^c = A^c \cap B^c, \tag{1.3}$$

$$(A \cap B)^c = A^c \cup B^c. \tag{1.4}$$

The various intersecting regions for two and three sets can be seen in figures 1.5 and 1.6.

25 Example Let $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{1, 3, 5, 7, 9\}$. Then

$$A \cup B = \{1, 2, 3, 4, 5, 6, 7, 9\}, \qquad A \cap B = \{1, 3, 5\}, \qquad A \setminus B = \{2, 4, 6\}, \qquad B \setminus A = \{7, 9\}.$$

In the following problem we will use the notation $\lfloor x \rfloor$ to denote the *floor* of x, which is x if $x \in \mathbb{Z}$ is an integer, or the integer just left of x if $x \notin \mathbb{Z}$. For example, $\lfloor 4 \rfloor = 4$, $\lfloor 4.1 \rfloor = 4$, $\lfloor 4.7 \rfloor = 4$, $\lfloor -\pi \rfloor = -4$.

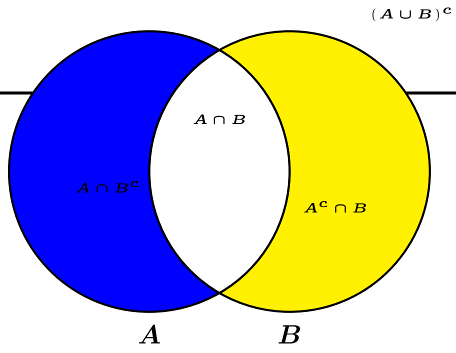


Figure 1.5: Two sets.

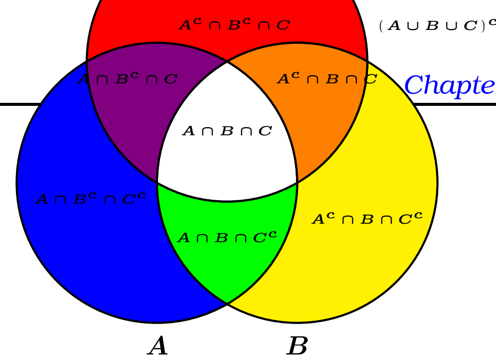


Figure 1.6: Three sets.

26 Example Consider the two sets

$$A = \{2, 7, 12, \ldots, 302\}, \qquad B = \{1, 7, 13, \ldots, 397\},$$

whose elements are in arithmetic progression. Find $A \cap B$.

▸**Solution:** *The progression in A has common difference 5 and the one in B has common difference 6. Observe that the smallest element they share is 7, and hence, they will share every $\mathrm{lcm}[5, 6] = 30$ elements, starting with 7. We now want the largest k so that*

$$7 + 30k \leq 302,$$

where we have chosen 302 since it is the minimum of 302 and 397. Solving,

$$k \leq \left\lfloor \frac{302 - 7}{7} \right\rfloor = 9.$$

Hence there are $9 + 1 = 10$ elements in the intersection. They are

$$A \cap B = \{7, 37, 67, 97, 127, 157, 187, 217, 247, 277\}.$$

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27 Example Let A, B, C be events. Then, as a function of A, B, C,

❶ The event that only A happens is $A \cap B^c \cap C^c$.

❷ The event that only A and C happen, but not B is $A \cap B^c \cap C$.

❸ The event that all three happen is $A \cap B \cap C$.

❹ The event that at least one of the three events occurs is $A \cup B \cup C$.

❺ The event that none of the events occurs is

$$(A \cup B \cup C)^c = A^c \cap B^c \cap C^c,$$

where the equality comes from the De Morgan's Laws.

❻ The event that exactly two of A, B, C occur is

$$(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C).$$

❼ The event that no more than two of A, B, C occur is $(A \cap B \cap C)^c$.

Homework

Problem 1.3.1 *In how many ways can $\{1, 2, 3\}$ be written as the union of two or more non-empty and disjoint subsets?*

Problem 1.3.2 *What is a simpler name for $(A^c)^c$?*

Problem 1.3.3 *What is a simpler name for $(A \cup B) \cap B$?*

Problem 1.3.4 *What is a simpler name for $(A \cup B^c) \cap B$?*

Problem 1.3.5 *Write $(A \cup B)$ as the union of two disjoint sets.*

Problem 1.3.6 *Write $(A \cup B)$ as the union of three disjoint sets.*

Problem 1.3.7 *Let A, B be events of some sample space Ω. Write in symbols the event "exactly one of A or B occurs."*

Problem 1.3.8 *Let A, B be events of some sample space Ω. If $A \cap B = \varnothing$, what is $(A^c \cup B^c)^c$?*

Problem 1.3.9 *Let A, B, C be events of some sample space Ω. Write in symbols*

❶ *the event that at least two of the three events occurs.*

❷ *the event that at most one of the three events occurs.*

Problem 1.3.10 *Given sets X, Y, Z as follows.*

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\},$$

$$Y = \{2, 4, 6, 8, 10, 12, 14, 16\},$$

$$Z = \{2, 3, 5, 7, 11, 13, 17\},$$

❶ *Determine $X \setminus Z$.*

❷ *Determine $Y \setminus Z$.*

❸ *Determine $(X \setminus Z) \cap (Y \setminus Z)$.*

Problem 1.3.11 *Consider the two sets*

$$A = \{3, 13, 23, \ldots, 456\}, \qquad B = \{1, 13, \ldots, 361\},$$

whose elements are in arithmetic progression. Find $A \cap B$.

Problem 1.3.12 *Let A, B be events of the same sample space Ω. What conclusion can you reach if $A \cup B = A$?*

Problem 1.3.13 *Let A, B be events of the same sample space Ω. What conclusion can you reach if $A \cap B = A$?*

Problem 1.3.14 *Let A, B, C be events of the same sample space Ω. What conclusion can you reach if*

$$A \cup B \cup C = A?$$

1.4 Functions

28 Definition By a *function* $f : \mathbf{Dom}(f) \to \mathbf{Target}(f)$ we mean the collection of the following ingredients:

❶ a *name* for the function. Usually we use the letter f.

❷ a set of inputs called the *domain* of the function. The domain of f is denoted by $\mathbf{Dom}(f)$.

❸ an *input parameter* , also called *independent variable* or *dummy variable*. We usually denote a typical input by the letter x.

❹ a set of possible outputs of the function, called the *target set* of the function. The target set of f is denoted by $\mathbf{Target}(f)$.

❺ an *assignment rule* or *formula*, assigning to **every input** a **unique** output. This assignment rule for f is usually denoted by $x \mapsto f(x)$. The output of x under f is also referred to as the *image of x under f*, and is denoted by $f(x)$.

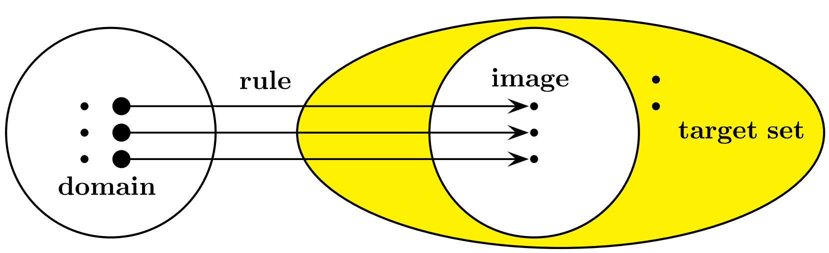

Figure 1.7: The main ingredients of a function.

The notation[4]

$$f: \begin{array}{ccc} \mathbf{Dom}(f) & \to & \mathbf{Target}(f) \\ x & \mapsto & f(x) \end{array}$$

read "the function f, with domain $\mathbf{Dom}(f)$, target set $\mathbf{Target}(f)$, and assignment rule f mapping x to $f(x)$" conveys all the above ingredients. See figure 1.7.

29 Definition The *image* $\mathbf{Im}(f)$ of a function f is its set of actual outputs. In other words,

$$\mathbf{Im}(f) = \{f(a) : a \in \mathbf{Dom}(f)\}.$$

Observe that we always have $\mathbf{Im}(f) \subseteq \mathbf{Target}(f)$.

30 Example Find all functions with domain $\{a, b\}$ and target set $\{c, d\}$.

▶**Solution:** *There are* $2^2 = 4$ *such functions, namely:*

❶ f_1 *given by* $f_1(a) = f_1(b) = c$. *Observe that* $\mathbf{Im}(f_1) = \{c\}$.

❷ f_2 *given by* $f_2(a) = f_2(b) = d$. *Observe that* $\mathbf{Im}(f_2) = \{d\}$.

❸ f_3 *given by* $f_3(a) = c$, $f_3(b) = d$. *Observe that* $\mathbf{Im}(f_3) = \{c, d\}$.

❹ f_4 *given by* $f_4(a) = d$, $f_4(b) = c$. *Observe that* $\mathbf{Im}(f_4) = \{c, d\}$.

◀

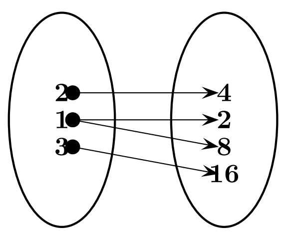

Figure 1.8: Not a function.

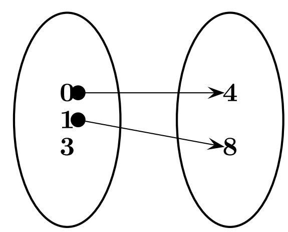

Figure 1.9: Not a function.

It must be emphasised that the uniqueness of the image of an element of the domain is crucial. For example, the diagram in figure 1.8 *does not* represent a function. The element 1 in the domain is assigned to more than one element of the target set. Also important in the definition of a function is the fact that *all the elements* of the domain must be operated on. For example, the diagram in 1.9 *does not* represent a function. The element 3 in the domain is not assigned to any element of the target set.

31 Example Consider the sets $A = \{1, 2, 3\}$, $B = \{1, 4, 9\}$, and the rule f given by $f(x) = x^2$, which means that f takes an input and squares it. Figures 1.10 through 1.11 give three ways of representing the function $f : A \to B$.

32 Definition A function is *injective* or *one-to-one* whenever two different values of its domain generate two different values in its image. A function is *surjective* or *onto* if every element of its target set is hit, that is, the target set is the same as the image of the function. A function is *bijective* if it is both injective and surjective.

[4]Notice the difference in the arrows. The straight arrow $\longrightarrow$ is used to mean that a certain set is associated with another set, whereas the arrow $\mapsto$ (read "maps to") is used to denote that an input becomes a certain output.

$$f : \begin{array}{ccc} \{1,2,3\} & \to & \{1,4,9\} \\ x & \mapsto & x^2 \end{array}$$

Figure 1.10: Example 31.

$$f : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \end{pmatrix}$$

Figure 1.11: Example 31.

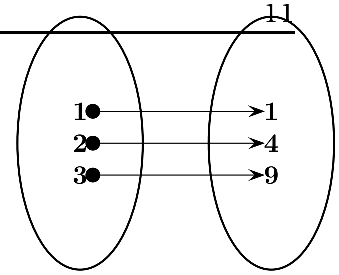

Figure 1.12: Example 31.

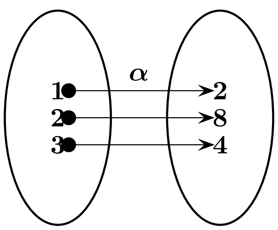

Figure 1.13: An injection.

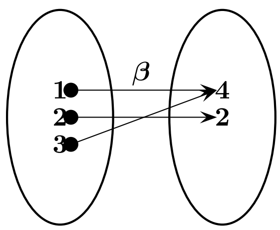

Figure 1.14: Not an injection

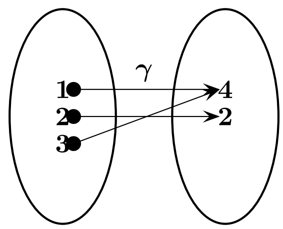

Figure 1.15: A surjection

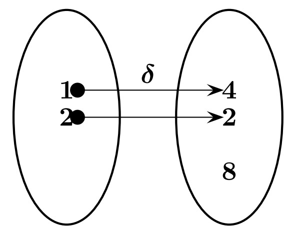

Figure 1.16: Not a surjection

33 Example The function α in the diagram 1.13 is an injective function. The function represented by the diagram 1.14, however is not injective, since $\beta(3) = \beta(1) = 4$, but $3 \neq 1$. The function γ represented by diagram 1.15 is surjective. The function δ represented by diagram 1.16 is not surjective since 8 is part of the target set but not of the image of the function.

> **34 Theorem** Let $f : A \to B$ be a function, and let A and B be finite. If f is injective, then $\mathbf{card}\,(A) \leq \mathbf{card}\,(B)$. If f is surjective then $\mathbf{card}\,(B) \leq \mathbf{card}\,(A)$. If f is bijective, then $\mathbf{card}\,(A) = \mathbf{card}\,(B)$.

Proof: *Put* $n = \mathbf{card}\,(A)$, $A = \{x_1, x_2, \ldots, x_n\}$ *and* $m = \mathbf{card}\,(B)$, $B = \{y_1, y_2, \ldots, y_m\}$.

If f were injective then $f(x_1), f(x_2), \ldots, f(x_n)$ are all distinct, and among the y_k. Hence $n \leq m$.

If f were surjective then each y_k is hit, and for each, there is an x_i with $f(x_i) = y_k$. Thus there are at least m different images, and so $n \geq m$. ❑

35 Definition A *permutation* is a function from a finite set to itself which reorders the elements of the set.

☞ *By necessity then, permutations are bijective.*

36 Example The following are permutations of $\{a, b, c\}$:

$$f_1 : \begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix} \qquad f_2 : \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}.$$

The following are *not* permutations of $\{a, b, c\}$:

$$f_3 : \begin{pmatrix} a & b & c \\ a & a & c \end{pmatrix} \qquad f_4 : \begin{pmatrix} a & b & c \\ b & b & a \end{pmatrix}.$$

Homework

Problem 1.4.1 *Find all functions from* $\{0, 1, 2\}$ *to* $\{-1, 1\}$. *How many are injective? How many are surjective?*

Problem 1.4.2 *Find all functions from* $\{-1, 1\}$ *to* $\{0, 1, 2\}$. *How many are injective? How many are surjective?*

Problem 1.4.3 *List all the permutations of* $\{1, 2\}$ *to itself.*

Problem 1.4.4 *List all the permutations of* $\{1, 2, 3\}$ *to itself.*

2 Counting

2.1 Inclusion-Exclusion

In this section we investigate a tool for counting unions of events. It is known as *The Principle of Inclusion-Exclusion* or Sylvester-Poincaré Principle.

Observe that $\mathbf{card}\,(A) + \mathbf{card}\,(B) \leq \mathbf{card}\,(A \cup B)$, because the sets A and B might overlap. What the difference of the dextral and sinistral quantities to the inequalities is, is the subject of the following theorem.

37 Theorem (Two set Inclusion-Exclusion)

$$\mathbf{card}\,(A \cup B) = \mathbf{card}\,(A) + \mathbf{card}\,(B) - \mathbf{card}\,(A \cap B)$$

Proof: *We have*

$$A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B),$$

and this last expression is a union of disjoint sets. Hence

$$\mathbf{card}\,(A \cup B) = \mathbf{card}\,(A \setminus B) + \mathbf{card}\,(B \setminus A) + \mathbf{card}\,(A \cap B)\,.$$

But

$$A \setminus B = A \setminus (A \cap B) \implies \mathbf{card}\,(A \setminus B) = \mathbf{card}\,(A) - \mathbf{card}\,(A \cap B)\,,$$
$$B \setminus A = B \setminus (A \cap B) \implies \mathbf{card}\,(B \setminus A) = \mathbf{card}\,(B) - \mathbf{card}\,(A \cap B)\,,$$

from where we deduce the result. ❑

In the Venn diagram 2.1, we mark by R_1 the number of elements which are simultaneously in both sets (i.e., in $A \cap B$), by R_2 the number of elements which are in A but not in B (i.e., in $A \setminus B$), and by R_3 the number of elements which are B but not in A (i.e., in $B \setminus A$). We have $R_1 + R_2 + R_3 = \mathbf{card}\,(A \cup B)$, which illustrates the theorem.

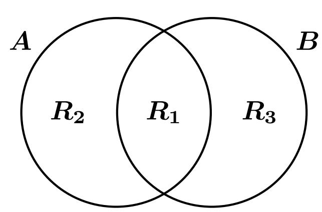

Figure 2.1: Two-set Inclusion-Exclusion

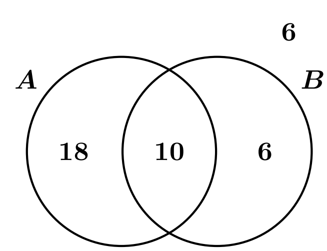

Figure 2.2: Example 38.

38 Example Of 40 people, 28 smoke and 16 chew tobacco. It is also known that 10 both smoke and chew. How many among the 40 neither smoke nor chew?

▶**Solution:** *Let A denote the set of smokers and B the set of chewers. Then*

$$\text{card}(A \cup B) = \text{card}(A) + \text{card}(B) - \text{card}(A \cap B) = 28 + 16 - 10 = 34,$$

meaning that there are 34 people that either smoke or chew (or possibly both). Therefore the number of people that neither smoke nor chew is $40 - 34 = 6$.

Aliter*: We fill up the Venn diagram in figure 2.2 as follows. Since $\text{card}(A \cap B) = 10$, we put a 10 in the intersection. Then we put a $28 - 10 = 18$ in the part that A does not overlap B and a $16 - 10 = 6$ in the part of B that does not overlap A. We have accounted for $10 + 18 + 6 = 34$ people that are in at least one of the set. The remaining $40 - 34 = 6$ are outside these sets.* ◀

39 Example How many integers between 1 and 1000 inclusive, do not share a common factor with 1000, that is, are relatively prime to 1000?

▶**Solution:** *Observe that $1000 = 2^3 5^3$, and thus from the 1000 integers we must weed out those that have a factor of 2 or of 5 in their prime factorisation. If A_2 denotes the set of those integers divisible by 2 in the interval $[1; 1000]$ then clearly $\text{card}(A_2) = \lfloor \frac{1000}{2} \rfloor = 500$. Similarly, if A_5 denotes the set of those integers divisible by 5 then $\text{card}(A_5) = \lfloor \frac{1000}{5} \rfloor = 200$. Also $\text{card}(A_2 \cap A_5) = \lfloor \frac{1000}{10} \rfloor = 100$. This means that there are $\text{card}(A_2 \cup A_5) = 500 + 200 - 100 = 600$ integers in the interval $[1; 1000]$ sharing at least a factor with 1000, thus there are $1000 - 600 = 400$ integers in $[1; 1000]$ that do not share a factor prime factor with 1000.* ◀

We now deduce a formula for counting the number of elements of a union of three events.

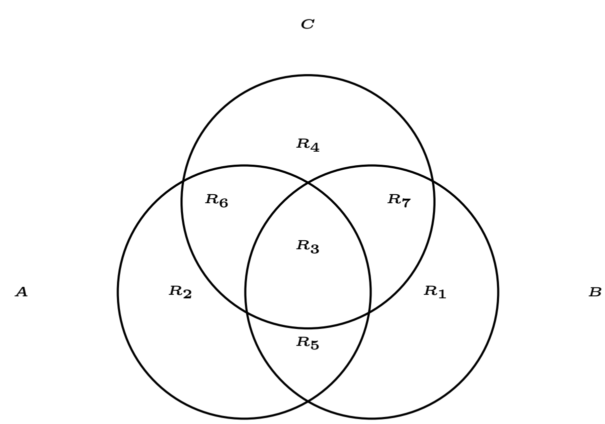

Figure 2.3: Three-set Inclusion-Exclusion

40 Theorem (Three set Inclusion-Exclusion) Let A, B, C be events of the same sample space Ω. Then

$$\begin{aligned}\text{card}(A \cup B \cup C) &= \text{card}(A) + \text{card}(B) + \text{card}(C) \\ &\quad - \text{card}(A \cap B) - \text{card}(B \cap C) - \text{card}(C \cap A) + \text{card}(A \cap B \cap C)\end{aligned}$$

Proof: *Using the associativity and distributivity of unions of sets, we see that*

$$\begin{aligned}
\mathrm{card}\,(A \cup B \cup C) &= \mathrm{card}\,(A \cup (B \cup C)) \\
&= \mathrm{card}\,(A) + \mathrm{card}\,(B \cup C) - \mathrm{card}\,(A \cap (B \cup C)) \\
&= \mathrm{card}\,(A) + \mathrm{card}\,(B \cup C) - \mathrm{card}\,((A \cap B) \cup (A \cap C)) \\
&= \mathrm{card}\,(A) + \mathrm{card}\,(B) + \mathrm{card}\,(C) - \mathrm{card}\,(B \cap C) \\
&\quad -\mathrm{card}\,(A \cap B) - \mathrm{card}\,(A \cap C) \\
&\quad +\mathrm{card}\,((A \cap B) \cap (A \cap C)) \\
&= \mathrm{card}\,(A) + \mathrm{card}\,(B) + \mathrm{card}\,(C) - \mathrm{card}\,(B \cap C) \\
&\quad - (\mathrm{card}\,(A \cap B) + \mathrm{card}\,(A \cap C) - \mathrm{card}\,(A \cap B \cap C)) \\
&= \mathrm{card}\,(A) + \mathrm{card}\,(B) + \mathrm{card}\,(C) \\
&\quad -\mathrm{card}\,(A \cap B) - \mathrm{card}\,(B \cap C) - \mathrm{card}\,(C \cap A) \\
&\quad +\mathrm{card}\,(A \cap B \cap C).
\end{aligned}$$

This gives the Inclusion-Exclusion Formula for three sets. See also figure 2.3.

❑

☞ *In the Venn diagram in figure 2.3 there are 8 disjoint regions: the 7 that form $A \cup B \cup C$ and the outside region, devoid of any element belonging to $A \cup B \cup C$.*

41 Example How many integers between 1 and 600 inclusive are not divisible by neither 3, nor 5, nor 7?

►**Solution:** *Let A_k denote the numbers in $[1;600]$ which are divisible by k. Then*

$$\begin{aligned}
\mathrm{card}\,(A_3) &= \left\lfloor \frac{600}{3} \right\rfloor = 200, \\
\mathrm{card}\,(A_5) &= \left\lfloor \frac{600}{5} \right\rfloor = 120, \\
\mathrm{card}\,(A_7) &= \left\lfloor \frac{600}{7} \right\rfloor = 85, \\
\mathrm{card}\,(A_{15}) &= \left\lfloor \frac{600}{15} \right\rfloor = 40 \\
\mathrm{card}\,(A_{21}) &= \left\lfloor \frac{600}{21} \right\rfloor = 28 \\
\mathrm{card}\,(A_{35}) &= \left\lfloor \frac{600}{35} \right\rfloor = 17 \\
\mathrm{card}\,(A_{105}) &= \left\lfloor \frac{600}{105} \right\rfloor = 5
\end{aligned}$$

By Inclusion-Exclusion there are $200 + 120 + 85 - 40 - 28 - 17 + 5 = 325$ integers in $[1;600]$ divisible by at least one of 3, 5, or 7. Those not divisible by these numbers are a total of $600 - 325 = 275$. ◄

42 Example In a group of 30 people, 8 speak English, 12 speak Spanish and 10 speak French. It is known that 5 speak English and Spanish, 5 Spanish and French, and 7 English and French. The number of people speaking all three languages is 3. How many do not speak any of these languages?

▶**Solution:** *Let A be the set of all English speakers, B the set of Spanish speakers and C the set of French speakers in our group. We fill-up the Venn diagram in figure 2.4 successively. In the intersection of all three we put 8. In the region common to A and B which is not filled up we put $5-2=3$. In the region common to A and C which is not already filled up we put $5-3=2$. In the region common to B and C which is not already filled up, we put $7-3=4$. In the remaining part of A we put $8-2-3-2=1$, in the remaining part of B we put $12-4-3-2=3$, and in the remaining part of C we put $10-2-3-4=1$. Each of the mutually disjoint regions comprise a total of $1+2+3+4+1+2+3=16$ persons. Those outside these three sets are then $30-16=14$.* ◀

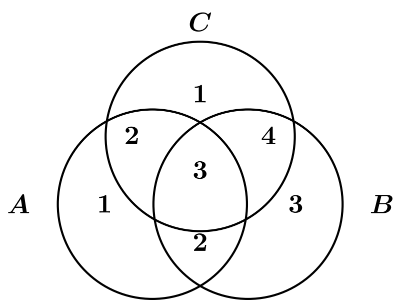

Figure 2.4: Example 42.

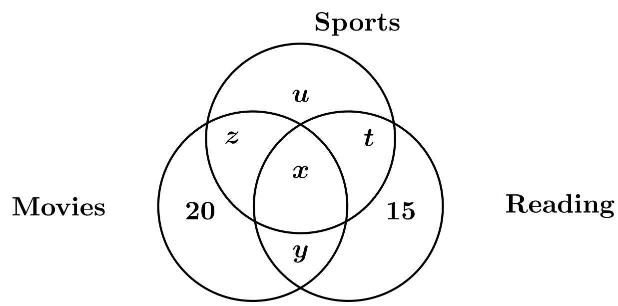

Figure 2.5: Example 43.

43 Example A survey shews that 90% of high-schoolers in Philadelphia like at least one of the following activities: going to the movies, playing sports, or reading. It is known that 45% like the movies, 48% like sports, and 35% like reading. Also, it is known that 12% like both the movies and reading, 20% like only the movies, and 15% only reading. What percent of high-schoolers like all three activities?

▶**Solution:** *We make the Venn diagram in as in figure 2.5. From it we gather the following system of equations*

$$\begin{array}{ccccccccccccccccc}
x & + & y & + & z & & & & & & & + & 20 & = & 45 \\
x & & & + & z & + & t & + & u & & & & & = & 48 \\
x & + & y & & & + & t & & & + & 15 & & & = & 35 \\
x & + & y & & & & & & & & & & & = & 12 \\
x & + & y & + & z & + & t & + & u & + & 15 & + & 20 & = & 90
\end{array}$$

The solution of this system is seen to be $x=5$, $y=7$, $z=13$, $t=8$, $u=22$. Thus the percent wanted is 5%. ◀

Homework

Problem 2.1.1 *Consider the set*

$$A = \{2, 4, 6, \dots, 114\}.$$

❶ *How many elements are there in A?*

❷ *How many are divisible by 3?*

❸ *How many are divisible by 5?*

❹ *How many are divisible by 15?*

❺ *How many are divisible by either 3, 5 or both?*

❻ *How many are neither divisible by 3 nor 5?*

❼ *How many are divisible by exactly one of 3 or 5?*

Problem 2.1.2 *Consider the set of the first 100 positive integers:*

$$A = \{1, 2, 3, \dots, 100\}.$$

❶ *How many are divisible by 2?*

❷ *How many are divisible by 3?*

❸ *How many are divisible by 7?*

❹ *How many are divisible by 6?*

❺ *How many are divisible by 14?*

❻ *How many are divisible by 21?*

❼ *How many are divisible by 42?*

❽ *How many are relatively prime to 42?*

❾ *How many are divisible by 2 and 3 but not by 7?*

❿ *How many are divisible by exactly one of 2, 3 and 7?*

Problem 2.1.3 *A survey of a group's viewing habits over the last year revealed the following information:*

❶ 28% *watched gymnastics*

❷ 29% *watched baseball*

❸ 19% *watched soccer*

❹ 14% *watched gymnastics and baseball*

❺ 12% *watched baseball and soccer*

❻ 10% *watched gymnastics and soccer*

❼ 8% *watched all three sports.*

Calculate the percentage of the group that watched none of the three sports during the last year.

Problem 2.1.4 *At* Medieval High *there are forty students. Amongst them, fourteen like Mathematics, sixteen like theology, and eleven like alchemy. It is also known that seven like Mathematics and theology, eight like theology and alchemy and five like Mathematics and alchemy. All three subjects are favoured by four students. How many students like neither Mathematics, nor theology, nor alchemy?*

Problem 2.1.5 *How many strictly positive integers less than or equal to 1000 are*

❶ *perfect squares?*

❷ *perfect cubes?*

❸ *perfect fifth powers?*

❹ *perfect sixth powers?*

❺ *perfect tenth powers?*

❻ *perfect fifteenth powers?*

❼ *perfect thirtieth powers?*

❽ *neither perfect squares, perfect cubes, perfect fifth powers?*

Problem 2.1.6 *An auto insurance company has $10,000$ policyholders. Each policy holder is classified as*

- *young or old,*
- *male or female, and*
- *married or single.*

Of these policyholders, 3000 are young, 4600 are male, and 7000 are married. The policyholders can also be classified as 1320 young males, 3010 married males, and 1400 young married persons. Finally, 600 of the policyholders are young married males. How many of the company's policyholders are young, female, and single?

Problem 2.1.7 (AHSME 1988) *X, Y, and Z are pairwise disjoint sets of people. The average ages of people in the sets X, Y, Z, $X \cup Y$, $X \cup Y$, and $Y \cup Z$ are given below:*

Set	X	Y	Z	$X \cup Y$	$X \cup Z$	$Y \cup Z$
Average Age	37	23	41	29	39.5	33

What is the average age of the people in the set $X \cup Y \cup Z$?

Problem 2.1.8 *Each of the students in the maths class twice attended a concert. It is known that $25, 12$, and 23 students attended concerts A, B, and C respectively. How many students are there in the maths class? How many of them went to concerts A and B, B and C, or B and C?*

Problem 2.1.9 *The films A, B, and C were shewn in the cinema for a week. Out of 40 students (each of which saw either all the three films, or one of them, 13 students saw film A, 16 students saw film B, and 19 students saw film C. How many students saw all three films?*

Problem 2.1.10 *Would you believe a market investigator that reports that of 1000 people, 816 like candy, 723 like ice cream, 645 cake, while 562 like both candy and ice cream, 463 like both candy and cake, 470 both ice cream and cake, while 310 like all three? State your reasons!*

Problem 2.1.11 (AHSME 1991) *For a set S, let* $\text{card}\left(2^S\right)$ *denote the number of subsets of S. If A, B, C, are sets for which*

$$\text{card}\left(2^A\right) + \text{card}\left(2^B\right) + \text{card}\left(2^C\right) = \text{card}\left(2^{A \cup B \cup C}\right)$$

and

$$\mathrm{card}\,(A) = \mathrm{card}\,(B) = 100,$$

then what is the minimum possible value of $\mathrm{card}\,(A \cap B \cap C)$*?*

Problem 2.1.12 *Find the sum of all the integers from* 1 *to* 1000 *inclusive, which are not multiples of* 3 *or* 5*. You may use the formula* $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Problem 2.1.13 (Lewis Carroll in *A Tangled Tale*.) *In a very hotly fought battle, at least* 70% *of the combatants lost an eye, at least* 75% *an ear, at least* 80% *an arm, and at least* 85% *a leg. What can be said about the percentage who lost all four members?*

2.2 The Product Rule

44 Rule (Product Rule) Suppose that an experiment E can be performed in k stages: E_1 first, E_2 second, $\ldots$, E_k last. Suppose moreover that E_i can be done in n_i different ways, and that the number of ways of performing E_i is not influenced by any predecessors $E_1, E_2, \ldots, E_{i-1}$. Then E_1 **and** E_2 **and** $\ldots$ **and** E_k can occur simultaneously in $n_1 n_2 \cdots n_k$ ways.

45 Example In a group of 8 men and 9 women we can pick one man **and** one woman in $8 \cdot 9 = 72$ ways. Notice that we are choosing two persons.

46 Example A red die and a blue die are tossed. In how many ways can they land?

▶**Solution:** *From example 20 we know that there are* 36 *possible outcomes. This can be confirmed he red die can land in any of* 6 *ways,*

$$\begin{array}{|c|c|}\hline 6 & \\ \hline\end{array}$$

and also, the blue die may land in any of 6 *ways*

$$\begin{array}{|c|c|}\hline 6 & 6 \\ \hline\end{array}.$$

◀

47 Example A multiple-choice test consists of 20 questions, each one with 4 choices. There are 4 ways of answering the first question, 4 ways of answering the second question, etc., hence there are $4^{20} = 1099511627776$ ways of answering the exam.

48 Example There are $9 \cdot 10 \cdot 10 = 900$ positive 3-digit integers:

$$100, 101, 102, \ldots, 998, 999.$$

For, the leftmost integer cannot be 0 and so there are only 9 choices $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for it,

$$\begin{array}{|c|c|c|}\hline 9 & & \\ \hline\end{array}.$$

There are 10 choices for the second digit

$$\begin{array}{|c|c|c|}\hline 9 & 10 & \\ \hline\end{array},$$

and also 10 choices for the last digit

$$\begin{array}{|c|c|c|}\hline 9 & 10 & 10 \\ \hline\end{array}.$$

49 Example There are $9 \cdot 10 \cdot 5 = 450$ even positive 3-digit integers:

$$100, 102, 104, \ldots, 996, 998.$$

For, the leftmost integer cannot be 0 and so there are only 9 choices $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for it,

$$\begin{array}{|c|c|c|}\hline 9 & & \\ \hline \end{array}.$$

There are 10 choices for the second digit

$$\begin{array}{|c|c|c|}\hline 9 & 10 & \\ \hline \end{array}.$$

Since the integer must be even, the last digit must be one of the 5 choices $\{0, 2, 4, 6, 8\}$

$$\begin{array}{|c|c|c|}\hline 9 & 10 & 5 \\ \hline \end{array}.$$

50 Definition A *palindromic integer* or *palindrome* is a positive integer whose decimal expansion is symmetric and that is not divisible by 10. In other words, one reads the same integer backwards or forwards.[1]

For example, the following integers are all palindromes:

$$1, 8, 11, 99, 101, 131, 999, 1234321, 9987899.$$

51 Example How many palindromes are there of 5 digits?

▶**Solution:** *There are 9 ways of choosing the leftmost digit.*

$$\begin{array}{|c|c|c|c|c|}\hline 9 & & & & \\ \hline \end{array}.$$

Once the leftmost digit is chosen, the last digit must be identical to it, so we have

$$\begin{array}{|c|c|c|c|c|}\hline 9 & & & & 1 \\ \hline \end{array}.$$

There are 10 choices for the second digit from the left

$$\begin{array}{|c|c|c|c|c|}\hline 9 & 10 & & & 1 \\ \hline \end{array}.$$

Once this digit is chosen, the second digit from the right must be identical to it, so we have only 1 choice for it,

$$\begin{array}{|c|c|c|c|c|}\hline 9 & 10 & & 1 & 1 \\ \hline \end{array}.$$

Finally, there are 10 choices for the third digit from the right,

$$\begin{array}{|c|c|c|c|c|}\hline 9 & 10 & 10 & 1 & 1 \\ \hline \end{array},$$

which give us 900 palindromes of 5-digits. ◀

52 Example How many palindromes of 5 digits are even?

[1] A palindrome in common parlance, is a word or phrase that reads the same backwards to forwards. The Philadelphia street name *Camac* is a palindrome. So are the phrases (if we ignore punctuation) (a) "A man, a plan, a canal, Panama!" (b) "Sit on a potato pan!, Otis." (c) "Able was I ere I saw Elba." This last one is attributed to Napoleon, though it is doubtful that he knew enough English to form it. The website http://www.palindromelist.com/ has very interesting palindromes.

▶**Solution:** *A five digit even palindrome has the form $ABCBA$, where A belongs to $\{2,4,6,8\}$, and B,C belong to $\{0,1,2,3,4,5,6,7,8,9\}$. Thus there are 4 choices for the first digit, 10 for the second, and 10 for the third. Once these digits are chosen, the palindrome is completely determined. Therefore, there are $4 \times 10 \times 10 = 400$ even palindromes of 5 digits.* ◀

53 Example How many positive divisors does 300 have?

▶**Solution:** *We have $300 = 3 \cdot 2^2 5^2$. Thus every factor of 300 is of the form $3^a 2^b 5^c$, where $0 \le a \le 1$, $0 \le b \le 2$, and $0 \le c \le 2$. Thus there are 2 choices for a, 3 for b and 3 for c. This gives $2 \cdot 3 \cdot 3 = 18$ positive divisors.* ◀

54 Example How many paths consisting of a sequence of horizontal and/or vertical line segments, each segment connecting a pair of adjacent letters in figure 2.6 spell $BIPOLAR$?

$$\begin{array}{ccccccccccccc}
 & & & & & & B & & & & & & \\
 & & & & & B & I & B & & & & & \\
 & & & & B & I & P & I & B & & & & \\
 & & & B & I & P & O & P & I & B & & & \\
 & & B & I & P & O & L & O & P & I & B & & \\
 & B & I & P & O & L & A & L & O & P & I & B & \\
B & I & P & O & L & A & R & A & L & O & P & I & B
\end{array}$$

Figure 2.6: Problem 54.

$$\begin{array}{ccccccc}
 & & & & & & B \\
 & & & & & B & I \\
 & & & & B & I & P \\
 & & & B & I & P & O \\
 & & B & I & P & O & L \\
 & B & I & P & O & L & A \\
B & I & P & O & L & A & R
\end{array}$$

Figure 2.7: Problem 54.

▶**Solution:** *Split the diagram, as in figure 2.7. Since every required path must use the R, we count paths starting from R and reaching up to a B. Since there are six more rows that we can travel to, and since at each stage we can go either up or left, we have $2^6 = 64$ paths. The other half of the figure will provide 64 more paths. Since the middle column is shared by both halves, we have a total of $64 + 64 - 1 = 127$ paths.* ◀

We now prove that if a set A has n elements, then it has 2^n subsets. To motivate the proof, consider the set $\{a,b,c\}$. To each element we attach a binary code of length 3. We write 0 if a particular element is not in the set and 1 if it is. We then have the following associations:

$\varnothing \leftrightarrow 000$,	$\{a,b\} \leftrightarrow 110$,
$\{a\} \leftrightarrow 100$,	$\{a,c\} \leftrightarrow 101$,
$\{b\} \leftrightarrow 010$,	$\{b,c\} \leftrightarrow 011$,
$\{c\} \leftrightarrow 001$,	$\{a,b,c\} \leftrightarrow 111$.

Thus there is a one-to-one correspondence between the subsets of a finite set of 3 elements and binary sequences of length 3.

55 Theorem (Cardinality of the Power Set) Let A be a finite set with $\mathbf{card}(A) = n$. Then A has 2^n subsets.

Proof: *We attach a binary code to each element of the subset, 1 if the element is in the subset and 0 if the element is not in the subset. The total number of subsets is the total number of such binary codes, and there are 2^n in number.* ❑

56 Theorem Let A, B be finite sets with $\mathbf{card}(A) = n$ and $\mathbf{card}(B) = m$. Then

- the number of functions from A to B is m^n.
- if $n \leq m$, the number of injective functions from A to B is $m(m-1)(m-2)\cdots(m-n+1)$. If $n > m$ there are no injective functions from A to B.

Proof: *Each of the n elements of A must be assigned an element of B, and hence there are $\underbrace{m \cdot m \cdots m}_{n \text{ factors}} = m^n$ possibilities, and thus m^n functions.If a function from A to B is injective then we must have $n \leq m$ in view of Theorem 34. If to different inputs we must assign different outputs then to the first element of A we may assign any of the m elements of B, to the second any of the $m-1$ remaining ones, to the third any of the $m-2$ remaining ones, etc., and so we have $m(m-1)\cdots(m-n+1)$ injective functions.* ❑

57 Example Let $A = \{a, b, c\}$ and $B = \{1, 2, 3, 4\}$. Then according to Theorem 56, there are $4^3 = 64$ functions from A to B and of these, $4 \cdot 3 \cdot 2 = 24$ are injective. Similarly, there are $3^4 = 81$ functions from B to A, and none are injective.

Homework

Problem 2.2.1 *A true or false exam has ten questions. How many possible answer keys are there?*

Problem 2.2.2 *Out of nine different pairs of shoes, in how many ways could I choose a right shoe and a left shoe, which should not form a pair?*

Problem 2.2.3 *In how many ways can the following prizes be given away to a class of twenty boys: first and second Classical, first and second Mathematical, first Science, and first French?*

Problem 2.2.4 *Under old hardware, a certain programme accepted passwords of the form*

$$eell$$

where

$$e \in \{0, 2, 4, 6, 8\}, \qquad l \in \{a, b, c, d, u, v, w, x, y, z\}.$$

The hardware was changed and now the software accepts passwords of the form

$$eeelll.$$

How many more passwords of the latter kind are there than of the former kind?

Problem 2.2.5 *A license plate is to be made according to the following provision: it has four characters, the first two characters can be any letter of the English alphabet and the last two characters can be any digit. One is allowed to repeat letters and digits. How many different license plates can be made?*

Problem 2.2.6 *In problem 2.2.5, how many different license plates can you make if (i) you may repeat letters but not digits?, (ii) you may repeat digits but not letters?, (iii) you may repeat neither letters nor digits?*

Problem 2.2.7 *An alphabet consists of the* **five** *consonants* $\{p, v, t, s, k\}$ *and the* **three** *vowels* $\{a, e, o\}$. *A license plate is to be made using* **four** *letters of this alphabet.*

❶ *How many letters does this alphabet have?*

❷ *If a license plate is of the form* $CCVV$ *where* C *denotes a consonant and* V *denotes a vowel, how many possible license plates are there, assuming that you may repeat both consonants and vowels?*

❸ *If a license plate is of the form* $CCVV$ *where* C *denotes a consonant and* V *denotes a vowel, how many possible license plates are there, assuming that you may repeat consonants but not vowels?*

❹ *If a license plate is of the form* $CCVV$ *where* C *denotes a consonant and* V *denotes a vowel, how many possible license plates are there, assuming that you may repeat vowels but not consonants?*

❺ *If a license plate is of the form* $LLLL$ *where* L *denotes any letter of the alphabet, how many possible license plates are there, assuming that you may not repeat letters?*

Problem 2.2.8 *A man lives within reach of three boys' schools and four girls' schools. In how many ways can he send his three sons and two daughters to school?*

Problem 2.2.9 *How many distinct four-letter words can be made with the letters of the set* $\{c, i, k, t\}$

❶ *if the letters are not to be repeated?*

❷ *if the letters can be repeated?*

Problem 2.2.10 *How many distinct six-digit numbers that are multiples of* 5 *can be formed from the list of digits* $\{1, 2, 3, 4, 5, 6\}$ *if we allow repetition?*

Problem 2.2.11 *Telephone numbers in* Land of the Flying Camels *have* 7 *digits, and the only digits available are* $\{0, 1, 2, 3, 4, 5, 7, 8\}$. *No telephone number may begin in* 0, 1 *or* 5. *Find the number of telephone numbers possible that meet the following criteria:*

❶ *You may repeat all digits.*

❷ *You may not repeat any of the digits.*

❸ *You may repeat the digits, but the phone number must be even.*

❹ *You may repeat the digits, but the phone number must be odd.*

❺ *You may not repeat the digits and the phone numbers must be odd.*

Problem 2.2.12 *How many* 5*-lettered words can be made out of* 26 *letters, repetitions allowed, but not consecutive repetitions (that is, a letter may not follow itself in the same word)?*

Problem 2.2.13 *How many positive integers are there having* $n \geq 1$ *digits?*

Problem 2.2.14 *How many* n*-digits integers* ($n \geq 1$) *are there which are even?*

Problem 2.2.15 *How many* n*-digit nonnegative integers do not contain the digit* 5?

Problem 2.2.16 *How many* n*-digit numbers do not have the digit* 0?

Problem 2.2.17 *There are* m *different roads from town A to town B. In how many ways can Dwayne travel from town A to town B and back if (a) he may come back the way he went?, (b) he must use a different road of return?*

Problem 2.2.18 *How many positive divisors does* $2^8 3^9 5^2$ *have? What is the sum of these divisors?*

Problem 2.2.19 *How many factors of* 2^{95} *are larger than* $1,000,000$?

Problem 2.2.20 *How many positive divisors does* 360 *have? How many are even? How many are odd? How many are perfect squares?*

Problem 2.2.21 (AHSME 1988) *At the end of a professional bowling tournament, the top 5 bowlers have a play-off. First # 5 bowls #4. The loser receives the 5th prize and the winner bowls # 3 in another game. The loser of this game receives the 4th prize and the winner bowls # 2. The loser of this game receives the 3rd prize and the winner bowls # 1. The loser of this game receives the 2nd prize and the winner the 1st prize. In how many orders can bowlers #1 through #5 receive the prizes?*

Problem 2.2.22 *The number* 3 *can be expressed as a sum of one or more positive integers in four ways, namely, as* 3, $1+2$, $2+1$, *and* $1+1+1$. *Shew that any positive integer* n *can be so expressed in* 2^{n-1} *ways.*

Problem 2.2.23 *Let* $n = 2^{31}3^{19}$. *How many positive integer divisors of* n^2 *are less than* n *but do not divide* n?

Problem 2.2.24 *Let* $n \geq 3$. *Find the number of* n*-digit ternary sequences that contain at least one* 0, *one* 1 *and one* 2.

Problem 2.2.25 *In how many ways can one decompose the set*

$$\{1, 2, 3, \ldots, 100\}$$

into subsets A, B, C *satisfying*

$$A \cup B \cup C = \{1, 2, 3, \ldots, 100\} \quad \text{and} \quad A \cap B \cap C = \varnothing$$

2.3 The Sum Rule

58 Rule (Sum Rule: Disjunctive Form) Let $E_1, E_2, \ldots, E_k$, be pairwise mutually exclusive events. If E_i can occur in n_i ways, then either E_1 **or** E_2 **or**, ..., **or** E_k can occur in

$$n_1 + n_2 + \cdots n_k$$

ways.

☞ *Notice that the "**or**" here is exclusive.*

59 Example In a group of 8 men and 9 women we can pick one man **or** one woman in $8 + 9 = 17$ ways. Notice that we are choosing one person.

60 Example There are five Golden retrievers, six Irish setters, and eight Poodles at the pound. In how many ways can two dogs be chosen if they are not the same kind?

▶**Solution:** *We choose: a Golden retriever* **and** *an Irish setter* **or** *a Golden retriever* **and** *a Poodle* **or** *an Irish setter* **and** *a Poodle.*

One Golden retriever and one Irish setter can be chosen in $5 \cdot 6 = 30$ ways; one Golden retriever and one Poodle can be chosen in $5 \cdot 8 = 40$ ways; one Irish setter and one Poodle can be chosen in $6 \cdot 8 = 48$ ways. By the sum rule, there are $30 + 40 + 48 = 118$ combinations. ◀

61 Example To write a book 1890 digits were utilised. How many pages does the book have?

▶**Solution:** *A total of*

$$1 \cdot 9 + 2 \cdot 90 = 189$$

digits are used to write pages 1 to 99, inclusive. We have of $1890 - 189 = 1701$ digits at our disposition which is enough for $1701/3 = 567$ extra pages (starting from page 100). The book has $99 + 567 = 666$ pages. ◀

62 Example The sequence of palindromes—starting with 1—is written in ascending order

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, \ldots$$

Find the 1984-th positive palindrome.

▶**Solution:** *It is easy to see that there are 9 palindromes of 1-digit, 9 palindromes with 2-digits, 90 with 3-digits, 90 with 4-digits, 900 with 5-digits and 900 with 6-digits. The last palindrome with 6 digits, 999999, constitutes the $9 + 9 + 90 + 90 + 900 + 900 = 1998$th palindrome. Hence, the 1997th palindrome is 998899, the 1996th palindrome is 997799, the 1995th palindrome is 996699, the 1994th is 995599, etc., until we find the 1984th palindrome to be 985589.* ◀

63 Example The integers from 1 to 1000 are written in succession. Find the sum of all the digits.

▶**Solution:** *When writing the integers from 000 to 999 (with three digits), $3 \times 1000 = 3000$ digits are used. Each of the 10 digits is used an equal number of times, so each digit is used 300 times. The the sum of the digits in the interval 000 to 999 is thus*

$$(0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9)(300) = 13500.$$

Therefore, the sum of the digits when writing the integers from 000 to 1000 is $13500 + 1 = 13501$.

Aliter: *Pair up the integers from* 0 *to* 999 *as*

$$(0, 999),\ (1, 998),\ (2, 997),\ (3, 996),\ \ldots, (499, 500).$$

Each pair has sum of digits 27 *and there are* 500 *such pairs. Adding* 1 *for the sum of digits of* 1000*, the required total is*

$$27 \cdot 500 + 1 = 13501.$$

◀

64 Example How many 4-digit integers can be formed with the set of digits $\{0, 1, 2, 3, 4, 5\}$ such that no digit is repeated and the resulting integer is a multiple of 3?

▶**Solution:** *The integers desired have the form* $D_1D_2D_3D_4$ *with* $D_1 \neq 0$*. Under the stipulated constraints, we must have*

$$D_1 + D_2 + D_3 + D_4 \in \{6, 9, 12\}.$$

We thus consider three cases.

Case I: $D_1 + D_2 + D_3 + D_4 = 6$*. Here we have* $\{D_1, D_2, D_3, D_4\} = \{0, 1, 2, 3\}, D_1 \neq 0$*. There are then* 3 *choices for* D_1*. After* D_1 *is chosen,* D_2 *can be chosen in* 3 *ways,* D_3 *in* 2 *ways, and* D_1 *in* 1 *way. There are thus* $3 \times 3 \times 2 \times 1 = 3 \cdot 3! = 18$ *integers satisfying case I.*

Case II: $D_1 + D_2 + D_3 + D_4 = 9$*. Here we have* $\{D_1, D_2, D_3, D_4\} = \{0, 2, 3, 4\}, D_1 \neq 0$ *or* $\{D_1, D_2, D_3, D_4\} = \{0, 1, 3, 5\}, D_1 \neq 0$*. Like before, there are* $3 \cdot 3! = 18$ *numbers in each possibility, thus we have* $2 \times 18 = 36$ *numbers in case II.*

Case III: $D_1 + D_2 + D_3 + D_4 = 12$*. Here we have* $\{D_1, D_2, D_3, D_4\} = \{0, 3, 4, 5\}, D_1 \neq 0$ *or* $\{D_1, D_2, D_3, D_4\} = \{1, 2, 4, 5\}$*. In the first possibility there are* $3 \cdot 3! = 18$ *numbers, and in the second there are* $4! = 24$*. Thus we have* $18 + 24 = 42$ *numbers in case III.*

The desired number is finally $18 + 36 + 42 = 96$*.* ◀

Homework

Problem 2.3.1 *How many different sums can be thrown with two dice, the faces of each die being numbered* $0, 1, 3, 7, 15, 31$*?*

Problem 2.3.2 *How many different sums can be thrown with three dice, the faces of each die being numbered* $1, 4, 13, 40, 121, 364$*?*

Problem 2.3.3 *How many two or three letter initials for people are available if at least one of the letters must be a D and one allows repetitions?*

Problem 2.3.4 *How many strictly positive integers have all their digits distinct?*

Problem 2.3.5 *The Morse code consists of points and dashes. How many letters can be in the Morse code if no letter contains more than four signs, but all must have at least one?*

Problem 2.3.6 *An* $n \times n \times n$ *wooden cube is painted blue and then cut into* n^3 $1 \times 1 \times 1$ *cubes. How many cubes (a) are painted on exactly three sides, (b) are painted in exactly two sides, (c) are painted in exactly one side, (d) are not painted?*

Problem 2.3.7 (AIME 1993) *How many even integers between* 4000 *and* 7000 *have four different digits?*

Problem 2.3.8 *All the natural numbers, starting with* 1*, are listed consecutively*

$$123456789101112131415161718192021\ldots$$

Which digit occupies the 1002*nd place?*

Problem 2.3.9 *All the positive integers are written in succession.*

$$123456789101112131415161718192021222324\ldots$$

Which digit occupies the 206790*th place?*

Problem 2.3.10 *All the positive integers with initial digit* 2 *are written in succession:*

$$2, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 200, 201, \ldots,$$

Find the 1978*-th digit written.*

Problem 2.3.11 (AHSME 1998) *Call a* 7*-digit telephone number* $d_1d_2d_3 - d_4d_5d_6d_7$ memorable *if the prefix sequence* $d_1d_2d_3$ *is exactly the same as either of the sequences* $d_4d_5d_6$ *or* $d_5d_6d_7$ *or possibly both. Assuming that each* d_i *can be any of the ten decimal digits* $0, 1, 2, \ldots, 9$*, find the number of different memorable telephone numbers.*

Problem 2.3.12 *Three-digit numbers are made using the digits* $\{1, 3, 7, 8, 9\}$.

❶ *How many of these integers are there?*

❷ *How many are even?*

❸ *How many are palindromes?*

❹ *How many are divisible by* 3*?*

Problem 2.3.13 (AHSME 1989) *Five people are sitting at a round table. Let* $f \geq 0$ *be the number of people sitting next to at least one female, and let* $m \geq 0$ *be the number of people sitting next to at least one male. Find the number of possible ordered pairs* (f, m).

Problem 2.3.14 *How many integers less than* 10000 *can be made with the eight digits* $0, 1, 2, 3, 4, 5, 6, 7$*?*

Problem 2.3.15 (ARML 1999) *In how many ways can one arrange the numbers* $21, 31, 41, 51, 61, 71,$ *and* 81 *such that the sum of every four consecutive numbers is divisible by* 3*?*

Problem 2.3.16 *The sequence of palindromes is written in increasing order*

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, 33, \ldots.$$

Thus 11 *occupies the tenth position,* 22 *the eleventh, etc. Which position is occupied by* 1003001*?*

Problem 2.3.17 *When writing all the integers from 1 to 2007, inclusive, how many* 0*'s are used?*

Problem 2.3.18 *How many pairs of integers* (x, y) *are there for which* $x^2 - y^2 = 81$*?*

Problem 2.3.19 *A die consists of a cube which has a different color on each of* 6 *faces. How many distinguishably different kinds of dice can be made?*

Problem 2.3.20 *Each of the six faces of a cube is painted in a different color. The cube has now a fixed color scheme. A die can be formed by painting the numbers* $\{1, 2, 3, 4, 5, 6\}$ *in such a way that the opposing faces add up to* 7*. How many different dice can be formed?*

Problem 2.3.21 *Let* S *be the set of all natural numbers whose digits are chosen from the set* $\{1, 3, 5, 7\}$ *such that no digits are repeated. Find the sum of the elements of* S.

Problem 2.3.22 *Find the number of ways to choose a pair* $\{a, b\}$ *of distinct numbers from the set* $\{1, 2, \ldots, 50\}$ *such that*

❶ $|a - b| = 5$

❷ $|a - b| \leq 5$.

Problem 2.3.23 (AIME 1994) *Given a positive integer* n*, let* $p(n)$ *be the product of the non-zero digits of* n*. (If* n *has only one digit, then* $p(n)$ *is equal to that digit.) Let*

$$S = p(1) + p(2) + \cdots + p(999).$$

Find S.

Problem 2.3.24 n *equally spaced points* $1, 2, \ldots, n$ *are marked on a circumference. If* 15 *directly opposite to* 49*, how many points are there total?*

Problem 2.3.25 *An urn has* 900 *chips, numbered* 100 *through* 999*. Chips are drawn at random and without replacement from the urn, and the sum of their digits is noted. What is the smallest number of chips that must be drawn in order to guarantee that at least three of these digital sums be equal?*

Problem 2.3.26 *Little Dwayne has* 100 *cards where the integers from* 1 *through* 100 *are written. He also has an unlimited supply of cards with the signs* $+$ *and* $=$*. How many true equalities can he make, if he uses each card no more than once?*

Problem 2.3.27 (AIME 1993) *How many ordered fourtuples of integers* (a, b, c, d) *with*

$$0 < a < b < c < d < 500$$

satisfy satisfy $a + d = b + c$ *and* $bc - ad = 93$*?*

Problem 2.3.28 $\mathscr{A}$ *is a set of one hundred distinct natural numbers such that any triplet* a, b, c *of* $\mathscr{A}$ *(repetitions are allowed in a triplet) gives a non-obtuse triangle whose sides measure* $a, b,$ *and* c*. Let* $S(\mathscr{A})$ *be the sum of the perimeters obtained by adding all the triplets in* $\mathscr{A}$*. Find the smallest value of* $S(\mathscr{A})$*. Note: we count repetitions in the sum* $S(\mathscr{A})$*, thus all permutations of a triplet* (a, b, c) *appear in* $S(\mathscr{A})$.

Problem 2.3.29 *Prove that the sum of the digits appearing in the integers*

$$1, 2, 3, \ldots, \underbrace{99\ldots9}_{n\ 9's}$$

is $\dfrac{9n10^n}{2}$.

Problem 2.3.30 (The Locker-room Problem) *A locker room contains n lockers, numbered 1 through n. Initially all doors are open. Person number 1 enters and closes all the doors. Person number 2 enters and opens all the doors whose numbers are multiples of 2. Person number 3 enters and if a door whose number is a multiple of 3 is open then he closes it; otherwise he opens it. Person number 4 enters and changes the status (from open to closed and viceversa) of all doors whose numbers are multiples of 4, and so forth till person number n enters and changes the status of door number n. Which lockers are now closed?*

Problem 2.3.31 (AHSME 1992) *For how many integers between 1 and 100 does*

$$x^2 + x - n$$

factor into the product of two linear factors with integer coefficients?

Problem 2.3.32 *How many triplets (a, b, c) with $a, b, c \in \{1, 2, \ldots, 101\}$ simultaneously satisfy $a < b$ and $a < c$?*

Problem 2.3.33 (Putnam 1987) *The sequence of digits*

$$123456789101112131415161718192021 22\ldots$$

is obtained by writing the positive integers in order. If the 10^n digit of this sequence occurs in the part in which the m-digit numbers are placed, define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = m$. For example $f(2) = 2$, because the hundredth digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, $f(1987)$.

2.4 Permutations without Repetitions

65 Definition We define the symbol ! (factorial), as follows: $0! = 1$, and for integer $n \geq 1$,

$$n! = 1 \cdot 2 \cdot 3 \cdots n.$$

$n!$ is read n *factorial*.

66 Example We have

$$\begin{aligned} 1! &= 1, \\ 2! &= 1 \cdot 2 = 2, \\ 3! &= 1 \cdot 2 \cdot 3 = 6, \\ 4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24, \\ 5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120. \end{aligned}$$

67 Example We have

$$\begin{aligned} \frac{7!}{4!} &= \frac{7 \cdot 6 \cdot 5 \cdot 4!}{4!} = 210, \\ \frac{(n+2)!}{n!} &= \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1), \\ \frac{(n-2)!}{(n+1)!} &= \frac{(n-2)!}{(n+1)(n)(n-1)(n-2)!} = \frac{1}{(n+1)(n)(n-1)}. \end{aligned}$$

68 Definition Let $x_1, x_2, \ldots, x_n$ be n distinct objects. A *permutation* of these objects is simply a rearrangement of them.

69 Example There are 24 permutations of the letters in $MATH$, namely

$$\begin{array}{cccccc} MATH & MAHT & MTAH & MTHA & MHTA & MHAT \\ AMTH & AMHT & ATMH & ATHM & AHTM & AHMT \\ TAMH & TAHM & TMAH & TMHA & THMA & THAM \\ HATM & HAMT & HTAM & HTMA & HMTA & HMAT \end{array}$$

70 Theorem Let $x_1, x_2, \ldots, x_n$ be n distinct objects. Then there are $n!$ permutations of them.

Proof: *The first position can be chosen in n ways, the second object in $n-1$ ways, the third in $n-2$, etc. This gives*

$$n(n-1)(n-2)\cdots 2\cdot 1 = n!.$$

❑

71 Example The number of permutations of the letters of the word $RETICULA$ is $8! = 40320$.

72 Example A bookshelf contains 5 German books, 7 Spanish books and 8 French books. Each book is different from one another.

❶ How many different arrangements can be done of these books?

❷ How many different arrangements can be done of these books if books of each language must be next to each other?

❸ How many different arrangements can be done of these books if all the French books must be next to each other?

❹ How many different arrangements can be done of these books if no two French books must be next to each other?

►**Solution:**

❶ *We are permuting $5+7+8 = 20$ objects. Thus the number of arrangements sought is* $20! = 2432902008176640000$.

❷ *"Glue" the books by language, this will assure that books of the same language are together. We permute the 3 languages in $3!$ ways. We permute the German books in $5!$ ways, the Spanish books in $7!$ ways and the French books in $8!$ ways. Hence the total number of ways is* $3!5!7!8! = 146313216000$.

❸ *Align the German books and the Spanish books first. Putting these $5+7 = 12$ books creates $12+1 = 13$ spaces (we count the space before the first book, the spaces between books and the space after the last book). To assure that all the French books are next each other, we "glue" them together and put them in one of these spaces. Now, the French books can be permuted in $8!$ ways and the non-French books can be permuted in $12!$ ways. Thus the total number of permutations is*

$$(13)8!12! = 251073478656000.$$

❹ *Align the German books and the Spanish books first. Putting these $5+7 = 12$ books creates $12+1 = 13$ spaces (we count the space before the first book, the spaces between books and the space after the last book). To assure that no two French books are next to each other, we put them into these spaces. The first French book can be put into any of 13 spaces, the second into any of 12, etc., the eighth French book can be put into any 6 spaces. Now, the non-French books can be permuted in $12!$ ways. Thus the total number of permutations is*

$$(13)(12)(11)(10)(9)(8)(7)(6)12!,$$

which is 24856274386944000.

◀

Homework

Problem 2.4.1 *How many changes can be rung with a peal of five bells?*

Problem 2.4.2 *A bookshelf contains* 3 *Russian novels,* 4 *German novels, and* 5 *Spanish novels. In how many ways may we align them if*

❶ *there are no constraints as to grouping?*

❷ *all the Spanish novels must be together?*

❸ *no two Spanish novels are next to one another?*

Problem 2.4.3 *How many permutations of the word* **IMPURE** *are there? How many permutations start with* **P** *and end in* **U**? *How many permutations are there if the* **P** *and the* **U** *must always be together in the order* **PU**? *How many permutations are there in which no two vowels (***I, U, E***) are adjacent?*

Problem 2.4.4 *How many arrangements can be made of out of the letters of the word* **DRAUGHT**, *the vowels never separated?*

Problem 2.4.5 (AIME 1991) *Given a rational number, write it as a fraction in lowest terms and calculate the product of the resulting numerator and denominator. For how many rational numbers between* 0 *and* 1 *will* 20! *be the resulting product?*

Problem 2.4.6 (AMC12 2001) *A spider has one sock and one shoe for each of its eight legs. In how many different orders can the spider put on its socks and shoes, assuming that, on each leg, the sock must be put on before the shoe?*

Problem 2.4.7 *How many trailing* 0*'s are there when* 1000! *is multiplied out?*

Problem 2.4.8 *In how many ways can* 8 *people be seated in a row if*

❶ *there are no constraints as to their seating arrangement?*

❷ *persons X and Y must sit next to one another?*

❸ *there are* 4 *women and* 4 *men and no* 2 *men or* 2 *women can sit next to each other?*

❹ *there are* 4 *married couples and each couple must sit together?*

❺ *there are* 4 *men and they must sit next to each other?*

2.5 Permutations with Repetitions

We now consider permutations with repeated objects.

73 Example In how many ways may the letters of the word

$$MASSACHUSETTS$$

be permuted?

▶**Solution:** *We put subscripts on the repeats forming*

$$MA_1S_1S_2A_2CHUS_3ET_1T_2S_4.$$

There are now 13 *distinguishable objects, which can be permuted in* 13! *different ways by Theorem 70. For each of these* 13! *permutations, A_1A_2 can be permuted in* 2! *ways, $S_1S_2S_3S_4$ can be permuted in* 4! *ways, and T_1T_2 can be permuted in* 2! *ways. Thus the over count* 13! *is corrected by the total actual count*

$$\frac{13!}{2!4!2!} = 64864800.$$

◀

A reasoning analogous to the one of example 73, we may prove

74 Theorem Let there be k types of objects: n_1 of type 1; n_2 of type 2; etc. Then the number of ways in which these $n_1 + n_2 + \cdots + n_k$ objects can be rearranged is

$$\frac{(n_1 + n_2 + \cdots + n_k)!}{n_1!n_2!\cdots n_k!}.$$

75 Example In how many ways may we permute the letters of the word $MASSACHUSETTS$ in such a way that $MASS$ is always together, in this order?

►**Solution:** *The particle $MASS$ can be considered as one block and the 9 letters A, C, H, U, S, E, T, T, S. In A, C, H, U, S, E, T, T, S there are four S's and two T's and so the total number of permutations sought is*

$$\frac{10!}{2!2!} = 907200.$$

◄

76 Example In how many ways may we write the number 9 as the sum of three strictly positive integer summands? Here order counts, so, for example, $1 + 7 + 1$ is to be regarded different from $7 + 1 + 1$.

►**Solution:** *We first look for answers with*

$$a + b + c = 9, \qquad 1 \leq a \leq b \leq c \leq 7$$

and we find the permutations of each triplet. We have

(a, b, c)	Number of permutations
$(1, 1, 7)$	$\frac{3!}{2!} = 3$
$(1, 2, 6)$	$3! = 6$
$(1, 3, 5)$	$3! = 6$
$(1, 4, 4)$	$\frac{3!}{2!} = 3$
$(2, 2, 5)$	$\frac{3!}{2!} = 3$
$(2, 3, 4)$	$3! = 6$
$(3, 3, 3)$	$\frac{3!}{3!} = 1$

Thus the number desired is

$$3 + 6 + 6 + 3 + 3 + 6 + 1 = 28.$$

◄

77 Example In how many ways can the letters of the word **MURMUR** be arranged without letting two letters which are alike come together?

►Solution: *If we started with, say , **MU** then the **R** could be arranged as follows:*

$$\begin{array}{|c|c|c|c|c|c|}\hline \mathbf{M} & \mathbf{U} & \mathbf{R} & & \mathbf{R} & \\ \hline \end{array}\,,$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \mathbf{M} & \mathbf{U} & \mathbf{R} & & & \mathbf{R} \\ \hline \end{array}\,,$$

$$\begin{array}{|c|c|c|c|c|c|}\hline \mathbf{M} & \mathbf{U} & & \mathbf{R} & & \mathbf{R} \\ \hline \end{array}\,.$$

*In the first case there are $2! = 2$ of putting the remaining **M** and **U**, in the second there are $2! = 2$ and in the third there is only $1!$. Thus starting the word with **MU** gives $2 + 2 + 1 = 5$ possible arrangements. In the general case, we can choose the first letter of the word in 3 ways, and the second in 2 ways. Thus the number of ways sought is $3 \cdot 2 \cdot 5 = 30$.* ◄

78 Example In how many ways can the letters of the word **AFFECTION** be arranged, keeping the vowels in their natural order and not letting the two **F**'s come together?

►Solution: *There are $\frac{9!}{2!}$ ways of permuting the letters of **AFFECTION**. The 4 vowels can be permuted in $4!$ ways, and in only one of these will they be in their natural order. Thus there are $\frac{9!}{2!4!}$ ways of permuting the letters of **AFFECTION** in which their vowels keep their natural order.*

*Now, put the 7 letters of **AFFECTION** which are not the two **F**'s. This creates 8 spaces in between them where we put the two **F**'s. This means that there are $8 \cdot 7!$ permutations of **AFFECTION** that keep the two **F**'s together. Hence there are $\frac{8 \cdot 7!}{4!}$ permutations of **AFFECTION** where the vowels occur in their natural order.*

In conclusion, the number of permutations sought is

$$\frac{9!}{2!4!} - \frac{8 \cdot 7!}{4!} = \frac{8!}{4!}\left(\frac{9}{2} - 1\right) = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4!}{4!} \cdot \frac{7}{2} = 5880$$

◄

79 Example How many arrangements of five letters can be made of the letters of the word **PALLMALL**?

►Solution: *We consider the following cases:*

❶ *there are four **L**'s and a different letter. The different letter can be chosen in 3 ways, so there are $\frac{3 \cdot 5!}{4!} = 15$ permutations in this case.*

❷ *there are three **L**'s and two **A**'s. There are $\frac{5!}{3!2!} = 10$ permutations in this case.*

❸ *there are three **L**'s and two different letters. The different letters can be chosen in 3 ways (either **P** and **A**; or **P** and **M**; or **A** and **M**), so there are $\frac{3 \cdot 5!}{3!} = 60$ permutations in this case.*

❹ *there are two **L**'s, two **A**'s and a different letter from these two. The different letter can be chosen in 2 ways. There are $\frac{2 \cdot 5!}{2!2!} = 60$ permutations in this case.*

❺ *there are two **L**'s and three different letters. The different letters can be chosen in 1 way. There are $\frac{1 \cdot 5!}{2!} = 60$ permutations in this case.*

❻ *there is one* $\mathbf{L}$. *This forces having two* $\mathbf{A}$*'s and two other different letters. The different letters can be chosen in* 1 *way. There are* $\frac{1 \cdot 5!}{2!} = 60$ *permutations in this case.*

The total number of permutations is thus seen to be

$$15 + 10 + 60 + 60 + 60 + 60 = 265.$$

◀

Homework

Problem 2.5.1 *In how many ways may one permute the letters of the word* **MEPHISTOPHELES**?

Problem 2.5.2 *How many arrangements of four letters can be made out of the letters of* **KAFFEEKANNE** *without letting the three* **E**'s *come together?*

Problem 2.5.3 *How many numbers can be formed with the digits*

$$1, 2, 3, 4, 3, 2, 1$$

so that the odd digits occupy the odd places?

Problem 2.5.4 *The password of the anti-theft device of a car is a four digit number, where one can use any digit in the set*

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

A. ❶ *How many such passwords are possible?*

❷ *How many of the passwords have all their digits distinct?*

B. *After an electrical failure, the owner must reintroduce the password in order to deactivate the anti-theft device. He knows that the four digits of the code are* $2, 0, 0, 3$ *but does not recall the order.*

❶ *How many such passwords are possible using only these digits?*

❷ *If the first attempt at the password fails, the owner must wait two minutes before a second attempt, if the second attempt fails he must wait four minutes before a third attempt, if the third attempt fails he must wait eight minutes before a fourth attempt, etc. (the time doubles from one attempt to the next). How many passwords can the owner attempt in a period of* 24 *hours?*

Problem 2.5.5 *An urn has* 2 *white marbles,* 3 *red marbles, and* 5 *blue marbles. Marbles are drawn one by one and without replacement. Urns of each colour are indistinguishable.*

1. *In how many ways may one draw the marbles out of the urn?*
2. *In how many ways may one draw the marbles out of the urn if the second, fourth, sixth, eighth and tenth marbles are blue?*
3. *In how many instances will all red marbles come before any of the blue marbles?*

Problem 2.5.6 *In this problem you will determine how many different signals, each consisting of* 10 *flags hung in a line, can be made from a set of* 4 *white flags,* 3 *red flags,* 2 *blue flags, and* 1 *orange flag, if flags of the same colour are identical.*

❶ *How many are there if there are no constraints on the order?*

❷ *How many are there if the orange flag must always be first?*

❸ *How many are there if there must be a white flag at the beginning and another white flag at the end?*

Problem 2.5.7 *In how many ways may we write the number* 10 *as the sum of three positive integer summands? Here order counts, so, for example,* $1 + 8 + 1$ *is to be regarded different from* $8 + 1 + 1$.

Problem 2.5.8 *Three distinguishable dice are thrown. In how many ways can they land and give a sum of* 9?

Problem 2.5.9 *In how many ways can* 15 *different recruits be divided into three equal groups? In how many ways can they be drafted into three different regiments?*

2.6 Combinations without Repetitions

80 Definition Let n, k be non-negative integers with $0 \leq k \leq n$. The symbol $\binom{n}{k}$ (read "n *choose* k") is defined and denoted by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1) \cdot (n-2) \cdots (n-k+1)}{1 \cdot 2 \cdot 3 \cdots k}.$$

☞ *Observe that in the last fraction, there are* k *factors in both the numerator and denominator. Also, observe the boundary conditions*

$$\binom{n}{0} = \binom{n}{n} = 1, \quad \binom{n}{1} = \binom{n}{n-1} = n.$$

81 Example We have

$$\begin{aligned}\binom{6}{3} &= \frac{6\cdot 5\cdot 4}{1\cdot 2\cdot 3} = 20,\\ \binom{11}{2} &= \frac{11\cdot 10}{1\cdot 2} = 55,\\ \binom{12}{7} &= \frac{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = 792,\\ \binom{110}{109} &= 110,\\ \binom{110}{0} &= 1.\end{aligned}$$

☞ *Since $n-(n-k)=k$, we have for integer n,k, $0\le k\le n$, the symmetry identity*

$$\boxed{\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!(n-(n-k))!} = \binom{n}{n-k}}.$$

This can be interpreted as follows: if there are n different tickets in a hat, choosing k of them out of the hat is the same as choosing $n-k$ of them to remain in the hat.

82 Example

$$\binom{11}{9} = \binom{11}{2} = 55,$$
$$\binom{12}{5} = \binom{12}{7} = 792.$$

83 Definition Let there be n distinguishable objects. A k-*combination* is a selection of k, $(0\le k\le n)$ objects from the n made without regards to order.

84 Example The 2-combinations from the list $\{X,Y,Z,W\}$ are

$$XY, XZ, XW, YZ, YW, WZ.$$

85 Example The 3-combinations from the list $\{X,Y,Z,W\}$ are

$$XYZ, XYW, XZW, YWZ.$$

86 Theorem Let there be n distinguishable objects, and let k, $0\le k\le n$. Then the numbers of k-combinations of these n objects is $\binom{n}{k}$.

Proof: *Pick any of the k objects. They can be ordered in $n(n-1)(n-2)\cdots(n-k+1)$, since there are n ways of choosing the* first, *$n-1$ ways of choosing the* second, *etc. This particular choice of k objects can be permuted in $k!$ ways. Hence the total number of k-combinations is*

$$\frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} = \binom{n}{k}.$$

❑

87 Example From a group of 10 people, we may choose a committee of 4 in $\binom{10}{4} = 210$ ways.

88 Example In a group of 2 camels, 3 goats, and 10 sheep in how many ways may one choose 6 animals if

❶ there are no constraints in species?

❷ the two camels must be included?

❸ the two camels must be excluded?

❹ there must be at least 3 sheep?

❺ there must be at most 2 sheep?

❻ Joe Camel, Billy Goat and Samuel Sheep hate each other and they will not work in the same group. How many compatible committees are there?

▶**Solution:**

❶ *There are* $2 + 3 + 10 = 15$ *animals and we must choose* 6, *whence* $\binom{15}{6} = 5005$

❷ *Since the* 2 *camels are included, we must choose* $6 - 2 = 4$ *more animals from a list of* $15 - 2 = 13$ *animals, so* $\binom{13}{4} = 715$

❸ *Since the* 2 *camels must be excluded, we must choose* 6 *animals from a list of* $15 - 2 = 13$, *so* $\binom{13}{6} = 1716$

❹ *If* k *sheep are chosen from the* 10 *sheep,* $6 - k$ *animals must be chosen from the remaining* 5 *animals, hence*

$$\binom{10}{3}\binom{5}{3}+\binom{10}{4}\binom{5}{2}+\binom{10}{5}\binom{5}{1}+\binom{10}{6}\binom{5}{0},$$

which simplifies to 4770.

❺ $\binom{10}{2}\binom{5}{4} + \binom{10}{1}\binom{5}{5} = 235$

❻ *A compatible group will either exclude all these three animals—which can be done in* $\binom{12}{6} = 924$ *ways—or include exactly one of them—which can be done in* $\binom{3}{1}\binom{12}{5} = 2376$. *Thus the total is* $2376 + 924 = 3300$.

◀

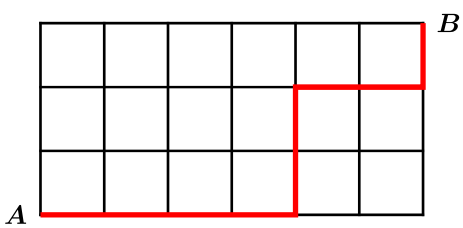

Figure 2.8: Example 89.

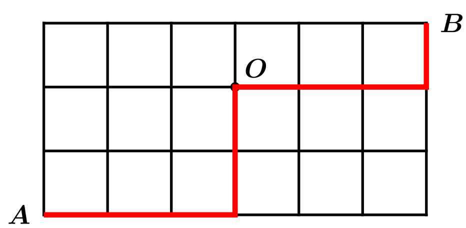

Figure 2.9: Example 90.

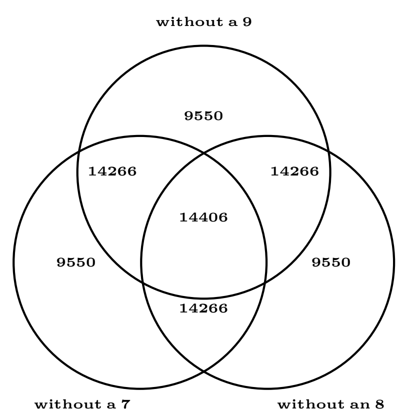

Figure 2.10: Example 91.

89 Example To count the number of shortest routes from A to B in figure 2.8 observe that any shortest path must consist of 6 horizontal moves and 3 vertical ones for a total of $6 + 3 = 9$ moves. Of these 9 moves once we choose the 6 horizontal ones the 3 vertical ones are determined. Thus there are $\binom{9}{6} = 84$ paths.

90 Example To count the number of shortest routes from A to B in figure 2.9 that pass through point O we count the number of paths from A to O (of which there are $\binom{5}{3} = 20$) and the number of paths from O to B (of which there are $\binom{4}{3} = 4$). Thus the desired number of paths is $\binom{5}{3}\binom{4}{3} = (20)(4) = 80$.

91 Example Consider the set of 5-digit positive integers written in decimal notation.

1. How many are there?
2. How many do not have a 9 in their decimal representation?
3. How many have at least one 9 in their decimal representation?
4. How many have exactly one 9?
5. How many have exactly two 9's?
6. How many have exactly three 9's?
7. How many have exactly four 9's?
8. How many have exactly five 9's?
9. How many have neither an 8 nor a 9 in their decimal representation?
10. How many have neither a 7, nor an 8, nor a 9 in their decimal representation?
11. How many have either a 7, an 8, or a 9 in their decimal representation?

►**Solution:**

1. *There are 9 possible choices for the first digit and 10 possible choices for the remaining digits. The number of choices is thus* $9 \cdot 10^4 = 90000$.
2. *There are 8 possible choices for the first digit and 9 possible choices for the remaining digits. The number of choices is thus* $8 \cdot 9^4 = 52488$.
3. *The difference* $90000 - 52488 = 37512$.
4. *We condition on the first digit. If the first digit is a 9 then the other four remaining digits must be different from 9, giving* $9^4 = 6561$ *such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have* $\binom{4}{1} = 4$ *ways of choosing where the 9 will be, and we have* 9^3 *ways of filling the 3 remaining spots. Thus in this case there are* $8 \cdot 4 \cdot 9^3 = 23328$ *such numbers. In total there are* $6561 + 23328 = 29889$ *five-digit positive integers with exactly one 9 in their decimal representation.*
5. *We condition on the first digit. If the first digit is a 9 then one of the remaining four must be a 9, and the choice of place can be accomplished in* $\binom{4}{1} = 4$ *ways. The other three remaining digits must be different from 9, giving* $4 \cdot 9^3 = 2916$ *such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have* $\binom{4}{2} = 6$ *ways of choosing where the two 9's will be, and we have* 9^2 *ways of filling the two remaining spots. Thus in this case there are* $8 \cdot 6 \cdot 9^2 = 3888$ *such numbers. Altogether there are* $2916 + 3888 = 6804$ *five-digit positive integers with exactly two 9's in their decimal representation.*
6. *Again we condition on the first digit. If the first digit is a 9 then two of the remaining four must be 9's, and the choice of place can be accomplished in* $\binom{4}{2} = 6$ *ways. The other two remaining digits must be different from 9, giving* $6 \cdot 9^2 = 486$ *such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have* $\binom{4}{3} = 4$ *ways of choosing where the three 9's will be, and we have 9 ways of filling the remaining spot. Thus in this case there are* $8 \cdot 4 \cdot 9 = 288$ *such numbers. Altogether there are* $486 + 288 = 774$ *five-digit positive integers with exactly three 9's in their decimal representation.*
7. *If the first digit is a 9 then three of the remaining four must be 9's, and the choice of place can be accomplished in* $\binom{4}{3} = 4$ *ways. The other remaining digit must be different from 9, giving* $4 \cdot 9 = 36$ *such numbers. If the first digit is not a 9, then there are 8 choices for this first digit. Also, we have* $\binom{4}{4} = 4$ *ways of choosing where the four 9's will be, thus filling all the spots. Thus in this case there are* $8 \cdot 1 = 8$ *such numbers. Altogether there are* $36 + 8 = 44$ *five-digit positive integers with exactly three 9's in their decimal representation.*
8. *There is obviously only 1 such positive integer.*

   ☞ *Observe that* $37512 = 29889 + 6804 + 774 + 44 + 1$.
9. *We have 7 choices for the first digit and 8 choices for the remaining 4 digits, giving* $7 \cdot 8^4 = 28672$ *such integers.*

10. *We have* 6 *choices for the first digit and* 7 *choices for the remaining* 4 *digits, giving* $6 \cdot 7^4 = 14406$ *such integers.*

11. *We use inclusion-exclusion. From figure 2.10, the numbers inside the circles add up to* 85854*. Thus the desired number is* $90000 - 85854 = 4146$*.*

◀

92 Example Find the number of surjections from $A = \{a, b, c, d\}$ to $B = \{1, 2, 3\}$.

▶**Solution:** *The trick here is that we know how to count the number of functions from one finite set to the other (Theorem 56). What we do is over count the number of functions, and then sieve out those which are not surjective by means of Inclusion-Exclusion. By Theorem 56, there are* $3^4 = 81$ *functions from* A *to* B*. There are* $\binom{3}{1}2^4 = 48$ *functions from* A *to* B *that miss one element from* B*. There are* $\binom{3}{2}1^4 = 3$ *functions from* A *to* B *that miss two elements from* B*. There are* $\binom{3}{0}0^4 = 0$ *functions from* A *to* B *that miss three elements from* B*. By Inclusion-Exclusion there are*

$$81 - 48 + 3 = 36$$

surjective functions from A *to* B*.* ◀

In analogy to example 92, we may prove the following theorem, which complements Theorem 56 by finding the number of surjections from one set to another set.

93 Theorem Let A and B be two finite sets with $\mathbf{card}(A) = n$ and $\mathbf{card}(B) = m$. If $n < m$ then there are no surjections from A to B. If $n \geq m$ then the number of surjective functions from A to B is

$$m^n - \binom{m}{1}(m-1)^n + \binom{m}{2}(m-2)^n - \binom{m}{3}(m-3)^n + \cdots + (-1)^{m-1}\binom{m}{m-1}(1)^n.$$

Homework

Problem 2.6.1 *Verify the following.*

❶ $\binom{20}{3} = 1140$

❷ $\binom{12}{4}\binom{12}{6} = 457380$

❸ $\dfrac{\binom{n}{1}}{\binom{n}{n-1}} = 1$

❹ $\binom{n}{2} = \dfrac{n(n-1)}{2}$

❺ $\binom{6}{1} + \binom{6}{3} + \binom{6}{5} = 2^5$

❻ $\binom{7}{0} + \binom{7}{2} + \binom{7}{4} = 2^6 - \binom{7}{6}$

Problem 2.6.2 *A publisher proposes to issue a set of dictionaries to translate from any one language to any other. If he confines his system to seven languages, how many dictionaries must be published?*

Problem 2.6.3 *From a group of* 12 *people—*7 *of which are men and* 5 *women—in how many ways may choose a committee of* 4 *with* 1 *man and* 3 *women?*

Problem 2.6.4 *N friends meet and shake hands with one another. How many handshakes?*

Problem 2.6.5 *How many 4-letter words can be made by taking 4 letters of the word* **RETICULA** *and permuting them?*

Problem 2.6.6 (AHSME 1989) *Mr. and Mrs. Zeta want to name baby Zeta so that its monogram (first, middle and last initials) will be in alphabetical order with no letters repeated. How many such monograms are possible?*

Problem 2.6.7 *In how many ways can $\{1, 2, 3, 4\}$ be written as the union of two non-empty, disjoint subsets?*

Problem 2.6.8 *How many lists of 3 elements taken from the set $\{1, 2, 3, 4, 5, 6\}$ list the elements in increasing order?*

Problem 2.6.9 *How many times is the digit 3 listed in the numbers 1 to 1000?*

Problem 2.6.10 *How many subsets of the set $\{a, b, c, d, e\}$ have exactly 3 elements?*

Problem 2.6.11 *How many subsets of the set $\{a, b, c, d, e\}$ have an odd number of elements?*

Problem 2.6.12 (AHSME 1994) *Nine chairs in a row are to be occupied by six students and Professors Alpha, Beta and Gamma. These three professors arrive before the six students and decide to choose their chairs so that each professor will be between two students. In how many ways can Professors Alpha, Beta and Gamma choose their chairs?*

Problem 2.6.13 *There are E (different) English novels, F (different) French novels, S (different) Spanish novels, and I (different) Italian novels on a shelf. How many different permutations are there if*

❶ *if there are no restrictions?*

❷ *if all books of the same language must be together?*

❸ *if all the Spanish novels must be together?*

❹ *if no two Spanish novels are adjacent?*

❺ *if all the Spanish novels must be together, and all the English novels must be together, but no Spanish novel is next to an English novel?*

Problem 2.6.14 *How many committees of seven with a given chairman can be selected from twenty people?*

Problem 2.6.15 *How many committees of seven with a given chairman and a given secretary can be selected from twenty people? Assume the chairman and the secretary are different persons.*

Problem 2.6.16 (AHSME 1990) *How many of the numbers*

$$100, 101, \ldots, 999,$$

have three different digits in increasing order or in decreasing order?

Problem 2.6.17 *There are twenty students in a class. In how many ways can the twenty students take five different tests if four of the students are to take each test?*

Problem 2.6.18 *In how many ways can a deck of playing cards be arranged if no two hearts are adjacent?*

Problem 2.6.19 *Given a positive integer n, find the number of quadruples $(a, b, c, d,)$ such that*

$$0 \leq a \leq b \leq c \leq d \leq n.$$

Problem 2.6.20 *There are T books on Theology, L books on Law and W books on Witchcraft on Dr. Faustus' shelf. In how many ways may one order the books*

❶ *there are no constraints in their order?*

❷ *all books of a subject must be together?*

❸ *no two books on Witchcraft are juxtaposed?*

❹ *all the books on Witchcraft must be together?*

Problem 2.6.21 *From a group of 20 students, in how many ways may a professor choose at least one in order to work on a project?*

Problem 2.6.22 *From a group of 20 students, in how many ways may a professor choose an even number number of them, but at least four in order to work on a project?*

Problem 2.6.23 *How many permutations of the word*

CHICHICUILOTE

are there

❶ *if there are no restrictions?*

❷ *if the word must start in an* **I** *and end also in an* **I**?

❸ *if the word must start in an* **I** *and end in a* **C**?

❹ *if the two* **H***'s are adjacent?*

❺ *if the two* **H***'s are not adjacent?*

❻ *if the particle* **LOTE** *must appear, with the letters in this order?*

Problem 2.6.24 *There are M men and W women in a group. A committee of C people will be chosen. In how many ways may one do this if*

❶ *there are no constraints on the sex of the committee members?*

❷ *there must be exactly T women?*

❸ *A committee must always include George and Barbara?*

❹ *A committee must always exclude George and Barbara?*

Assume George and Barbara form part of the original set of people.

Problem 2.6.25 *There are M men and W women in a group. A committee of C people will be chosen. In how many ways may one do this if George and Barbara are feuding and will not work together in a committee? Assume George and Barbara form part of the original set of people.*

Problem 2.6.26 *Out of 30 consecutive integers, in how many ways can three be selected so that their sum be even?*

Problem 2.6.27 *In how many ways may we choose three distinct integers from $\{1, 2, \ldots, 100\}$ so that one of them is the average of the other two?*

Problem 2.6.28 *How many vectors $(a_1, a_2, \ldots, a_k)$ with integral*

$$a_i \in \{1, 2, \ldots, n\}$$

are there satisfying

$$1 \leq a_1 \leq a_2 \leq \cdots \leq a_k \leq n?$$

Problem 2.6.29 *A square chessboard has 16 squares (4 rows and 4 columns). One puts 4 checkers in such a way that only one checker can be put in a square. Determine the number of ways of putting these checkers if*

❶ *there must be exactly one checker per row and column.*

❷ *there must be exactly one column without a checker.*

❸ *there must be at least one column without a checker.*

Problem 2.6.30 *A box contains 4 red, 5 white, 6 blue, and 7 magenta balls. In how many of all possible samples of size 5, chosen without replacement, will every colour be represented?*

Problem 2.6.31 *In how many ways can eight students be divided into four indistinguishable teams of two each?*

Problem 2.6.32 *How many ways can three boys share fifteen different sized pears if the youngest gets seven pears and the other two boys get four each?those in which the digit 1 occurs or those in which it does not occur?*

Problem 2.6.33 *Four writers must write a book containing seventeen chapters. The first and third writers must each write five chapters, the second must write four chapters, and the fourth must write three chapters. How many ways can the book be divided between the authors? What if the first and third had to write ten chapters combined, but it did not matter which of them wrote how many (i.e. the first could write ten and the third none, the first could write none and the third one, etc.)?*

Problem 2.6.34 *In how many ways can a woman choose three lovers or more from seven eligible suitors?*

Problem 2.6.35 (AIME 1988) *One commercially available ten-button lock may be opened by depressing—in any order—the correct five buttons. Suppose that these locks are redesigned so that sets of as many as nine buttons or as few as one button could serve as combinations. How many additional combinations would this allow?*

Problem 2.6.36 *From a set of $n \geq 3$ points on the plane, no three collinear,*

❶ *how many straight lines are determined?*

❷ *how many straight lines pass through a particular point?*

❸ *how many triangles are determined?*

❹ *how many triangles have a particular point as a vertex?*

Problem 2.6.37 *In how many ways can you pack twelve books into four parcels if one parcel has one book, another has five books, and another has two books, and another has four books?*

Problem 2.6.38 *In how many ways can a person invite three of his six friends to lunch every day for twenty days if he has the option of inviting the same or different friends from previous days?*

Problem 2.6.39 *A committee is to be chosen from a set of nine women and five men. How many ways are there to form the committee if the committee has three men and three women?*

Problem 2.6.40 *At a dance there are b boys and g girls. In how many ways can they form c couples consisting of different sexes?*

Problem 2.6.41 *From three Russians, four Americans, and two Spaniards, how many selections of people can be made, taking at least one of each kind?*

Problem 2.6.42 *The positive integer r satisfies*

$$\frac{1}{\binom{9}{r}} - \frac{1}{\binom{10}{r}} = \frac{11}{6\binom{11}{r}}.$$

Find r.

Problem 2.6.43 *If* $11\binom{28}{2r} = 225\binom{24}{2r-4}$, *find* r.

Problem 2.6.44 *Compute the number of ten-digit numbers which contain only the digits* $1, 2,$ *and* 3 *with the digit* 2 *appearing in each number exactly twice.*

Problem 2.6.45 *Prove* Pascal's Identity*:*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k},$$

for integers $1 \leq k \leq n$.

Problem 2.6.46 *Give a combinatorial interpretation of* **Newton's Identity***:*

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k} \tag{2.1}$$

for $0 \leq k \leq r \leq n$.

Problem 2.6.47 *Give a combinatorial proof that for integer* $n \geq 1$,

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2.$$

Problem 2.6.48 *Give a combinatorial proof of* Vandermonde's Convolution Identity:

$$\sum_{k=0}^{n}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

for positive integers $r, s \geq n$.

Problem 2.6.49 *In each of the* 6*-digit numbers*

$$333333, 225522, 118818, 707099,$$

each digit in the number appears at least twice. Find the number of such 6*-digit natural numbers.*

Problem 2.6.50 *In each of the* 7*-digit numbers*

$$1001011, 5550000, 3838383, 7777777,$$

each digit in the number appears at least thrice. Find the number of such 7*-digit natural numbers.*

Problem 2.6.51 (AIME 1983) *The numbers* $1447, 1005$ *and* 1231 *have something in common: each is a four-digit number beginning with* 1 *that has exactly two identical digits. How many such numbers are there?*

Problem 2.6.52 *If there are fifteen players on a baseball team, how many ways can the coach choose nine players for the starting lineup if it does not matter which position the players play (i.e., no distinction is made between player A playing shortstop, left field, or any other positions as long as he is on the field)? How many ways are there if it does matter which position the players play?*

Problem 2.6.53 (AHSME 1989) *A child has a set of* 96 *distinct blocks. Each block is one of two materials (*plastic, wood*), three sizes (*small, medium, large*), four colours (*blue, green, red, yellow*), and four shapes (*circle, hexagon, square, triangle*). How many blocks in the set are different from the "*plastic medium red circle*" in exactly two ways? (The "*wood medium red square*" is such a block.)*

Problem 2.6.54 *There are four different kinds of sweets at a sweets store. I want to buy up to four sweets (I'm not sure if I want none, one, two, three, or four sweets) and I refuse to buy more than one of any kind of sweet. How many ways can I do this?*

Problem 2.6.55 *Suppose five people are in a lift. There are eight floors that the lift stops at. How many distinct ways can the people exit the lift if either one or zero people exit at each stop?*

Problem 2.6.56 *If the natural numbers from* 1 *to* 222222222 *are written down in succession, how many* 0*'s are written?*

Problem 2.6.57 *In how many ways can we distribute* k *identical balls into* n *different boxes so that each box contains at most one ball and no two consecutive boxes are empty?*

Problem 2.6.58 *In a row of* n *seats in the doctor's waiting-room* k *patients sit down in a particular order from left to right. They sit so that no two of them are in adjacent seats. In how many ways could a suitable set of* k *seats be chosen?*

Problem 2.6.59 (Derangements) *Ten different letters are taken from their envelopes, read, and then randomly replaced in the envelopes. In how many ways can this replacing be done so that none of the letters will be in the correct envelope?*

2.7 Combinations with Repetitions

94 Theorem (De Moivre) Let n be a positive integer. The number of positive integer solutions to

$$x_1 + x_2 + \cdots + x_r = n$$

is

$$\binom{n-1}{r-1}.$$

Proof: *Write n as*

$$n = 1 + 1 + \cdots + 1 + 1,$$

where there are n 1s and $n-1$ +s. To decompose n in r summands we only need to choose $r-1$ pluses from the $n-1$, which proves the theorem. ❑

95 Example In how many ways may we write the number 9 as the sum of three positive integer summands? Here order counts, so, for example, $1+7+1$ is to be regarded different from $7+1+1$.

▶**Solution:** *Notice that this is example 76. We are seeking integral solutions to*

$$a + b + c = 9, \quad a > 0, b > 0, c > 0.$$

By Theorem 94 this is

$$\binom{9-1}{3-1} = \binom{8}{2} = 28.$$

◀

96 Example In how many ways can 100 be written as the sum of four positive integer summands?

▶**Solution:** *We want the number of positive integer solutions to*

$$a + b + c + d = 100,$$

which by Theorem 94 is

$$\binom{99}{3} = 156849.$$

◀

97 Corollary Let n be a positive integer. The number of non-negative integer solutions to

$$y_1 + y_2 + \cdots + y_r = n$$

is

$$\binom{n+r-1}{r-1}.$$

Proof: *Put $x_r - 1 = y_r$. Then $x_r \geq 1$. The equation*

$$x_1 - 1 + x_2 - 1 + \cdots + x_r - 1 = n$$

is equivalent to

$$x_1 + x_2 + \cdots + x_r = n + r,$$

which from Theorem 94, has

$$\binom{n+r-1}{r-1}$$

solutions. ❑

98 Example Find the number of quadruples (a, b, c, d) of integers satisfying

$$a + b + c + d = 100, \; a \geq 30, b > 21, c \geq 1, d \geq 1.$$

▶**Solution:** *Put $a' + 29 = a, b' + 20 = b$. Then we want the number of positive integer solutions to*

$$a' + 29 + b' + 21 + c + d = 100,$$

or

$$a' + b' + c + d = 50.$$

By Theorem 94 this number is

$$\binom{49}{3} = 18424.$$

◀

99 Example There are five people in a lift of a building having eight floors. In how many ways can they choose their floor for exiting the lift?

▶**Solution:** *Let x_i be the number of people that floor i receives. We are looking for non-negative solutions of the equation*

$$x_1 + x_2 + \cdots + x_8 = 5.$$

Putting $y_i = x_i + 1$, then

$$\begin{aligned} x_1 + x_2 + \cdots + x_8 = 5 \quad &\Longrightarrow \quad (y_1 - 1) + (y_2 - 1) + \cdots + (y_8 - 1) = 5 \\ &\Longrightarrow \quad y_1 + y_2 + \cdots + y_8 = 13, \end{aligned}$$

whence the number sought is the number of positive solutions to

$$y_1 + y_2 + \cdots + y_8 = 13$$

which is $\binom{12}{7} = 792$. ◀

100 Example Find the number of quadruples (a, b, c, d) of non-negative integers which satisfy the inequality

$$a + b + c + d \leq 2001.$$

▶**Solution:** *The number of non-negative solutions to*

$$a + b + c + d \leq 2001$$

equals the number of solutions to

$$a + b + c + d + f = 2001$$

where f is a non-negative integer. This number is the same as the number of positive integer solutions to

$$a_1 - 1 + b_1 - 1 + c_1 - 1 + d_1 - 1 + f_1 - 1 = 2001,$$

which is easily seen to be $\binom{2005}{4}$. ◀

101 Example

How many integral solutions to the equation

$$a + b + c + d = 100,$$

are there given the following constraints:

$$1 \le a \le 10, \qquad b \ge 0, \qquad c \ge 2, \qquad 20 \le d \le 30?$$

►**Solution:** *We use Inclusion-Exclusion. There are* $\binom{80}{3} = 82160$ *integral solutions to*

$$a + b + c + d = 100, \qquad a \ge 1, \qquad b \ge 0, \qquad c \ge 2, \qquad d \ge 20.$$

Let A be the set of solutions with

$$a \ge 11, \qquad b \ge 0, \qquad c \ge 2, \qquad d \ge 20$$

and B be the set of solutions with

$$a \ge 1, \qquad b \ge 0, \qquad c \ge 2, \qquad d \ge 31.$$

Then $\text{card}\,(A) = \binom{70}{3}$, $\text{card}\,(B) = \binom{69}{3}$, $\text{card}\,(A \cap B) = \binom{59}{3}$ *and so*

$$\text{card}\,(A \cup B) = \binom{70}{3} + \binom{69}{3} - \binom{59}{3} = 74625.$$

The total number of solutions to

$$a + b + c + d = 100$$

with

$$1 \le a \le 10, \qquad b \ge 0, \qquad c \ge 2, \qquad 20 \le d \le 30$$

is thus

$$\binom{80}{3} - \binom{70}{3} - \binom{69}{3} + \binom{59}{3} = 7535.$$

◄

Homework

Problem 2.7.1 *How many positive integral solutions are there to*

$$a + b + c = 10?$$

Problem 2.7.2 *Three fair dice, one red, one white, and one blue are thrown. In how many ways can they land so that their sum be* 10 *?*

Problem 2.7.3 *Adena has twenty indistinguishable pieces of sweet-meats that she wants to divide amongst her five stepchildren. How many ways can she divide the sweet-meats so that each stepchild gets at least two pieces of sweet-meats?*

Problem 2.7.4 *How many integral solutions are there to the equation*

$$x_1 + x_2 + \cdots + x_{100} = n$$

subject to the constraints

$$x_1 \ge 1, x_2 \ge 2, x_3 \ge 3, \ldots, x_{99} \ge 99, x_{100} \ge 100?$$

Problem 2.7.5 (AIME 1998) *Find the number of ordered quadruplets (a, b, c, d) of positive odd integers satisfying $a + b + c + d = 98$.*

Problem 2.7.6 *A* lattice point *is a coordinate point (x, y) with both x and y integers. How many lattice points (x, y) satisfy $|x| + |y| < 100$?*

2.8 Binomial Theorem

102 Theorem (Binomial Theorem) For $n \in \mathbb{Z}, n \geq 0$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

Proof: *Observe that expanding*

$$\underbrace{(x+y)(x+y)\cdots(x+y)}_{n \text{ times}}$$

consists of adding up all the terms obtained from multiplying either an x or a y from the first set of parentheses times either an x or a y from the second set of parentheses etc. To get x^k, x must be chosen from exactly k of the sets of parentheses. Thus the number of x^k terms is $\binom{n}{k}$. It follows that

$$\begin{aligned}(x+y)^n &= \binom{n}{0}x^0y^n + \binom{n}{1}xy^{n-1} + \binom{n}{2}x^2y^{n-2} + \cdots + \binom{n}{n}x^ny^0 \\ &= \sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}.\end{aligned} \tag{2.2}$$

❑

☞ *By setting $x = y = 1$ in 2.2 we obtain*

$$\boxed{2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n},} \tag{2.3}$$

which is another proof that the number of subsets of a set with n elements is 2^n, since the dextral side counts how many subsets there are with $0, 1, 2, \ldots, n$ elements, respectively.

103 Example Expand $(2-x)^5$.

►**Solution:** *By the Binomial Theorem,*

$$(2-x)^5 = \sum_{k=0}^{5} 2^{5-k}(-x)^k \binom{5}{k} = 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5.$$

◄

Here is another proof of Theorem 55.

104 Theorem Let $n \in \mathbb{N}$. If A is a finite set with n elements, then the power set of A has 2^n different elements, i.e., A has 2^n different subsets.

Proof: *A has exactly* $1 = \binom{n}{0}$ *subset with* 0 *elements, exactly* $n = \binom{n}{1}$ *subsets with* 1 *elements,. . . , and exactly* $1 = \binom{n}{n}$ *subset with* n *elements. By the Binomial Theorem,*

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n.$$

❑

105 Example (AIME 1989) Ten points are marked on a circle. How many distinct convex polygons of three or more sides can be drawn using some (or all) of the ten points as vertices? (Polygons are distinct unless they have exactly the same vertices.)

▶**Solution:** *Choosing* k *points* $3 \leq k \leq 10$ *points will determine a* k*-sided polygon, since the polygons are convex and thus have no folds. The answer is thus*

$$\sum_{k=3}^{10} \binom{10}{k} = 2^{10} - \binom{10}{0} - \binom{10}{1} - \binom{10}{2} = 1024 - 1 - 10 - 45 = 968.$$

◀

106 Example Simplify

$$\sum_{k=1}^{10} 2^k \binom{11}{k}.$$

▶**Solution:** *By the Binomial Theorem, the complete sum* $\sum_{k=0}^{11} \binom{11}{k} 2^k = 3^{11}$*. The required sum lacks the zeroth term,* $\binom{11}{0} 2^0 = 1$*, and the eleventh term,* $\binom{11}{11} 2^{11}$ *from this complete sum. The required sum is thus* $3^{11} - 2^{11} - 1$*.* ◀

107 Example Find the coefficient of x^{12} in the expansion of

$$(x^2 + 2x)^{10}.$$

▶**Solution:** *We have*

$$(x^2 + 2x)^{10} = \sum_{k=0}^{10} \binom{10}{k} (x^2)^k (2x)^{10-k} = \sum_{k=0}^{10} \binom{10}{k} 2^{10-k} x^{k+10}.$$

To obtain x^{12} *we need* $k = 2$*. Hence the coefficient sought is* $\binom{10}{2} 2^8 = 11520$ ◀

We will now derive some identities for later use.

108 Lemma

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}.$$

Proof:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n}{k} \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n}{k} \binom{n-1}{k-1}.$$

❑

109 Lemma

$$\binom{n}{k} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \binom{n-2}{k-2}.$$

Proof:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)}{k(k-1)} \cdot \frac{(n-2)!}{(k-2)!(n-k)!} = \frac{n}{k} \cdot \frac{n-1}{k-1} \cdot \binom{n-2}{k-2}.$$

❑

110 Theorem

$$\sum_{k=1}^{n} k\binom{n}{k} p^k(1-p)^{n-k} = np.$$

Proof: *We use the identity* $k\binom{n}{k} = n\binom{n-1}{k-1}$. *Then*

$$\begin{aligned}
\sum_{k=1}^{n} k\binom{n}{k} p^k(1-p)^{n-k} &= \sum_{k=1}^{n} n\binom{n-1}{k-1} p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n-1} n\binom{n-1}{k} p^{k+1}(1-p)^{n-1-k} \\
&= np\sum_{k=0}^{n-1} \binom{n-1}{k} p^k(1-p)^{n-1-k} \\
&= np(p+1-p)^{n-1} \\
&= np.
\end{aligned}$$

❑

111 Lemma

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} p^k(1-p)^{n-k} = n(n-1)p^2.$$

Proof: *We use the identity*

$$k(k-1)\binom{n}{k} = n(n-1)\binom{n-2}{k-2}.$$

Then

$$\begin{aligned}
\sum_{k=2}^{n} k(k-1)\binom{n}{k} p^k(1-p)^{n-k} &= \sum_{k=2}^{n} n(n-1)\binom{n-2}{k-2} p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n-2} n(n-1)\binom{n-2}{k} p^{k+2}(1-p)^{n-1-k} \\
&= n(n-1)p^2\sum_{k=0}^{n-2} \binom{n-1}{k} p^k(1-p)^{n-2-k} \\
&= n(n-1)p^2(p+1-p)^{n-2} \\
&= n(n-1)p^2.
\end{aligned}$$

❑

112 Theorem

$$\sum_{k=0}^{n}(k-np)^2\binom{n}{k}p^k(1-p)^{n-k} = np(1-p).$$

Proof: *We use the identity*

$$(k-np)^2 = k^2 - 2knp + n^2p^2 = k(k-1) + k(1-2np) + n^2p^2.$$

Then

$$\begin{aligned}
\sum_{k=0}^{n}(k-np)^2\binom{n}{k}p^k(1-p)^{n-k} &= \sum_{k=0}^{n}(k(k-1)+k(1-2np) \\
&\quad +n^2p^2)\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n}k(k-1)\binom{n}{k}p^k(1-p)^{n-k} \\
&\quad +(1-2np)\sum_{k=0}^{n}k\binom{n}{k}p^k(1-p)^{n-k} \\
&\quad +n^2p^2\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k} \\
&= n(n-1)p^2 + np(1-2np) + n^2p^2 \\
&= np(1-p).
\end{aligned}$$

❑

Homework

Problem 2.8.1 *Expand* $(a-2b)^5$.

Problem 2.8.2 *Expand* $(2a+3b)^4$.

Problem 2.8.3 *By alternately putting* $x = 1$ *and* $x = -1$ *in* 2.2 *and adding and subtracting the corresponding quantities, deduce the identities*

$$2^{n-1} = \binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots,$$

$$2^{n-1} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots,$$

3 Probability Axioms

3.1 Some History

Throughout time Man has wondered about experiences whose outcomes are uncertain. Games of chance are perhaps as old as civilisation. The ancient Greek historian Thucydides (c. 465-395 BC) utilises the idea of *chance* in his narrative, claiming that "it is impossible to calculate with precision the events that are the product of chance." For Aristotle (384-322 BC), the "probability" of an "event" is manifest in the "relative frequency of the event." In the Bible (Acts 1:23-1:26), the Apostles cast lots (a game of chance) in order to choose a substitute for Judas:

> And they appointed two, Joseph called Barsabas, who was surnamed Justus, and Matthias. And they prayed, and said, Thou, Lord, which knowest the hearts of all [men,] shew whether of these two thou hast chosen, That he may take part of this ministry and apostleship, from which Judas by transgression fell, that he might go to his own place. And they gave forth their lots; and the lot fell upon Matthias; and he was numbered with the eleven apostles.

Mathematical probability is really born in the XVI century with Girolamo Cardano (1501-1576), who first tried to attach a mathematical significance to winning or losing in games of chance in his *Liber de Ludo Alae*. Cardano had a very colourful life and was somewhat of a crackpot. He got in trouble with the Inquisition for writing a horoscope for Christ.

Most of the combinatorial ideas that we learned in the preceding chapter were born out of the correspondence in the XVII century between the great French mathematicians Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662). Pascal was a child prodigy, producing most of his mathematical output during his teen years. He later became a religious fanatic and quit Mathematics altogether, except for a one night exception: He was kept awake by a terrible toothache. In order to take his mind away from the toothache he worked on a mathematical problem. A while later the toothache was gone, a sign that Pascal took as the divine approval of Mathematics.

In the XVIII century, Jacques Bernoulli (654- 1705) applies probabilistic methods to social phenomena and proves the law of large numbers. Abraham De Moivre (1667-1754) refines his predecessors ideas and gives a better mathematical formulation of the idea of probability.

In the XIX century Pierre Simon, Marquis of Laplace (1749-1827) and Karl Friedrich Gauss (1777-1856) introduced and demonstrated the practical value of the normal curve. The Reverend Thomas Bayes (c. 1702-1761) discovered his theorem on a posteriori probability.

In the XX century probability is put on a firm axiomatic base by Andrei Kolmogorov (1903-1987), whose axioms we will study in the next section.

3.2 Probability Spaces

We would like now to formulate the notion of *probability*. We would like this notion to correspond to whatever intuition we may have of what a probability should be.

Let us consider some examples.

113 Example What is meant when a meteorologist announces that there is 20% probability of rain on 1 April 2007? Does it mean that, say, in the last 100 years, there has been rain twenty times on 1 April 2007? Does it mean that during the last few months, of every ten days there have been two days with rain?

114 Example Suppose that test for a disease gives false positives 90% of the time and false negatives 90% of the time. Suppose moreover, that 15% of the population has this disease and that there is an evil employer that wants to keep out of his company people afflicted with this disease. Suppose that your results on the test are positive? Is it fair for the employer to argue that you have 90% chance of having the disease and hence he should not hire you?

115 Example Suppose we flip a coin a large number of times. If the coin is somehow "fair" we would expect it to shew heads half of the time and tails half of the time. Thus we would like to define the probability of obtaining heads, which we will denote by $\mathbf{P}(H)$ to be $\mathbf{P}(H) = \frac{1}{2}$, and similarly we would like the probability of obtaining tails to be $\mathbf{P}(T) = \frac{1}{2}$. Now, we would expect only these two outcomes to be possible: there should be no way for our coin to land "standing up." We have thus defined a sample space $S = \{H, T\}$ with $\mathbf{P}(H) + \mathbf{P}(T) = 1$.

We now formalise what is meant by *probability*.

116 Definition A *probability* $\mathbf{P}()$ is a real valued function defined on subsets of a sample space Ω and satisfying the following axioms, called the *Kolmogorov Axioms*:

❶ $0 \leq \mathbf{P}(A) \leq 1$ for $A \subseteq \Omega$,

❷ $\mathbf{P}(\Omega) = 1$,

❸ for a finite or countably infinite sequence $A_1, A_2, \ldots \subseteq \Omega$ of disjoint events,

$$\mathbf{P}\left(\bigcup_{i=1}^{+\infty} A_i\right) = \sum_{i=1}^{+\infty} \mathbf{P}(A_i).$$

The number $\mathbf{P}(A)$ is called the *probability* of event A.

In words: a probability is a number between zero and one, the probability of the event that the sample space will occur is always one, and if a union can be decomposed into disjoint sets, then the probability of this union is the sum over the probabilities of the disjoint sets. This, of course, does not tell you anything meaningful in terms of clarifying your intuition of what probability is. But let us consider more examples.

☞ *If $A \subseteq \Omega$ is an event with only one outcome a, that is, $A = \{a\}$, the following will all mean the same:*

$$\mathbf{P}(A) = \mathbf{P}(\{a\}) = \mathbf{P}(a).$$

This last notation will be preferred for typographical convenience.

Axioms (2) and (3) are sometimes used as follows. Suppose that

$$\Omega = \{x_1, x_2, \ldots, x_n\}$$

is a sample space with a finite number of outcomes x_k. Then by Axioms (2) and (3),

$$\mathbf{P}(x_1) + \mathbf{P}(x_2) + \cdots + \mathbf{P}(x_n) = 1,$$

that is, the sum of the probabilities over all the outcomes of the sample space is always 1.

117 Example An Admissions Office of a large Midwestern university has an admission formula that classifies all applicants into three mutually exclusive groups, I, J, or K. This formula gives 10% preference to people in pool I over people in pool J, and 20% preference to people in pool J over people in pool K. What are the respective probabilities for the people belonging to a particular pool to be admitted?

►**Solution:** *By the Axioms (2) and (3) of the definition of probability,*

$$\mathbf{P}(I) + \mathbf{P}(J) + \mathbf{P}(K) = 1.$$

The data of the problem is

$$\mathbf{P}(I) = 1.1\mathbf{P}(J), \quad \mathbf{P}(J) = 1.2\mathbf{P}(K).$$

The trick is to express each probability in terms of just one. Let us express all probabilities in terms of K's probability. Hence we deduce

$$\mathbf{P}(I) = 1.1\mathbf{P}(J) = 1.1(1.2\mathbf{P}(K)) = 1.32\mathbf{P}(K).$$

Therefore,

$$\mathbf{P}(I) + \mathbf{P}(J) + \mathbf{P}(K) = 1 \implies 1.32\mathbf{P}(K) + 1.2\mathbf{P}(K) + \mathbf{P}(K) = 1 \implies \mathbf{P}(K) = \frac{1}{3.52} = \frac{25}{88},$$

and so,

$$\mathbf{P}(J) = 1.2\mathbf{P}(K) = \frac{6}{5} \cdot \frac{25}{88} = \frac{15}{44}, \qquad \mathbf{P}(I) = 1.32\mathbf{P}(K) = \frac{33}{25} \cdot \frac{25}{88} = \frac{3}{8}.$$

◄

☞ *Notice that in the preceding problem one indeed has*

$$\mathbf{P}(I) + \mathbf{P}(J) + \mathbf{P}(K) = \frac{3}{8} + \frac{15}{44} + \frac{25}{88} = \frac{33}{88} + \frac{30}{88} + \frac{25}{88} = 1.$$

We will now deduce some results that will facilitate the calculation of probabilities in the future.

118 Theorem Let $Y \subseteq X$ belong to the same sample space Ω. Then $\mathbf{P}(X \setminus Y) = \mathbf{P}(X) - \mathbf{P}(Y)$.

Proof: *Clearly $X = Y \cup (X \setminus Y)$, and $Y \cap (X \setminus Y) = \varnothing$. Thus by Axiom (3) of the definition of probability,*

$$\mathbf{P}(X) = \mathbf{P}(Y) + \mathbf{P}(X \setminus Y) \implies \mathbf{P}(X) - \mathbf{P}(Y) = \mathbf{P}(X \setminus Y).$$

❑

119 Corollary (Complementary Event Rule) Let A be an event. Then

$$\mathbf{P}(A^c) = 1 - \mathbf{P}(A).$$

Proof: *Since $\mathbf{P}(\Omega) = 1$, it is enough to take $X = \Omega, Y = A, X \setminus Y = A^c$ in the preceding theorem.* ❑

120 Corollary $\mathbf{P}(\varnothing) = 0$.

Proof: *Take $A = \varnothing$, $A^c = \Omega$ in the preceding corollary.* ❑

121 Theorem (Probabilistic two-set Inclusion-Exclusion) Let A, B be events. Then

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B).$$

Proof: *Observe that*

$$A \cup B = (A \setminus (A \cap B)) \cup (B \setminus (A \cap B)) \cup (A \cap B),$$

is a decomposition of $A \cup B$ *into three disjoint sets. Thus by Axiom (3) of the definition of probability,*

$$\mathbf{P}(A \cup B) = \mathbf{P}(A \setminus (A \cap B)) + \mathbf{P}(B \setminus (A \cap B)) + \mathbf{P}(A \cap B).$$

Since by Theorem 118 we have $\mathbf{P}(A \setminus (A \cap B)) = \mathbf{P}(A) - \mathbf{P}(A \cap B)$ *and* $\mathbf{P}(B \setminus (A \cap B)) = \mathbf{P}(B) - \mathbf{P}(A \cap B)$, *we deduce that*

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) - \mathbf{P}(A \cap B) + \mathbf{P}(B) - \mathbf{P}(A \cap B) + \mathbf{P}(A \cap B),$$

from where the result follows. ❑

122 Example There are two telephone lines A and B. Let E_1 be the event that line A is engaged and let E_2 be the event that line B is engaged. After a statistical study one finds that $\mathbf{P}(E_1) = 0.5$, that $\mathbf{P}(E_2) = 0.6$ and that $\mathbf{P}(E_1 \cap E_2) = 0.3$. Find the probability of the following events:

❶ F: "line B is free."

❷ G: "at least one line is engaged."

❸ H: "both lines are free."

►**Solution:** *The event that line A is free is* E_1^c, *similarly,* E_2^c *is the event that line B is free.*

❶ *Observe that* $F = E_2^c$ *and hence* $\mathbf{P}(F) = \mathbf{P}\left(E_2^c\right) = 1 - \mathbf{P}(E_2) = 1 - 0.6 = 0.4.$

❷ *Observe that the event that both lines are free is* $E_1^c \cap E_2^c$ *and hence* $G = (E_1^c \cap E_2^c)^c = (E_1^c)^c \cup (E_2^c)^c = E_1 \cup E_2$ *using the De Morgan Laws. Hence, by Inclusion-Exclusion,*

$$\begin{aligned}
\mathbf{P}(G) &= \mathbf{P}(E_1 \cup E_2) \\
&= \mathbf{P}(E_1) + \mathbf{P}(E_2) - \mathbf{P}(E_1 \cap E_2) \\
&= 0.5 + 0.6 - 0.3 \\
&= 0.8.
\end{aligned}$$

❸ *We need* $E_1^c \cap E_2^c$. *Observe that by the De Morgan Laws,*

$$\mathbf{P}\left(E_1^c \cup E_2^c\right) = \mathbf{P}\left((E_1 \cap E_2)^c\right) = 1 - \mathbf{P}(E_1 \cap E_2) = 1 - 0.3 = 0.7,$$

hence by Inclusion-Exclusion,

$$\begin{aligned}
\mathbf{P}(H) &= \mathbf{P}\left(E_1^c \cap E_2^c\right) \\
&= \mathbf{P}\left(E_1^c\right) + \mathbf{P}\left(E_2^c\right) - \mathbf{P}\left(E_1^c \cup E_2^c\right) \\
&= 0.5 + 0.4 - 0.7 \\
&= 0.2.
\end{aligned}$$

◄

Homework

Problem 3.2.1 *Let* $S = \{a, b, c, d\}$ *be a sample space with* a, b, c, d *being different outcomes. Outcome* a *is* 2 *times as likely as outcome* b*; outcome* b *is* 4 *times as likely as outcome* c*; outcome* c *is* 2 *times as likely as outcome* d*. Find*

$$\mathbf{P}(a), \mathbf{P}(b), \mathbf{P}(c), \mathbf{P}(d).$$

Problem 3.2.2 *Let* $S = \{a, b, c, d\}$ *be a sample space with* $\mathbf{P}(a) = 3\mathbf{P}(b)$, $\mathbf{P}(b) = 3\mathbf{P}(c)$, $\mathbf{P}(c) = 3\mathbf{P}(d)$. *Find the numerical value of* $\mathbf{P}(a)$, $\mathbf{P}(b)$, $\mathbf{P}(c)$, *and* $\mathbf{P}(d)$.

Problem 3.2.3 *Let* $\mathbf{P}(A) = 0.8$, $\mathbf{P}(B) = 0.5$ *and* $\mathbf{P}(A \cap B) = 0.4$. *Find* $\mathbf{P}(A^c \cap B^c)$ *and* $\mathbf{P}(A^c \cup B^c)$.

Problem 3.2.4 *Let* $\mathbf{P}(A) = 0.9$, $\mathbf{P}(B) = 0.6$. *Find the maximum and minimum possible values for* $\mathbf{P}(A \cap B)$.

Problem 3.2.5 *Among a large group of patients recovering from shoulder injuries, it is found that* 22% *visit both a physical therapist and a chiropractor, whereas* 12% *visit neither of these. The probability that a patient visits a chiropractor exceeds by* 0.14 *the probability that a patient visits a physical therapist. Determine the probability that a randomly chosen member of this group visits a physical therapist.*

Problem 3.2.6 *Let* $\mathbf{P}(A \cap B) = 0.2$, $\mathbf{P}(A) = 0.6$, $\mathbf{P}(B) = 0.5$. *Find* $\mathbf{P}(A^c \cup B^c)$.

Problem 3.2.7 *In a horse race, the odds in favour of* Rocinante *winning in an* 8*-horse race are* $2 : 5$. *The odds against* Babieca *winning are* $7 : 3$. *What is the probability that either Rocinante or Babieca will win this race?*

Problem 3.2.8 (Probabilistic three-set Inclusion-Exclusion) *Let* A_1, A_2, A_3 *be three events belonging to the same sample space* Ω. *Prove that*

$$\begin{aligned}\mathbf{P}(A_1 \cup A_2 \cup A_3) &= \mathbf{P}(A_1) + \mathbf{P}(A_2) + \mathbf{P}(A_3) \\ &\quad -\mathbf{P}(A_1 \cap A_2) \\ &\quad -\mathbf{P}(A_2 \cap A_3) \\ &\quad -\mathbf{P}(A_3 \cap A_1) \\ &\quad +\mathbf{P}(A_1 \cap A_2 \cap A_3).\end{aligned}$$

3.3 Random Variables

123 Definition A *random variable* $\mathbf{X}$ is a function that to each outcome point of the sample space (the inputs) assigns a real number output. This output is not fixed, but assigned with a certain probability. The *distribution function* $F_{\mathbf{X}}$ of the random variable $\mathbf{X}$ is defined for all $x \in \mathbb{R}$ by

$$F_{\mathbf{X}}(x) = \mathbf{P}(\mathbf{X} \leq x).$$

The *range* or *image* of $\mathbf{X}$ is the set of outputs assumed by $\mathbf{X}$.

124 Definition A random variable is said to be *discrete* if the cardinality of its image is either finite or countably infinite. The function $x \mapsto \mathbf{P}(\mathbf{X} = x)$ its call the *probability mass function* of $\mathbf{X}$.

If $\mathbf{X}$ is discrete and if its image is the countable set

$$\{x_1, x_2, x_3, \ldots,\}$$

then Axioms (2) and (3) of the definition of probability give

$$\sum_{k=1}^{+\infty} \mathbf{P}(\mathbf{X} = x_k) = 1. \tag{3.1}$$

125 Example A fair die is tossed. If the resulting number is even, you add 1 to your score and get that many dollars. If the resulting number is odd, you add 2 to your score and get that many dollars. Let $\mathbf{X}$ be the random variable counting your gain, in dollars. Then the range of $\mathbf{X}$ is $\{3, 5, 7\}$. By (3.1) we must have

$$\mathbf{P}(\mathbf{X} = 3) + \mathbf{P}(\mathbf{X} = 5) + \mathbf{P}(\mathbf{X} = 7) = 1.$$

126 Example A hand of three cards is chosen from a standard deck of cards. You get \$3 for each heart in your hand. Let Z be the random variable measuring your gain. Then the range of Z is $\{0, 3, 6, 9\}$. By (3.1) we must have

$$P(Z = 0) + P(Z = 3) + P(Z = 6) + P(Z = 9) = 1.$$

127 Definition Let X be a discrete random variable with range $\{x_1, x_2, \ldots, \}$. A *histogram* of X is a bar chart of X against $P(X)$.

We will usually situate the centre of the base of the j-th bar at $(x_j, 0)$. The height of the j-th bar is $P(X = x_j)$.

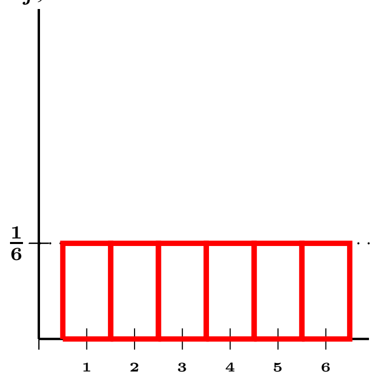

Figure 3.1: Histogram for example 128.

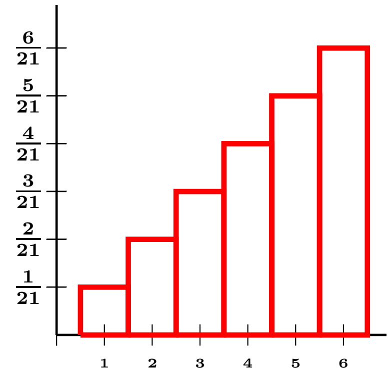

Figure 3.2: Histogram for example 129.

128 Example Consider a fair ordinary die. If X is the random variable counting the number of dots, then $P(X = k) = \frac{1}{6}$, for $k = 1, 2, \ldots, 6$. Observe that

$$P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) = 1$$

since

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1.$$

A histogram for X is given in figure 3.1.

129 Example The six faces of a die are numbered $1, 2, 3, 4, 5, 6$, but the die is loaded so that the the probability of obtaining a given number is proportional to the number of the dots. If X is the random variable counting the number of dots, find $P(X = k)$ for $k = 1, 2, \ldots, 6$.

►**Solution:** *The hypothesis implies that there is a constant α such that $P(X = k) = \alpha k$ for $1 \leq k \leq 6$. Then*

$$1 = P(X = 1) + \cdots + P(X = 6) = \alpha(1 + \cdots + 6) = 21\alpha$$

giving $\alpha = \frac{1}{21}$ and $P(X = k) = \frac{k}{21}$. Observe that

$$P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) = 1$$

since

$$\frac{1}{21} + \frac{2}{21} + \frac{3}{21} + \frac{4}{21} + \frac{5}{21} + \frac{6}{21} = 1.$$

A histogram for X is given in figure 3.2. ◄

130 Example Two fair dice, a red and a blue die, are tossed at random and their score added. Let S be the random variable of the sum of the dots displayed. Determine its probability mass function and draw its histogram.

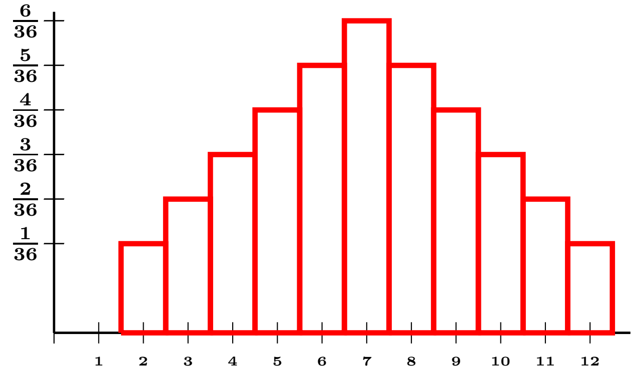

Figure 3.3: Histogram for example 130.

▶**Solution:** *From example 20, we know that the sample space for this experiment consists of* $6 \cdot 6 = 36$ *possible outcomes. Assuming each outcome is equally likely, we see that range of* S *is obtained as follows:*

S	(red, blue)
2	(1, 1)
3	(1, 2), (2, 1)
4	(1, 3), (3, 1), (2, 2)
5	(1, 4), (4, 1), (2, 3), (3, 2)
6	(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)
7	(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)
8	(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)
9	(3, 6), (6, 3), (4, 5), (5, 4)
10	(4, 6), (6, 4), (5, 5)
11	(5, 6), (6, 5)
12	(6, 6)

Therefore, the probability mass function is

$$\mathbf{P}(S = 2) = \frac{1}{36}, \qquad \mathbf{P}(S = 4) = \frac{3}{36} = \frac{1}{12}, \qquad \mathbf{P}(S = 6) = \frac{5}{36},$$

$$\mathbf{P}(S = 3) = \frac{2}{36} = \frac{1}{18}, \qquad \mathbf{P}(S = 5) = \frac{4}{36} = \frac{1}{9}, \qquad \mathbf{P}(S = 7) = \frac{6}{36} = \frac{1}{6},$$

$$P(S=8) = \frac{5}{36}, \qquad P(S=10) = \frac{3}{36} = \frac{1}{12}, \qquad P(S=12) = \frac{1}{36}.$$
$$P(S=9) = \frac{4}{36} = \frac{1}{9}, \qquad P(S=11) = \frac{2}{36} = \frac{1}{18},$$

A histogram for S *is found in figure 3.3.* ◄

Homework

Problem 3.3.1 *The six faces of a die are numbered* $1, 2, 3, 4, 5, 6$*, but the die is loaded so that the probability of obtaining a given number is proportional to the square of the number of the dots. If* X *is the random variable counting the number of dots, find* $P(X = k)$ *for* $k = 1, 2, \ldots, 6$.

Problem 3.3.2 *Three fair dice, a red, a white and a blue one are thrown. The* **sum** *of the dots is given by the random variable* Y*. What is the range of the random variable* Y*? Construct a histogram for* Y.

Problem 3.3.3 *Two fair dice, a red and a blue one are thrown. The* **product** *of the dots is given by the random variable* Y*. What is the range of the random variable* Y*? Construct a histogram for* Y.

Problem 3.3.4 *A fair die is tossed. If the resulting number is either* 2 *or* 3*, you multiply your score by* 2 *and get that many dollars. If the resulting number is either* 1 *or* 4*, you add* 1 *to your score and get that many dollars. If the resulting number is either* 5 *or* 6*, you get that many dollars. Let* X *be the random variable counting your gain, in dollars. Give the range of* X*. Construct a histogram for* X.

Problem 3.3.5 (AHSME 1994) *When* n *fair dice are rolled, the probability of obtaining a sum of* 1994 *is strictly positive and is the same as the probability of obtaining a sum of* S*. What is the smallest possible value of* S*? (Hint: In a fair die there are* $7 - a$ *dots on the face opposite* a *dots. Hence* $P(S = x) = P(S = 7n - x)$*.)*

3.4 Independence

131 Definition Two events A and B are said to be independent if

$$P(A \cap B) = P(A) \cdot P(B).$$

132 Example Recall 130. Two dice, a red one and a blue one, are thrown. If A is the event: "the red die lands on 4" and B is the event: "the sum on the dice is 9" then A and B are not independent. For $P(A) = \frac{1}{6}$, $P(B) = \frac{1}{9}$, and hence $P(A)P(B) = \frac{1}{54}$. On the other hand,

$$P(A \cap B) = P(\text{blue die shews } 5) = \frac{1}{6}.$$

☞ *More often than not independence is built into a problem physically, that is, an event* A *does not physically influence an event* B*. In particular, in problems where sampling is done* with replacement, *we should infer independence.*

133 Example Two dice, a red one and a blue one, are thrown. If A is the event: "the red die lands on an even number" and B is the event: "the blue die lands on a prime number" then A and B are independent, as they do not physically influence one another.

134 Example Let A, B be independent events with $P(A) = P(B)$ and $P(A \cup B) = \frac{1}{2}$. Find $P(A)$.

►**Solution:** *By Inclusion-Exclusion (Theorem 121),*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B),$$

which yields

$$\frac{1}{2} = 2\mathbf{P}(A) - (\mathbf{P}(A))^2 \implies 2x^2 - 4x + 1 = 0,$$

with $x = \mathbf{P}(A)$. *Solving this quadratic equation and bearing in mind that we must have* $0 < x < 1$, *we find* $\mathbf{P}(A) = x = 1 - \frac{\sqrt{2}}{2}$. ◀

135 Example A die is loaded so that if $\mathbf{D}$ is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{D} = k) = \frac{k}{21}$, where $k = 1, 2, 3, 4, 5, 6$. Another die is loaded differently, so that if $\mathbf{X}$ is the random variable giving the score on the die, then $\mathbf{P}(\mathbf{X} = k) = \frac{k^2}{91}$. Find $\mathbf{P}(\mathbf{D} + \mathbf{X} = 4)$.

▶**Solution:** *Clearly the value on which the first die lands does not influence the value on which the second die lands. Thus by independence*

$$\begin{aligned}
\mathbf{P}(\mathbf{D} + \mathbf{X} = 4) \quad &\iff \quad \mathbf{P}(\mathbf{D} = 1 \cap \mathbf{X} = 3) + \mathbf{P}(\mathbf{D} = 2 \cap \mathbf{X} = 2) \\
&\qquad + \mathbf{P}(\mathbf{D} = 3 \cap \mathbf{X} = 1) \\
&= \quad \mathbf{P}(\mathbf{D} = 1) \cdot \mathbf{P}(\mathbf{X} = 3) + \mathbf{P}(\mathbf{D} = 2) \cdot \mathbf{P}(\mathbf{X} = 2) \\
&\qquad + \mathbf{P}(\mathbf{D} = 3) \cdot \mathbf{P}(\mathbf{X} = 1) \\
&= \quad \frac{1}{91} \cdot \frac{3}{21} + \frac{4}{91} \cdot \frac{2}{21} + \frac{9}{91} \cdot \frac{1}{21} \\
&= \quad \frac{20}{1911}.
\end{aligned}$$

◀

136 Example Two men, A and B are shooting a target. The probability that A hits the target is $\mathbf{P}(A) = \frac{1}{3}$, and the probability that B shoots the target is $\mathbf{P}(B) = \frac{1}{5}$, one independently of the other. Find

❶ That A misses the target.

❷ That both men hit the target.

❸ That at least one of them hits the target.

❹ That none of them hits the target.

▶**Solution:** *The desired probabilities are plainly*

❶ $\mathbf{P}(A^c) = 1 - \frac{1}{3} = \frac{2}{3}$.

❷ $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B) = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$.

❸ $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = \frac{1}{3} + \frac{1}{5} - \frac{1}{15} = \frac{7}{15}$.

❹ $\mathbf{P}(A^c \cap B^c) = \mathbf{P}((A \cup B)^c) = 1 - \mathbf{P}(A \cup B) = 1 - \frac{7}{15} = \frac{8}{15}$.

◀

137 Example A certain type of missile hits its target 30% of the time. Determine the minimum number of missiles that must be shot at a certain target in order to obtain a change higher than 80% of hitting the target.

▶**Solution:** *The probability that n missiles miss the target is $(0.7)^n$. The probability that at least one of the n missiles hits the target is thus $1-(0.7)^n$. We need $1-(0.7)^n > 0.8$ and by a few calculations,*

$$1-(0.7)^1 = 0.3,$$

$$1-(0.7)^2 = 0.51,$$

$$1-(0.7)^3 = 0.657,$$

$$1-(0.7)^4 = .7599,$$

$$1-(0.7)^5 = .83193,$$

whence the minimum n is found to be $n = 5$. ◀

When we deal with more than two events, the following definition is pertinent.

138 Definition The events $A_1, A_2, \ldots, A_n$ are independent if for any choice of k $(2 \leq k \leq n)$ indexes $\{i_1, i_2, \ldots, l_k\}$ we have

$$\mathbf{P}\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}\right) = \mathbf{P}\left(A_{i_1}\right)\mathbf{P}\left(A_{i_2}\right)\cdots\mathbf{P}\left(A_{i_k}\right).$$

Considerations of independence are important in the particular case when trials are done in succession.

139 Example A biased coin with $\mathbf{P}(H) = \frac{2}{5}$ is tossed three times in a row. Find the probability that one will obtain HHT, in that order. What is the probability of obtaining two heads and one tail, in the three tosses?

▶**Solution:** *Each toss is physically independent from the other. The required probability is*

$$\mathbf{P}(HHT) = \mathbf{P}(H)\cdot\mathbf{P}(H)\cdot\mathbf{P}(T) = \frac{2}{5}\cdot\frac{2}{5}\cdot\frac{3}{5} = \frac{12}{125}.$$

For the second question, we want

$$\mathbf{P}(\{HHT, HTH, THH\}) = \mathbf{P}(HHT) + \mathbf{P}(HTH) + \mathbf{P}(THH) = 3\cdot\frac{12}{125} = \frac{36}{125}.$$

◀

140 Example An urn has 3 white marbles, 4 red marbles, and 5 blue marbles. Three marbles are drawn in succession from the urn *with replacement*, and their colour noted. What is the probability that a red, a white and another white marble will be drawn, in this order?

▶**Solution:** *Since the marbles are replaced, the probability of successive drawings is not affected by previous drawings. The probability sought is thus*

$$\frac{4}{12}\cdot\frac{3}{12}\cdot\frac{3}{12} = \frac{1}{48}.$$

◀

141 Example A box contains 20 white balls, 30 blue balls, and 50 red balls. Ten balls are selected, one at a time, with replacement. Find the probability that at least one colour will be missing from the ten selected balls.

▶**Solution:** *Let W be the event that the white balls are not represented among the ten selected balls, and similarly define R and B. Since selection is done with replacement, these events are independent.Then by Inclusion-Exclusion*

$$\begin{aligned}\mathbf{P}(W \cup B \cup R) &= \mathbf{P}(W) + \mathbf{P}(B) + \mathbf{P}(R) - \mathbf{P}(W \cap B) - \mathbf{P}(W \cap R) - \mathbf{P}(R \cap B) \\ &\quad + \mathbf{P}(W \cap R \cap B) \\ &= (0.8)^{10} + (0.7)^{10} + (0.5)^{10} - (0.5)^{10} - (0.3)^{10} - (0.2)^{10} + 0 \\ &\approx 0.1356.\end{aligned}$$

◀

Homework

Problem 3.4.1 *Suppose that a monkey is seated at a computer keyboard and randomly strikes the 26 letter keys and the space bar. Find the probability that its first 48 characters typed (including spaces) will be:* **"the slithy toves did gyre and gimble in the wabe"**[1].

Problem 3.4.2 *An urn has 3 white marbles, 4 red marbles, and 5 blue marbles. Three marbles are drawn in succession from the urn* with replacement, *and their colour noted. What is the probability that a red, a white and a blue marble will be drawn, in this order?*

Problem 3.4.3 *A fair coin is tossed three times in succession. What is the probability of obtaining exactly two heads?*

Problem 3.4.4 *Two cards are drawn in succession and with replacement from an ordinary deck of cards. What is the probability that the first card is a heart and the second one a queen?*

Problem 3.4.5 *Two numbers X and Y are chosen at random, and with replacement, from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Find the probability that $X^2 - Y^2$ be divisible by 2.*

Problem 3.4.6 *Events A and B are independent, events A and C are mutually exclusive, and events B and C are independent. If $\mathbf{P}(A) = \frac{1}{2}$, $\mathbf{P}(B) = \frac{1}{4}$, $\mathbf{P}(C) = \frac{1}{8}$, find $\mathbf{P}(A \cup B \cup C)$.*

Problem 3.4.7 *A population consists of 20% zeroes, 40% ones, and 40% twos. A random sample X, Y of size 2 is selected with replacement. Find $\mathbf{P}(|X - Y| = 1)$.*

Problem 3.4.8 *A book has 4 typos. After each re-reading, an uncorrected typo is corrected with probability $\frac{1}{3}$. The correction of different typos is each independent one from the other. Each of the re-readings is also independent one from the other. How many re-readings are necessary so that the probability that there be no more errors be greater than 0.9?*

Problem 3.4.9 *A die is rolled three times in succession. Find the probability of obtaining at least one six.*

Problem 3.4.10 *A, B, C are mutually independent events with $\mathbf{P}(A) = \mathbf{P}(B) = \mathbf{P}(C) = \frac{1}{3}$. Find $\mathbf{P}(A \cup B \cup C)$.*

Problem 3.4.11 *A pair of dice is tossed 10 successive times. What is the probability of observing neither a 7 nor an 11 in any of the 10 trials?*

3.5 Conditional Probability

In this section we will explore what happens when we are given extra information about the possibility of an event happening.

142 Definition Given an event B, the probability that event A happens given that event B has occurred is defined and denoted by

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}, \quad \mathbf{P}(B) \neq 0.$$

[1] From Lewis Carroll's *The Jabberwock.*

143 Example Ten cards numbered 1 through 10 are placed in a hat, mixed and then one card is pulled at random. If the card is an even numbered card, what is the probability that its number is divisible by 3?

▶**Solution:** *Let A be the event "the card's number is divisible by 3" and B be the event "the card is an even numbered card." We want* $\mathbf{P}(A|B)$. *Observe that* $\mathbf{P}(B) = \frac{5}{10} = \frac{1}{2}$. *Now the event $A \cap B$ is the event that the card's number is both even and divisible by 3, which happens only when the number of the card is 6. Hence* $\mathbf{P}(A \cap B) = \frac{1}{10}$. *The desired probability is*

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)} = \frac{\frac{1}{10}}{\frac{1}{2}} = \frac{1}{5}.$$

◀

144 Example A coin is tossed twice. What is the probability that in both tosses appear heads given that in at least one of the tosses appeared heads?

▶**Solution:** *Let* $E = \{(H,H)\}$ *and* $F = \{(H,H),(H,T),(T,H)\}$. *Then*

$$\mathbf{P}(E|F) = \frac{\mathbf{P}(E \cap F)}{\mathbf{P}(F)} = \frac{\mathbf{P}(\{(H,H)\})}{\mathbf{P}(\{(H,H),(H,T),(T,H)\})} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

◀

The conditional probability formula can be used to obtain probabilities of intersections of events. Thus

$$\mathbf{P}(A \cap B) = \mathbf{P}(B)\,\mathbf{P}(A|B) \tag{3.2}$$

Observe that the sinistral side of the above equation is symmetric. Thus we similarly have

$$\mathbf{P}(A \cap B) = \mathbf{P}(B \cap A) = \mathbf{P}(A)\,\mathbf{P}(B|A) \tag{3.3}$$

145 Example Darlene is undecided on whether taking Statistics or Philosophy. She knows that if she takes Statistics she will get an A with probability $\frac{1}{3}$, while if she takes Philosophy she will receive an A with probability $\frac{1}{2}$. Darlene bases her decision on the flip of a coin. What is the probability that Darlene will receive an A in Statistics?

▶**Solution:** *Let E be the event that Darlene takes Statistics and let F be the event that she receives an A in whatever course she decides to take. Then we want* $\mathbf{P}(E \cap F)$. *But*

$$\mathbf{P}(E \cap F) = \mathbf{P}(E)\,\mathbf{P}(F|E) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

◀

146 Example An urn contains eight black balls and three white balls. We draw two balls without replacement. What is the probability that both balls are black?

▶**Solution:** *Let B_1 be the event that the first ball is black and let B_2 be the event that the second ball is black. Clearly* $\mathbf{P}(B_1) = \frac{8}{11}$. *If a black ball is taken out, there remain 10 balls in* *the urn, 7 of which are black. Thus* $\mathbf{P}\left(B_2|B_1\right) = \frac{7}{10}$*. We conclude that*

$$\mathbf{P}\left(B_1 \cap B_2\right) = \mathbf{P}\left(B_1\right)\mathbf{P}\left(B_2|B_1\right) = \frac{8}{11} \cdot \frac{7}{10} = \frac{28}{55}.$$

◀

The formula for conditional probability can be generalised to any number of events. Thus if A_1, A_2, ..., A_n are events, then

$$\begin{aligned} \mathbf{P}\left(A_1 \cap A_2 \cap \ldots \cap A_n\right) &= \mathbf{P}\left(A_1\right) \\ &\cdot \mathbf{P}\left(A_2|A_1\right)\mathbf{P}\left(A_3|A_1 \cap A_2\right) \\ &\cdots \mathbf{P}\left(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1}\right) \end{aligned} \tag{3.4}$$

147 Example An urn contains 5 red marbles, 4 blue marbles, and 3 white marbles. Three marbles are drawn in succession, without replacement. Find the probability that the first two are white and the third one is blue.

▶**Solution:** *Let the required events be* W_1, W_2, B_3*. Then*

$$\mathbf{P}\left(W_1 \cap W_2 \cap B_3\right) = \mathbf{P}\left(W_1\right)\mathbf{P}\left(W_2|W_1\right)\mathbf{P}\left(B_3|W_1 \cap W_2\right) = \frac{3}{12} \cdot \frac{2}{11} \cdot \frac{4}{10} = \frac{1}{55}.$$

◀

Sometimes we may use the technique of *conditioning*, which consists in decomposing an event into mutually exclusive parts. Let E and F be events. Then

$$\begin{aligned} \mathbf{P}\left(E\right) &= \mathbf{P}\left(E \cap F\right) + \mathbf{P}\left(E \cap F^c\right) \\ &= \mathbf{P}\left(F\right)\mathbf{P}\left(E|F\right) + \mathbf{P}\left(F^c\right)\mathbf{P}\left(E|F^c\right). \end{aligned} \tag{3.5}$$

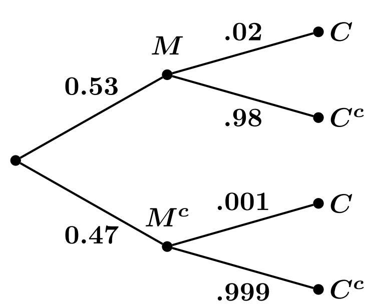

Figure 3.4: Example 148.

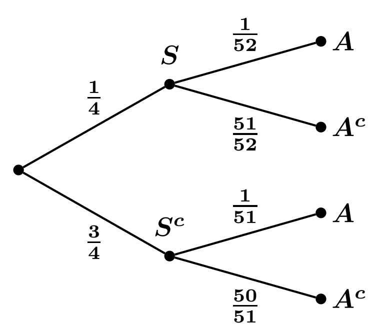

Figure 3.5: Example 149.

148 Example A population consists of 53% men. The probability of colour blindness is $.02$ for a man and $.001$ for a woman. Find the probability that a person picked at random is colour blind.

▶**Solution:** *We condition on the sex of the person. Let M be the event that the person is a man and let C be the event that the person is colour-blind. Then*

$$\mathbf{P}\left(C\right) = \mathbf{P}\left(C \cap M\right) + \mathbf{P}\left(C \cap M^c\right).$$

But $\mathbf{P}\left(C \cap M\right) = \mathbf{P}\left(M\right)\mathbf{P}\left(C|M\right) = (.53)(.02) = 0.106$ *and* $\mathbf{P}\left(C \cap M^c\right) = \mathbf{P}\left(M^c\right)\mathbf{P}\left(C|M^c\right) = (.47)(.001) = .00047$ *and so* $\mathbf{P}\left(C\right) = 0.10647.$ *A tree diagram explaining this calculation can be seen in figure 3.4.* ◀

149 Example Draw a card. If it is a spade, put it back and draw a second card. If the first card is not a spade, draw a second card without replacing the second one. Find the probability that the second card is the ace of spades.

▶**Solution:** *We condition on the first card. Let S be the event that the first card is a spade and let A be the event that the second card is the ace of spades. Then*

$$\mathbf{P}\left(A\right) = \mathbf{P}\left(A \cap S\right) + \mathbf{P}\left(A \cap S^c\right).$$

But $\mathbf{P}\left(A \cap S\right) = \mathbf{P}\left(S\right)\mathbf{P}\left(A|S\right) = \dfrac{1}{4}\cdot\dfrac{1}{52} = \dfrac{1}{108}$ *and* $\mathbf{P}\left(A \cap S^c\right) = \mathbf{P}\left(S^c\right)\mathbf{P}\left(A|S^c\right) = \dfrac{3}{4}\cdot\dfrac{1}{51} = \dfrac{1}{68}$. *We thus have*

$$\mathbf{P}\left(A\right) = \frac{1}{108} + \frac{1}{68} = \frac{11}{459}.$$

A tree diagram explaining this calculation can be seen in figure 3.5. ◀

150 Example A multiple-choice test consists of five choices per question. You think you know the answer for 75% of the questions and for the other 25% you guess at random. When you think you know the answer, you are right only 80% of the time. Find the probability of getting an arbitrary question right.

▶**Solution:** *We condition on whether you think you know the answer to the question. Let K be the event that you think you know the answer to the question and let R be the event that you get a question right. Then*

$$\mathbf{P}\left(R\right) = \mathbf{P}\left(K \cap R\right) + \mathbf{P}\left(K^c \cap R\right)$$

Now $\mathbf{P}\left(K \cap R\right) = \mathbf{P}\left(K\right)\cdot\mathbf{P}\left(R|K\right) = (.75)(.8) = .6$ *and*

$$\mathbf{P}\left(K^c \cap R\right) = \mathbf{P}\left(K^c\right)\cdot\mathbf{P}\left(R|K^c\right) = (.25)(.2) = .05.$$

Therefore $\mathbf{P}\left(R\right) = .6 + .05 = .65.$ ◀

If instead of conditioning on two disjoint sets we conditioned in n pairwise disjoint sets, we would obtain

151 Theorem (Law of Total Probability) Let $F = F_1 \cup F_2 \cup \cdots \cup F_n$, where $F_j \cap F_k = \varnothing$ if $j \neq k$, then

$$\mathbf{P}\left(E \cap F\right) = \mathbf{P}\left(F_1\right)\mathbf{P}\left(E|F_1\right) + \mathbf{P}\left(F_2\right)\mathbf{P}\left(E|F_2\right) + \cdots + \mathbf{P}\left(F_n\right)\mathbf{P}\left(E|F_n\right).$$

152 Example An urn contains 4 red marbles and 5 green marbles. A marble is selected at random and its colour noted, then this marble is put back into the urn. If it is red, then 2 more red marbles are put into the urn and if it is green 1 more green marble is put into the urn. A second marble is taken from the urn. Let R_1, R_2 be the events that we select a red marble on the first and second trials respectively, and let G_1, G_2 be the events that we select a green marble on the first and second trials respectively.

❶ Find $\mathbf{P}\left(R_2\right)$.

❷ Find $\mathbf{P}\left(R_2 \cap R_1\right)$.

❸ Find $\mathbf{P}\left(R_1 | R_2\right)$.

▶**Solution:** *Plainly,*

❶
$$\mathbf{P}\left(R_2\right) = \frac{4}{9} \cdot \frac{6}{11} + \frac{5}{9} \cdot \frac{3}{5} = \frac{19}{33}.$$

❷
$$\mathbf{P}\left(R_2 \cap R_1\right) = \frac{4}{9} \cdot \frac{6}{11} = \frac{8}{33}$$

❸
$$\mathbf{P}\left(R_1 | R_2\right) = \frac{\mathbf{P}\left(R_2 \cap R_1\right)}{\mathbf{P}\left(R_2\right)} = \frac{8}{19}.$$

◀

153 Example An urn contains 10 marbles: 4 red and 6 blue. A second urn contains 16 red marbles and an unknown number of blue marbles. A single marble is drawn from each urn. The probability that both marbles are the same colour is 0.44. Calculate the number of blue marbles in the second urn.

▶**Solution:** *Let b be the number of blue marbles in the second urn, let $R_k, k = 1, 2$ denote the event of drawing a red marble from urn k, and similarly define $B_k, k = 1, 2$. We want*

$$\mathbf{P}\left(\left(R_1 \cap R_2\right) \cup \left(B_1 \cap B_2\right)\right).$$

Observe that the events $R_1 \cap R_2$ and $B_1 \cap B_2$ are mutually exclusive, and that R_1 is independent of R_2 and B_1 is independent of B_2 (drawing a marble from the first urn does not influence drawing a second marble from the second urn). We then have

$$\begin{aligned} 0.44 &= \mathbf{P}\left(\left(R_1 \cap R_2\right) \cup \left(B_1 \cap B_2\right)\right) \\ &= \mathbf{P}\left(R_1 \cap R_2\right) + \mathbf{P}\left(B_1 \cap B_2\right) \\ &= \mathbf{P}\left(R_1\right)\mathbf{P}\left(R_2\right) + \mathbf{P}\left(B_1\right)\mathbf{P}\left(B_2\right) \\ &= \frac{4}{10} \cdot \frac{16}{b+16} + \frac{6}{10} \cdot \frac{b}{b+16}. \end{aligned}$$

Clearing denominators

$$0.44(10)(b+16) = 4(16) + 6b \implies b = 4.$$

◀

154 Example A sequence of independent trials is performed by rolling a pair of fair dice. What is the probability that an 8 will be rolled before rolling a 7?

▶**Solution:** *Let A be the event that an 8 occurs before a 7. Now, either: (i) the first trial will be an 8, which we will call event X, or (ii) the first trial will be a 7, which we will call event Y, or (iii) the first trial will be neither an 8 nor a 7, which we will call event Z. Since X , Y, Z partition A we have*

$$\mathbf{P}\left(A\right) = \mathbf{P}\left(A|X\right)\mathbf{P}\left(X\right) + \mathbf{P}\left(A|Y\right)\mathbf{P}\left(Y\right) + \mathbf{P}\left(A|Z\right)\mathbf{P}\left(Z\right).$$

Observe that

$$\mathbf{P}\left(A|X\right)\mathbf{P}\left(X\right) = 1\cdot\frac{5}{36}, \qquad \mathbf{P}\left(A|Y\right)\mathbf{P}\left(Y\right) = 0\cdot\frac{6}{36}, \qquad \mathbf{P}\left(A|Z\right)\mathbf{P}\left(Z\right) = \mathbf{P}\left(A\right)\cdot\frac{25}{36},$$

where the last equality follows because if the first outcome is neither an 8 nor a 7 we are in the situation as in the beginning of the problem. Thus

$$\mathbf{P}\left(A\right) = \frac{5}{36} + \frac{25}{36}\cdot\mathbf{P}\left(A\right) \implies \mathbf{P}\left(A\right) = \frac{5}{11}.$$

This will be considered again as example 182. ◀

155 Example Three people, **X, Y, Z**, in order, roll a fair die. The first one to roll an even number wins and the game is ended. What is the probability that **X** will eventually win?

▶**Solution:** *Either* $\mathbf{X}$ *wins on his first attempt, or it does not. Let F be the event that F wins on his first attempt and let* $\mathbf{P}\left(\mathbf{X}\right)$ *be the probability eventually wins. Then*

$$\mathbf{P}\left(\mathbf{X}\right) = \mathbf{P}\left(\mathbf{X}|F\right)\mathbf{P}\left(F\right) + \mathbf{P}\left(\mathbf{X}|F^c\right)\mathbf{P}\left(F^c\right) = 1\cdot\frac{1}{2} + \left(\frac{\mathbf{P}\left(\mathbf{X}\right)}{4}\right)\cdot\frac{1}{2} \implies \mathbf{P}\left(\mathbf{X}\right) = \frac{4}{7}.$$

Here we observe that

$$\mathbf{P}\left(\mathbf{X}|F^c\right) = \frac{1}{2}\cdot\frac{1}{2}\cdot\mathbf{P}\left(\mathbf{X}\right),$$

since if $\mathbf{X}$ *does not win on the first attempt but still wins, we need* $\mathbf{Y}$ *and* $\mathbf{Z}$ *to lose on the first attempts. This problem will be considered again in example 181.* ◀

156 Example (Monty Hall Problem) You are on a television shew where the host shews you three doors. Behind two of them are goats, and behind the remaining one a car. You choose one door, but the door is not yet opened. The host opens a door that has a goat behind it (he never opens the door that hides the car), and asks you whether you would like to switch your door to the unopened door. Should you switch?

▶**Solution:** *It turns out that by switching, the probability of getting the car increases from $\frac{1}{3}$ to $\frac{2}{3}$. Let us consider the following generalisation: an urn contains a white marbles and b black marbles with $a + b \geq 3$. You have two strategies:*

❶ *You may simply draw a marble at random. If it is white you win, otherwise you lose.*

❷ *You draw a marble at random without looking at it, and you dispose of it. The host removes a black marble from the urn. You now remove a marble from the urn. If it is white you win, otherwise you lose.*

In the first strategy your probability of winning is clearly $\frac{a}{a+b}$. To compute the probability of winning on the second strategy we condition on the colour of the marble that you first drew. The probability of winning is thus

$$\frac{a}{a+b}\cdot\frac{a-1}{a+b-2} + \frac{b}{a+b}\cdot\frac{a}{a+b-2} = \frac{a}{a+b}\left(1 + \frac{1}{a+b-2}\right).$$

This is greater than the probability on the first strategy, so the second strategy is better. ◀

157 Example A simple board game has four fields A, B, C, and D. Once you end up on field A you have won and once you end up on field B you have lost. From fields C and D you move to other fields by flipping a coin. If you are on field C and you throw a head, then you move to field A, otherwise to field D. From field D, you move to field C if you throw a head, and otherwise you mover to field B.

Suppose that you start in field D. What is the probability that you will win (i.e., what is the probability that you will end up on field A)?

▶**Solution:** *We want* $\mathbf{P}(A|D)$. *This can happen in two moves (from* D *to* C *to* A*) with probability* $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$, *or it can happen in* 4 *moves (from* D *to* C *to* D *to* C *to* A*) with probability* $\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{16}$, *or in six moves, ...,etc. We must sum thus the infinite geometric series*

$$\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots=\frac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{3}.$$

The required probability is therefore $\frac{1}{3}$. ◀

Homework

Problem 3.5.1 *Two cards are drawn in succession from a well-shuffled standard deck of cards. What is the probability of successively obtaining*

❶ *a red card and then a black card?*

❷ *two red cards?*

❸ *a knave and then a queen?*

❹ *two knaves?*

Problem 3.5.2 *Five cards are drawn at random from a standard deck of cards. It is noticed that there is at least one picture (A, J, Q, or K) card. Find the probability that this hand of cards has two knaves.*

Problem 3.5.3 *Five cards are drawn at random from a standard deck of cards. It is noticed that there is exactly one ace card. Find the probability that this hand of cards has two knaves.*

Problem 3.5.4 A *and* B *are two events from the same sample space satisfying*

$$\mathbf{P}(A)=\frac{1}{2};\qquad \mathbf{P}(B)=\frac{2}{3};\qquad \mathbf{P}(A|B)=\frac{1}{4}.$$

Find $\mathbf{P}(A^c\cap B^c)$.

Problem 3.5.5 *A cookie jar has* 3 *red marbles and* 1 *white marble. A shoebox has* 1 *red marble and* 1 *white marble. Three marbles are chosen at random without replacement from the cookie jar and placed in the shoebox. Then* 2 *marbles are chosen at random and without replacement from the shoebox. What is the probability that both marbles chosen from the shoebox are red?*

Problem 3.5.6 *A fair coin is tossed until a head appears. Given that the first head appeared on an even numbered toss, what is the conditional probability that the head appeared on the fourth toss?*

Problem 3.5.7 *Three fair standard dice are tossed, and the sum is found to be* 6. *What is the probability that none of the dice landed a* 1*?*

Problem 3.5.8 *An urn contains* 5 *red marbles and* 5 *green marbles. A marble is selected at random and its colour noted, then this marble is put back into the urn. If it is red, then* 2 *more red marbles are put into the urn and if it is green* 3 *more green marbles are put into the urn. A second marble is taken from the urn. Let* R_1, R_2 *be the events that we select a red marble on the first and second trials respectively, and let* G_1, G_2 *be the events that we select a green marble on the first and second trials respectively.*

1. *Find* $\mathbf{P}(R_1)$.
2. *Find* $\mathbf{P}(G_1)$.
3. *Find* $\mathbf{P}(R_2|R_1)$.
4. *Find* $\mathbf{P}(G_2|R_1)$.
5. *Find* $\mathbf{P}(G_2|G_1)$.
6. *Find* $\mathbf{P}(R_2|G_1)$.
7. *Find* $\mathbf{P}(R_2)$.
8. *Find* $\mathbf{P}(G_2)$.
9. *Find* $\mathbf{P}(R_2\cap R_1)$.
10. *Find* $\mathbf{P}(R_1|R_2)$.
11. *Find* $\mathbf{P}(G_2\cap R_1)$.
12. *Find* $\mathbf{P}(R_1|G_2)$.

Problem 3.5.9 *Five urns are numbered* $3, 4, 5, 6$*, and* 7*, respectively. Inside each urn is* n^2 *dollars where* n *is the number on the urn. You select an urn at random. If it is a prime number, you receive the amount in the urn. If* *the number is not a prime number, you select a second urn from the remaining four urns and you receive the total amount of money in the two urns selected. What is the probability that you end up with* $\$25$*?*

Problem 3.5.10 *A family has five children. Assuming that the probability of a girl on each birth was* $\frac{1}{2}$ *and that the five births were independent, what is the probability the family has at least one girl, given that they have at least one boy?*

Problem 3.5.11 *Events* S *and* T *have probabilities* $\mathbf{P}(S) = \mathbf{P}(T) = \frac{1}{3}$ *and* $\mathbf{P}(S|T) = \frac{1}{6}$*. What is* $\mathbf{P}(S^c \cap T^c)$*?*

Problem 3.5.12 *Peter writes to Paul and does not receive an answer. Assuming that one letter in* n *is lost in the mail, find the probability that Paul received the letter. (Assume that Paul would have answered the letter had he received it.)*

Problem 3.5.13 *A deck of cards is shuffled and then divided into two halves of* 26 *cards each. A card is drawn from one of the halves; it turns out to be an ace. The ace is then placed in the second half-deck. This half is then shuffled, and a card drawn from it. Find the probability that this drawn card is an ace.*

Problem 3.5.14 *An insurance company examines its pool of auto insurance customers and gathers the following information:*

- *All customers insure at least one car.*
- *70% of the customers insure more than one car.*
- *20% of the customers insure a sports car.*
- *Of those customers who insure more than one car, 15% insure a sports car.*

Calculate the probability that a randomly selected customer insures exactly one car and that car is not a sports car.

Problem 3.5.15 *Let* A *and* B *be independent events with probabilities* $\mathbf{P}(A) = 0.2$ *and* $\mathbf{P}(B) = 0.3$*. Let* C *denote the event that "both* A *and* B *occur," and let* D *be the event "either* A *or* B*, but not both, occur."*

1. *Express* D *in terms of* A *and* B *using set-theoretic notation and compute* $\mathbf{P}(D)$
2. *Find* $\mathbf{P}(A|D)$.
3. *Are* C *and* D *independent?*

3.6 Bayes' Rule

Suppose $\Omega = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_j \cap A_k = \varnothing$ if $j \neq k$ is a partition of the sample space. Then

$$\mathbf{P}(A_k|B) = \frac{\mathbf{P}(A_k \cap B)}{\mathbf{P}(B)}.$$

By the Law of Total Probability Theorem 151, $\mathbf{P}(B) = \mathbf{P}(A_1)\mathbf{P}(B|A_1) + \mathbf{P}(A_2)\mathbf{P}(B|A_2) + \cdots + \mathbf{P}(A_n)\mathbf{P}(B|A_n)$. This gives

158 Theorem (Bayes' Rule) . Let $A_1, A_2, \ldots, A_n$ be pairwise disjoint with union Ω. Then

$$\mathbf{P}(A_k|B) = \frac{\mathbf{P}(A_k \cap B)}{\mathbf{P}(B)} = \frac{\mathbf{P}(A_k \cap B)}{\sum_{k=1}^{n} \mathbf{P}(A_k)\mathbf{P}(B|A_k)}.$$

159 Example A supermarket buys its eggs from three different chicken ranches. They buy $1/3$ of their eggs from Eggs'R Us, $1/2$ of their eggs from The Yolk Ranch, and $1/6$ of their eggs from Cheap Eggs. The supermarket determines that 1% of the eggs from Eggs'R Us are cracked, 2% of the eggs from the Yolk Ranch are cracked, and 5% of the eggs from Cheap Eggs are cracked. What is the probability that an egg chosen at random is from Cheap Eggs, given that the egg is cracked?

▶**Solution:** *See figure 3.6 for a tree diagram. We have*

$$\begin{aligned}
\mathbf{P}(\text{cracked}) &= \mathbf{P}(\text{cracked}|\text{R'Us})\,\mathbf{P}(\text{R'Us}) + \mathbf{P}(\text{cracked}|\text{YR})\,\mathbf{P}(\text{YR}) + \mathbf{P}(\text{cracked}|\text{ChE})\,\mathbf{P}(\text{ChE}) \\
&= \frac{1}{3}\cdot\frac{1}{100} + \frac{1}{2}\cdot\frac{2}{100} + \frac{1}{6}\cdot\frac{5}{100} \\
&= \frac{13}{600}
\end{aligned}$$

and so,

$$\begin{aligned}
\mathbf{P}(\text{ChE}|\text{cracked}) &= \frac{\mathbf{P}(\text{ChE}\cap\text{cracked})}{\mathbf{P}(\text{cracked})} \\
&= \frac{\mathbf{P}(\text{cracked}|\text{ChE})\cdot\mathbf{P}(\text{ChE})}{\mathbf{P}(\text{cracked})} \\
&= \frac{\frac{5}{100}\cdot\frac{1}{6}}{\frac{13}{600}} \\
&= \frac{5}{13}
\end{aligned}$$

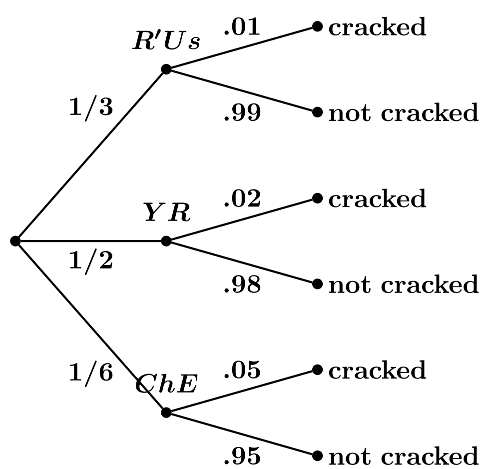

Figure 3.6: Example 159.

◀

160 Example 6% of Type A spark plugs are defective, 4% of Type B spark plugs are defective, and 2% of Type C spark plugs are defective. A spark plug is selected at random from a batch of spark plugs containing 50 Type A plugs, 30 Type B plugs, and 20 Type C plugs. The selected plug is found to be defective. What is the probability that the selected plug was of Type A?

▶**Solution:** *Let A, B, C denote the events that the plug is type A, B, C respectively, and D the event that the plug is defective. We have*

$$\begin{aligned}
\mathbf{P}(D) &= \mathbf{P}(D|A)\cdot\mathbf{P}(A) + \mathbf{P}(D|B)\cdot\mathbf{P}(B) + \mathbf{P}(D|C)\cdot\mathbf{P}(C) \\
&= \frac{6}{100}\cdot\frac{50}{100} + \frac{4}{100}\cdot\frac{30}{100} + \frac{2}{100}\cdot\frac{20}{100} \\
&= \frac{23}{500}.
\end{aligned}$$

Hence

$$
\begin{aligned}
\mathbf{P}\left(A|D\right) &= \frac{\mathbf{P}\left(A\cap D\right)}{\mathbf{P}\left(D\right)} \\
&= \frac{\mathbf{P}\left(D|A\right)\cdot\mathbf{P}\left(A\right)}{\mathbf{P}\left(D\right)} \\
&= \frac{\frac{6}{100}\cdot\frac{50}{100}}{\frac{23}{500}} \\
&= \frac{15}{23}.
\end{aligned}
$$

◀

Homework

Problem 3.6.1 *There are three coins in a box. When tossed, one of the coins comes up heads only 30% of the time, one of the coins is fair, and the third comes up heads 80% of the time. A coin is selected at random from the box and tossed three times. If two heads and a tails come up—in this order—what is the probability that the coin was the fair coin?*

Problem 3.6.2 *On a day when Tom operates the machinery, 70% of its output is high quality. On a day when Sally operates the machinery, 90% of its output is high quality. Tom operates the machinery 3 days out of 5. Three pieces of a random day's output were selected at random and 2 of them were found to be of high quality. What is the probability that Tom operated the machinery that day?*

Problem 3.6.3 *Two distinguishable dice have probabilities p, and 1 respectively of throwing a 6. One of the dice is chosen at random and thrown. A 6 appeared.*

1. *Find the probability of throwing a 6.*
2. *What is the probability that one simultaneously chooses die I and one throws a 6?*
3. *What is the probability that the die chosen was the first one?*

Problem 3.6.4 *Three boxes identical in appearance contain the following coins: Box I has two quarters and a dime; Box II has 1 quarter and 2 dimes; Box III has 1 quarter and 1 dime. A coin drawn at random from a box selected is a quarter.*

1. *Find the probability of obtaining a quarter.*
2. *What is the probability that one simultaneously choosing box III and getting a quarter?*
3. *What is the probability that the quarter came from box III?*

Problem 3.6.5 *There are three urns, A, B, and C. Urn A has a red marbles and b green marbles, urn B has c red marbles and d green marbles, and urn C has a red marbles and c green marbles. Let A be the event of choosing urn A, B of choosing urn B and, C of choosing urn C. Let R be the event of choosing a red marble and G be the event of choosing a green marble. An urn is chosen at random, and after that, from this urn, a marble is chosen at random.*

1. *Find* $\mathbf{P}\left(G\right)$.
2. *Find* $\mathbf{P}\left(G|\mathrm{C}\right)$.
3. *Find* $\mathbf{P}\left(C|G\right)$.
4. *Find* $\mathbf{P}\left(R\right)$.
5. *Find* $\mathbf{P}\left(R|\mathrm{A}\right)$.
6. *Find* $\mathbf{P}\left(\mathrm{A}|R\right)$.

Problem 3.6.6 *Three dice have the following probabilities of throwing a 6: p, q, r, respectively. One of the dice is chosen at random and thrown. A 6 appeared. What is the probability that the die chosen was the first one?*

Problem 3.6.7 *Three boxes identical in appearance contain the following coins: Box A has two quarters; Box B has 1 quarter and 2 dimes; Box C has 1 quarter and 1 dime. If a coin drawn at random from a box selected is a quarter, what is the probability that the randomly selected box contains at least one dime?*

Problem 3.6.8 *An urn contains 6 red marbles and 3 green marbles. One marble is selected at random and is replaced by a marble of the other colour. A second marble is then drawn. What is the probability that the first marble selected was red given that the second one was also red?*

Problem 3.6.9 *There are three dice. Die I is an ordinary fair die, so if $\mathbf{F}$ is the random variable giving the score on this die, then $\mathbf{P}\left(\mathbf{F}=k\right)=\frac{1}{6}$, Die II is loaded so that if $\mathbf{D}$ is the random variable giving the score on the die, then $\mathbf{P}\left(\mathbf{D}=k\right)=\frac{k}{21}$, where $k=1,2,3,4,5,6$. Die is loaded differently, so that if $\mathbf{X}$ is the random variable giving the* *score on the die, then* $\mathbf{P}(\mathbf{X} = k) = \frac{k^2}{91}$*. A die is chosen at random and a* 5 *appears. What is the probability that it was Die II?*

Problem 3.6.10 *There are* 3 *urns each containing* 5 *white marbles and* 2 *black marbles, and* 2 *urns each containing* 1 *white marble and* 4 *black marbles. A black marble having been drawn, find the chance that it came from the first group of urns.*

Problem 3.6.11 *There are four marbles in an urn, but it is not known of what colours they are. One marble is drawn and found to be white. Find the probability that all the marbles are white.*

Problem 3.6.12 *In an urn there are six marbles of unknown colours. Three marbles are drawn and found to be black. Find the chance that no black marble is left in the urn.*

Problem 3.6.13 *John speaks the truth* 3 *out of* 4 *times. Peter speaks the truth* 5 *out of* 6 *times. What is the probability that they will contradict each other in stating the same fact?*

Problem 3.6.14 *Adolf is taking a multiple choice exam in which each question has* 5 *possible answers, exactly one of which is correct. If Adolf knows the answer, he selects the correct answer. Otherwise he selects one answer at random from the* 5 *possible answers. Suppose that, for each question, there is a* 70% *chance that Adolf knows the answer.*

1. *Compute the probability that, on a randomly chosen question, Adolf gets the correct answer.*
2. *Compute the probability that Adolf knows the answer to a question given that she has answered the question correctly.*

Problem 3.6.15 *Hugh has just found out that he has probability* $\frac{1}{3}$ *that he has contracted a viral infection from his last incursion into* The Twilight Zone. *People who contract this disease have probability* $\frac{1}{4}$ *of having children who are cyclops. (Assume that for the uninfected population, the probability is zero of spawning cyclops.) Hugh marries Leigh and they have three children, in sequence. What is the probability that the first two children will not be cyclops? What is the probability that their third child will be a cyclops, given that the first two were not?*

Problem 3.6.16 *In some faraway country, families have either one, two, or three children only, with probability* $\frac{1}{3}$*. Boys and girls appear with equal probability. Given that David has no brothers, what is the probability that he is an only child? Suppose now that David has no sisters, what is the probability that he is an only child?*

Problem 3.6.17 *Four coins* A, B, C, D *have the following probabilities of landing heads:*

$$\mathbf{P}(A = H) = \frac{1}{5}; \qquad \mathbf{P}(B = H) = \frac{2}{5};$$

$$\mathbf{P}(C = H) = \frac{3}{5}; \qquad \mathbf{P}(D = H) = \frac{4}{5},$$

and they land tails otherwise. A coin is chosen at random and flipped three times. On the first and second flips it lands heads, on the third, tails. Which of the four coins is it the most likely to be?

4 Discrete Random Variables

4.1 Uniform Random Variables

Consider a non-empty finite set Ω with $\text{card}(\Omega)$ number of elements and let A, B be disjoint subsets of Ω. It is clear that

❶ $0 \leq \dfrac{\text{card}(A)}{\text{card}(\Omega)} \leq 1$,

❷ $\dfrac{\text{card}(\Omega)}{\text{card}(\Omega)} = 1$,

❸ $\dfrac{\text{card}(A \cup B)}{\text{card}(\Omega)} = \dfrac{\text{card}(A)}{\text{card}(\Omega)} + \dfrac{\text{card}(B)}{\text{card}(\Omega)}$ when $A \cap B = \varnothing$.

Thus the quantity $\dfrac{\text{card}(A)}{\text{card}(\Omega)}$ on the subsets of Ω is a probability (satisfies definition 116), and we put

$$\mathbf{P}(A) = \frac{\text{card}(A)}{\text{card}(\Omega)}. \tag{4.1}$$

Observe that in this model the probability of any single outcome is $\dfrac{1}{\text{card}(\Omega)}$, that is, every outcome is *equally likely*.

161 Definition Let

$$\Omega = \{x_1, x_2, \ldots, x_n\}$$

be a finite sample space. A *uniform discrete random variable* X *defined on* Ω is a function that achieves the distinct values x_k with equal probability:

$$\mathbf{P}(\mathrm{X} = x_k) = \frac{1}{\text{card}(\Omega)}.$$

Since

$$\sum_{k=1}^{n} \mathbf{P}(\mathrm{X} = x_k) = \sum_{k=1}^{n} \frac{1}{\text{card}(\Omega)} = \frac{\text{card}(\Omega)}{\text{card}(\Omega)} = 1,$$

this is a bonafide random variable.

162 Example If the experiment is flipping a fair coin, then $\Omega = \{H, T\}$ is the sample space (H for heads, T for tails) and $E = \{H\}$ is the event of obtaining a head. Then

$$\mathbf{P}(H) = \frac{1}{2} = \mathbf{P}(T).$$

163 Example Consider a standard deck of cards. One card is drawn at random.

❶ Find the size of the sample space of this experiment.

❷ Find the probability $\mathbf{P}(K)$ of drawing a king.

❸ Find the probability $\mathbf{P}(J)$ of drawing a knave[1].

❹ Find the probability $\mathbf{P}(R)$ of drawing a red card.

❺ Find the probability $\mathbf{P}(K \cap R)$ of drawing a red king.

❻ Find the probability $\mathbf{P}(K \cup R)$ of drawing either a king or a red card.

❼ Find the probability $\mathbf{P}(K \setminus R)$ of drawing a king which is not red.

❽ Find the probability $\mathbf{P}(R \setminus K)$ of drawing a red card which is not a king.

❾ Find the probability $\mathbf{P}(K \cap J)$ of drawing a king which is also a knave.

▶**Solution:**

❶ *The size of the sample space for this experiment is* $\mathbf{card}(S) = \binom{52}{1} = 52$.

❷ *Since there are* 4 *kings,* $\mathbf{card}(K) = 4$. *Hence* $\mathbf{P}(K) = \frac{4}{52} = \frac{1}{13}$.

❸ *Since there are* 4 *knaves,* $\mathbf{card}(J) = 4$. *Hence* $\mathbf{P}(J) = \frac{4}{52} = \frac{1}{13}$.

❹ *Since there are* 26 *red cards,* $\mathbf{card}(R) = 26$. *Hence* $\mathbf{P}(R) = \frac{26}{52} = \frac{1}{2}$.

❺ *Since a card is both a king and red in only two instances (when it is* $K\heartsuit$ *or* $K\diamondsuit$*), we have* $\mathbf{P}(K \cap R) = \frac{2}{52} = \frac{1}{26}$.

❻ *By Inclusion-Exclusion we find*

$$\mathbf{P}(K \cup R) = \mathbf{P}(K) + \mathbf{P}(R) - \mathbf{P}(K \cap R) = \frac{7}{13}.$$

❼ *Since of the* 4 *kings two are red we have* $\mathbf{P}(K \setminus R) = \frac{2}{52} = \frac{1}{26}$.

❽ *Since of the* 26 *red cards two are kings,* $\mathbf{P}(R \setminus K) = \frac{24}{52} = \frac{6}{13}$.

❾ *Since no card is simultaneously a king and a knave,* $\mathbf{P}(K \cap J) = \mathbf{P}(\varnothing) = 0$.

◀

164 Example A number is chosen at random from the set

$$\{1, 2, \ldots, 1000\}.$$

What is the probability that it is a palindrome?

▶**Solution:** *There are* 9 *palindromes with* 1*-digit,* 9 *with* 2 *digits and* 90 *with three digits. Thus the number of palindromes in the set is* $9 + 9 + 90 = 108$. *The probability sought is* $\frac{108}{1000} = \frac{27}{250}$. ◀

165 Example A fair die is rolled three times and the scores added. What is the probability that the sum of the scores is 6?

▶**Solution:** *Let* A *be the event of obtaining a sum of* 6 *in three rolls, and let* Ω *be the sample space created when rolling a die thrice. The sample space has* $6^3 = 216$ *elements, since the first roll can land in* 6 *different ways, as can the second and third roll. To obtain a sum of* 6 *in three rolls, the die must have the following outcomes:*

$$\begin{aligned} A &= \{(2,2,2),(4,1,1),(1,4,1),(1,1,4),(1,2,3), \\ &\quad (1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\} \end{aligned}$$

and so $\mathbf{card}(A) = 10$. *Hence* $\mathbf{P}(A) = \frac{10}{216} = \frac{5}{108}$. ◀

166 Example Consider a standard deck of cards. Four cards are chosen at random without regards to order and without replacement. Then

[1] A *knave* is what refined people call a *jack*. Cf. Charles Dickens' *Great Expectations*.

❶ The sample space for this experiment has size

$$\binom{52}{4} = 270725.$$

❷ The probability of choosing the four kings is

$$\frac{\binom{4}{4}}{\binom{52}{4}} = \frac{1}{270725}.$$

❸ The probability of choosing four cards of the same face is

$$\frac{\binom{13}{1}\binom{4}{4}}{\binom{52}{4}} = \frac{13}{270725} = \frac{1}{20825}.$$

❹ The probability of choosing four cards of the same colour is

$$\frac{\binom{2}{1}\binom{26}{4}}{\binom{52}{4}} = \frac{(2)(14950)}{270725} = \frac{92}{833}.$$

❺ The probability of choosing four cards of the same suit is

$$\frac{\binom{4}{1}\binom{13}{4}}{\binom{52}{4}} = \frac{(4)(715)}{270725} = \frac{44}{4165}.$$

167 Example Consider again the situation in example 167, but this time order is taken into account, that is, say, you shuffle the cards, draw them one by one without replacement, and align them from left to right. Then

❶ The sample space for this experiment has size $52 \cdot 51 \cdot 50 \cdot 49 = 6497400$.

❷ The probability of choosing the four kings is

$$\frac{4 \cdot 3 \cdot 2 \cdot 1}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{1}{270725},$$

as before.

❸ One chooses the face for a card in 13 ways, and thus the probability of choosing four cards of the same face, using the previous probability, is

$$13\left(\frac{1}{270725}\right) = \frac{13}{270725} = \frac{1}{20825}.$$

❹ To choose four cards of the same colour, first choose the colour in 2 ways, and the four cards in $26 \cdot 25 \cdot 24 \cdot 23$ ways. The probability of choosing four cards of the same colour is thus

$$\frac{2 \cdot 26 \cdot 25 \cdot 24 \cdot 23}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{92}{833},$$

as before.

❺ To choose four cards of the same suit, one first chooses the suit in 4 ways, and then the cards in $13 \cdot 12 \cdot 11 \cdot 10$ ways.The probability of choosing four cards of the same suit is

$$\frac{4 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{52 \cdot 51 \cdot 50 \cdot 49} = \frac{44}{4165},$$

as before.

168 Example A number X is chosen at random from the series

$$2, 5, 8, 11 \ldots, 299$$

and another number Y is chosen from the series

$$3, 7, 11, \ldots, 399.$$

What is the probability $\mathbf{P}(X = Y)$?

▶**Solution:** *There are* 100 *terms in each of the arithmetic progressions. Hence we may choose* X *in* 100 *ways and* Y *in* 100 *ways. The size of the sample space for this experiment is thus* $100 \cdot 100 = 10000$. *Now we note that* 11 *is the smallest number that belongs to both progressions. Since the first progression has common difference* 3 *and the second progression has common difference* 4, *and since the least common multiple of* 3 *and* 14 *is* 12, *the progressions have in common numbers of the form*

$$11 + 12k.$$

We need the largest integer k *with*

$$11 + 12k \leq 299 \implies k = 24.$$

Therefore, the 25 *numbers*

$$11 = 11 + 12 \cdot 0, 23 = 11 + 12 \cdot 1, 35 = 11 + 12 \cdot 2, \ldots, 299 = 11 + 12 \cdot 24$$

belong to both progressions and the probability sought is

$$\frac{25}{10000} = \frac{1}{400}.$$

◀

169 Example (Poker Hands) A poker hand consists of 5 cards from a standard deck of 52 cards, and so there are $\binom{52}{5} = 2598960$ ways of selecting a poker hand. Various hands, and their numbers, are shewn below.

❶ **1 pair** occurs when you have one pair of faces of any suit, and none of the other faces match. For example, $A♣, A♢, 2♡, 4♣, 6♢$ is a pair. The number of ways of getting a pair is

$$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1098240$$

and so the probability of getting a pair is $\frac{1098240}{2598960} \approx 0.422569$.

❷ **2 pairs** occurs when you have 2 *different* pairs of faces of any suit, and the remaining card of a different face than the two pairs. For example, $A♣, A♢, 3♡, 3♢, 7♡$ is a 2 pair. The number of ways of getting two pairs is $\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123552$ and so the probability of getting 2 pairs is $\frac{123552}{2598960} \approx 0.047539$.

❸ **3 of a kind** occurs when you have three cards of the same face and the other two cards are from a different face. For example, $A♣, A♢, A♠, 3♠, 7♢$. The number of ways of getting a 3 of a kind is $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54912$ and so the probability of this event is $\frac{54912}{2598960} \approx 0.021128$.

❹ **straight** occurs when the faces are consecutive, but no four cards belong to the same suit, as in $2♣, 3♡, 4♠, 5♠, 6♢$. The number of ways of getting a straight is $10(4^5 - 4) = 10200$ and so the probability of this event is $\frac{10200}{2598960} \approx 0.003925$.

❺ **straight flush** occurs when one gets five consecutive cards of the same suit, as in $2♣, 3♣, 4♣, 5♣, 6♣$. The number of ways of getting this is $\binom{4}{1}10 = 40$, and the probability of this event is $\frac{40}{2598960} \approx 0.000015$.

❻ **royal flush** occurs when you have the ace, king, queen, knave, and 10 in the same suit. The number of ways of obtaining a royal flush is $\binom{4}{1}(1) = 4$ and so the probability of this event is $\frac{4}{2598960} \approx 0.0000015390$.

❼ **flush** occurs when you have five non-consecutive cards of the same suit, but neither a royal nor a straight flush, as in $2♣, 4♣, 7♣, 8♣, 10♣$. The number of ways of obtaining a flush is $\binom{4}{1}\binom{13}{5} - 40 = 5068$ and so the probability of this event is $\frac{5068}{2598960} \approx 0.00195$.

❽ **full house** occurs when 3 cards have the same face and the other two cards have the same face (different from the first three cards), as in $8♣, 8♠, 8♢, 7♡, 7♣$. The number of ways of getting this is

$$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3774$$

and so the probability of this event is $\frac{3774}{2598960} \approx 0.001441$.

❾ **4 of a kind** occurs when a face appears four times, as in $8♣, 8♠, 8♢, 8♡, 7♣$. The number of ways of getting this is

$$\binom{13}{1}\binom{4}{4}\binom{12}{1}\binom{4}{1} = 624,$$

and the probability for this event is $\frac{624}{2598960} \approx 0.00024$.

170 Example (The Birthday Problem) If there are n people in a classroom, what is the probability that no pair of them celebrates their birthday on the same day of the year?

►**Solution:** *To simplify assumptions, let us discard 29 February as a possible birthday and let us assume that a year has* 365 *days. There are* 365^n n*-tuples, each slot being the possibility* *of a day of the year for each person. The number of ways in which no two people have the same birthday is*

$$365 \cdot 364 \cdot 363 \cdots (365 - n + 1),$$

as the first person can have his birthday in 365 *days, the second in* 364 *days, etc. Thus if* A *is the event that no two people have the same birthday, then*

$$\mathbf{P}\left(A\right) = \frac{365 \cdot 364 \cdot 363 \cdots (365 - n + 1)}{365^n}.$$

The probability sought is

$$\mathbf{P}\left(A^c\right) = 1 - \mathbf{P}\left(A\right) = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365 - n + 1)}{365^n}.$$

A numerical computation shews that for $n = 23$, $\mathbf{P}\left(A\right) < \frac{1}{2}$, *and so* $\mathbf{P}\left(A^c\right) > \frac{1}{2}$. *This means that if there are* 23 *people in a room, the probability is better than* $\frac{1}{2}$ *that two will have the same birthday.* ◀

171 Example An urn has five blue and eight red marbles. Jack and Jill draw marbles, alternately and without replacement, until the first blue marble is drawn, which is considered a win. What is the probability that Jill will win?

▶**Solution:** *For each* $1 \leq k \leq 9$, *let* A_k *denote the event that the first blue ball appears on the* k^{th} *attempt. Since Jill draws on the* 2^{nd}, 4^{th}, 6^{th}, *and* 8^{th} *attempts, the probability of winning at these attempts are*

$$\mathbf{P}\left(A_2\right) = \frac{8 \cdot 5}{13 \cdot 12} = \frac{40}{156} = \frac{10}{39},$$

$$\mathbf{P}\left(A_4\right) = \frac{8 \cdot 7 \cdot 6 \cdot 5}{13 \cdot 12 \cdot 11 \cdot 10} = \frac{1,680}{17,160} = \frac{14}{143},$$

$$\mathbf{P}\left(A_6\right) = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 5}{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8} = \frac{33,600}{1,235,520} = \frac{35}{1287},$$

$$\mathbf{P}\left(A_8\right) = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 5}{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6} = \frac{201,600}{51,891,840} = \frac{5}{1287}.$$

As these events are mutually exclusive, the probability of Jill winning is

$$\mathbf{P}\left(\bigcup_{i=1}^{4} A_{2i}\right) = \sum_{i=1}^{4} P(A_{2i}) = \frac{496}{1287} \approx 0.3854.$$

◀

172 Example (Derangements) A hat contains three tickets, numbered 1, 2 and 3. The tickets are drawn from the box one at a time. Find the probability that the ordinal number of at least one ticket coincides with its own number.

▶**Solution:** *Let* $A_k, k = 1, 2, 3$ *be the event that when drawn from the hat, ticket* k *is the* k*-th chosen. We want*

$$\mathbf{P}\left(A_1 \cup A_2 \cup A_3\right).$$

By Inclusion-Exclusion for three sets (problem 3.2.8),

$$\begin{aligned}\mathbf{P}\left(A_1 \cup A_2 \cup A_3\right) &= \mathbf{P}\left(A_1\right) + \mathbf{P}\left(A_2\right) + \mathbf{P}\left(A_3\right) \\ &\quad - \mathbf{P}\left(A_1 \cap A_2\right) - \mathbf{P}\left(A_2 \cap A_3\right) - \mathbf{P}\left(A_3 \cap A_1\right) \\ &\quad + \mathbf{P}\left(A_1 \cap A_2 \cap A_3\right)\end{aligned}$$

By symmetry,

$$\mathbf{P}\left(A_1\right) = \mathbf{P}\left(A_2\right) = \mathbf{P}\left(A_3\right) = \frac{2!}{3!} = \frac{1}{3},$$

$$\mathbf{P}\left(A_1 \cap A_2\right) = \mathbf{P}\left(A_2 \cap A_3\right) = \mathbf{P}\left(A_3 \cap A_1\right) = \frac{1!}{3!} = \frac{1}{6},$$

$$\mathbf{P}\left(A_1 \cap A_2 \cap A_3\right) = \frac{1}{3!} = \frac{1}{6}.$$

The probability sought is finally

$$\mathbf{P}\left(A_1 \cup A_2 \cup A_3\right) = 3 \cdot \frac{1}{3} - 3 \cdot \frac{1}{6} + \frac{1}{6} = \frac{2}{3}.$$

◀

Homework

Problem 4.1.1 *There are 100 cards: 10 of each red—numbered 1 through 10; 20 white—numbered 1 through 20; 30 blue—numbered 1 through 30; and 40 magenta—numbered 1 through 40.*

❶ *Let R be the event of picking a red card. Find* $\mathbf{P}\left(R\right)$.

❷ *Let B be the event of picking a blue card. Find* $\mathbf{P}\left(B\right)$.

❸ *Let E be the event of picking a card with face value 11. Find* $\mathbf{P}\left(E\right)$.

❹ *Find* $\mathbf{P}\left(B \cup R\right)$.

❺ *Find* $\mathbf{P}\left(E \cap R\right)$.

❻ *Find* $\mathbf{P}\left(E \cap B\right)$.

❼ *Find* $\mathbf{P}\left(E \cup R\right)$.

❽ *Find* $\mathbf{P}\left(E \cup B\right)$.

❾ *Find* $\mathbf{P}\left(E \setminus B\right)$.

❿ *Find* $\mathbf{P}\left(B \setminus E\right)$.

Problem 4.1.2 *Find the chance of throwing at least one ace in a single throw of two dice.*

Problem 4.1.3 *Suppose n ordinary dice are rolled. What is the chance that at least one number appears more than once?*

Problem 4.1.4 *Phone numbers in a certain town are 7-digit numbers that do not start in 0, 1, or 9. What is the probability of getting a phone number in this town that is divisible by 5?*

Problem 4.1.5 *A hat contains 20 tickets, each with a different number from 1 to 20. If 4 tickets are drawn at random, what is the probability that the largest number is 15 and the smallest number is 9?*

Problem 4.1.6 *A box contains four \$10 bills, six \$5 bills, and two \$1 bills. Two bills are taken at random from the box without replacement. What is the probability that both bills will be of the same denomination?*

Problem 4.1.7 *A number N is chosen at random from $\{1, 2, \ldots, 25\}$. Find the probability that $N^2 + 1$ be divisible by 10.*

Problem 4.1.8 *An urn has 3 white marbles, 4 red marbles, and 5 blue marbles. Three marbles are drawn at once from the urn, and their colour noted. What is the probability that a marble of each colour is drawn?*

Problem 4.1.9 *Two cards are drawn at random from a standard deck. What is the probability that both are queens?*

Problem 4.1.10 *Four cards are drawn at random from a standard deck. What is the probability that two are red queens and two are spades? What is the probability that there are no hearts?*

Problem 4.1.11 *A number X is chosen at random from the set $\{1, 2, \ldots, 25\}$. Find the probability that when divided by 6 it leaves remainder 1.*

Problem 4.1.12 *A $3 \times 3 \times 3$ wooden cube is painted red and cut into twenty-seven $1 \times 1 \times 1$ smaller cubes. These cubes are mixed in a hat and one of them chosen at random. What is the probability that it has exactly 2 of its sides painted red?*

Problem 4.1.13 *A box contains 100 numbered lottery tickets, of which 10 are winning tickets. You start drawing tickets one at a time, until you have found a winning ticket. (a) What is the probability that you need to draw exactly 5 tickets to obtain a winning ticket, if the drawing is done without replacement? What is the probability that you need to draw exactly 5 tickets to obtain a winning ticket if the drawing is performed with replacement?*

Problem 4.1.14 *Twelve married couples (men and wives) end up in an island populated by savage cannibals. The cannibals each twelve people for dinner. What is the probability that exactly one member of each family is eaten?*

Problem 4.1.15 *Ten married communist couples from the English Department go in pilgrimage to Lenin's Tomb. Suddenly, a gang of capitalists raids them, and only six communitsts are able to escape. What is the probability that there are no married couples among the six escapees? Exactly one married couple? Exactly two married couples? Three married couples?*

Problem 4.1.16 *Three fair dice, a red, a white, and a blue one are tossed, and their scores registered in the random variables R, W, B respectively. What is the probability that $R \leq W \leq B$?*

Problem 4.1.17 *From a group of A males and B females a committee of C people will be chosen.*

❶ *What is the probability that there are exactly T females?*

❷ *What is the probability that at least $C - 2$ males will be chosen?*

❸ *What is the probability that at most 3 females will be chosen?*

❹ *What is the probability that Mary and Peter will be serving together in a committee?*

❺ *What is the probability that Mary and Peter will not be serving together?*

Problem 4.1.18 *A school has 7 men and 5 women on its faculty. What is the probability that women will outnumber men on a randomly selected five-member committee?*

Problem 4.1.19 *Five (distinguishable) camels and five (distinguishable) goats are lined up at random. What is the probability that all the camels are grouped together and all the camels are grouped together? What is the probability that either the camels are grouped together or the goats are grouped together?*

Problem 4.1.20 *Of the 120 students in a class, 30 speak Chinese, 50 speak Spanish, 75 speak French, 12 speak Spanish and Chinese, 30 speak Spanish and French, and 15 speak Chinese and French. Seven students speak all three languages. What is the probability that a randomly chosen student speaks none of these languages?*

Problem 4.1.21 *A box contains 3 red balls, 4 white balls, and 3 blue balls. Balls are drawn from the box one at a time, at random, without replacement. What is the probability that all three red balls will be drawn before any white ball is obtained?*

Problem 4.1.22 *Three fair dice are thrown at random.*

❶ *Find the probability of getting no 5 on the faces.*

❷ *Find the probability of getting at least one 5 on the faces.*

❸ *Find the probability of obtaining at least two faces with the same number.*

❹ *Find the probability that the sum of the points on the faces is even.*

Problem 4.1.23 *Six cards are drawn without replacement from a standard deck of cards. What is the probability that*

❶ *three are red and three are black?*

❷ *two are queens, two are aces, and two are kings?*

❸ *four have the same face (number or letter)?*

❹ *exactly four are from the same suit?*

❺ *there are no queens?*

Problem 4.1.24 *An ordinary fair die and a die whose faces have $2, 3, 4, 6, 7, 9$ dots but is otherwise balanced are tossed and the total noted. What is the probability that the sum of the dots shewing on the dice exceeds 9?*

Problem 4.1.25 (AHSME 1976) *A point in the plane, both of whose rectangular coordinates are integers with absolute value less than or equal to four, is chosen at random, with all such points having an equal probability of being chosen. What is the probability that the distance from the point to the origin is at most two units?*

Problem 4.1.26 *What is the probability that three randomly-selected people were born on different days of the week? (Assume that the chance of someone being born on a given day of the week is $1/7$).*

Problem 4.1.27 *Let k, N be positive integers. Find the probability that an integer chosen at random from $\{1, 2, \ldots, N\}$ be divisible by k.*

Problem 4.1.28 *What is the probability that a random integer taken from $\{1, 2, 3, \ldots, 100\}$ has no factors in common with 100?*

Problem 4.1.29 *A number N is chosen at random from $\{1, 2, \ldots, 25\}$. Find the probability that $N^2 - 1$ be divisible by 10.*

Problem 4.1.30 *Three integers are drawn at random and* without replacement *from the set of twenty integers $\{1, 2, \ldots, 20\}$. What is the probability that their sum be divisible by 3?*

Problem 4.1.31 *There are twenty guns in a row, and it is known that exactly three will fire. A person fires the guns, one after the other. What is the probability that he will have to try exactly seventeen guns in order to know which three will fire?*

Problem 4.1.32 *Two different numbers* X *and* Y *are chosen from* $\{1, 2, \ldots, 10\}$. *Find the probability that* $X^2 + Y^2 \leq 27$.

Problem 4.1.33 *Ten different numbers are chosen at random from the set of* 30 *integers* $\{1, 2, \ldots, 30\}$. *Find the probability that*

❶ *all the numbers are odd.*

❷ *exactly* 5 *numbers be divisible by* 3.

❸ *exactly* 5 *numbers are even, and exactly one of them is divisible by* 10.

Problem 4.1.34 *Two numbers* X *and* Y *are chosen at random, and with replacement, from the set*

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

Find the probability that $X^2 - Y^2$ *be divisible by* 3.

Problem 4.1.35 *Ali Baba has a farm . In the farm he has a herd of* 20 *animals,* 15 *are camels and the rest are sheep. Ahmed, sheik of the* Forty Thieves *steals* 5 *animals at night, without knowing what they are. What is the probability that exactly three of the five stolen animals are camels?*

Problem 4.1.36 *A student knows how to do* 15 *out of the* 20 *core problems for a given chapter. If the TA chooses* 3 *of the core problems at random for a quiz, what is the probability that the student knows how to do exactly* 2 *of them?*

Problem 4.1.37 *Ten equally-qualified applicants,* 6 *men and* 4 *women, apply for* 3 *lab technician positions. Unable to justify choosing any of the applicants over the others, the personnel director decides to select* 3 *at random. What is the probability that one man and two women will be chosen?*

Problem 4.1.38 *An urn has seven red and five green marbles. Five marbles are drawn out of the urn, without replacement. What is the probability that the green marbles outnumber the red ones?*

Problem 4.1.39 (MMPC 1992) *From the set*

$$\{1, 2, \ldots, n\},$$

k *distinct integers are selected at random and arranged in numerical order (lowest to highest). Let* $\mathbf{P}\left(i, r, k, n\right)$ *denote the probability that integer* i *is in position* r. *For example, observe that* $\mathbf{P}\left(1, 2, k, n\right) = 0$ *and* $\mathbf{P}\left(2, 1, 6, 10\right) = 4/15$. *Find a general formula for* $\mathbf{P}\left(i, r, k, n\right)$.

Problem 4.1.40 *There are two winning tickets amongst ten tickets available. Determine the probability that (a) one, (b) both tickets will be among five tickets selected at random.*

Problem 4.1.41 *Find the chance of throwing more that* 15 *in a single throw of three dice.*

Problem 4.1.42 *Little Edna is playing with the four letters of her name, arranging them at random in a row. What is the probability that the two vowels come together?*

Problem 4.1.43 (Galileo's Paradox) *Three distinguishable fair dice are thrown (say, one red, one blue, and one white). Observe that*

$$\begin{aligned} 9 &= 1+2+6 \\ &= 1+3+5 \\ &= 1+4+4 \\ &= 2+2+5 \\ &= 2+3+4 \\ &= 3+3+3, \end{aligned}$$

and

$$\begin{aligned} 10 &= 1+3+6 \\ &= 1+4+5 \\ &= 2+2+6 \\ &= 2+3+5 \\ &= 2+4+4 \\ &= 3+3+4. \end{aligned}$$

The probability that a sum S *of* 9 *appears is lower than the probability that a sum of* 10 *appears. Explain why and find these probabilities.*

Problem 4.1.44 *Five people entered the lift cabin on the ground floor of an* 8*-floor building (this includes the ground floor). Suppose each of them, independently and with equal probability, can leave the cabin at any of the other seven floors. Find out the probability of all five people leaving at different floors.*

Problem 4.1.45 (AHSME 1984) *A box contains* 11 *balls, numbered* $1, 2, \ldots 11$. *If six balls are drawn simultaneously at random, find the probability that the sum of the numbers on the balls drawn is odd.*

Problem 4.1.46 *A hat contains* 7 *tickets numbered* 1 *through* 7. *Three are chosen at random. What is the probability that their product be an odd integer?*

Problem 4.1.47 (AHSME 1986) *Six distinct integers are chosen at random from $\{1, 2, 3, \ldots, 10\}$. What is the probability that, among those selected, the second smallest is 3?*

Problem 4.1.48 *An urn contains n black and n white balls. Three balls are chosen from the urn at random and without replacement. What is the value of n if the probability is $\frac{1}{12}$ that all three balls are white?*

Problem 4.1.49 *A standard deck is shuffled and the cards are distributed to four players, each one holding thirteen cards. What is the probability that each has an ace?*

Problem 4.1.50 *Twelve cards numbered 1 through 12 are thoroughly shuffled and distributed to three players so that each receives four cards. What is the probability that one of the players receives the three lowest cards (1, 2, and 3)?*

Problem 4.1.51 *A fair die is tossed twice in succession. Let $\mathbf{A}$ denote the first score and $\mathbf{B}$ the second score. Consider the quadratic equation*

$$x^2 + \mathbf{A}x + \mathbf{B} = 0.$$

Find the probability that

❶ *the equation has 2 distinct roots.*

❷ *the equation has a double root.*

❸ *$x = -3$ be a root of the equation,*

❹ *$x = 3$ be a root of the equation.*

Problem 4.1.52 *An urn contains $3n$ counters: n red, numbered 1 through n, n white, numbered 1 through n, and n blue, numbered 1 through n. Two counters are to be drawn at random without replacement. What is the probability that both counters will be of the same colour or bear the same number?*

Problem 4.1.53 (AIME 1984) *A gardener plants three maple trees, four oak trees and five birch trees in a row. He plants them in random order, each arrangement being equally likely. Let m/n in lowest terms be the probability that no two birch trees are next to each other. Find $m+n$.*

Problem 4.1.54 *Five fair dice are thrown. What is the probability that a full house in thrown (that is, where two dice shew one number and the other three dice shew a second number)?*

Problem 4.1.55 *If thirteen cards are randomly chosen without replacement from an ordinary deck of cards, what is the probability of obtaining exactly three aces?*

Problem 4.1.56 *Mrs. Flowers plants rosebushes in a row. Eight of the bushes are white and two are red, and she plants them in a random order. What is the probability that she will consecutively plant seven or more white bushes?*

Problem 4.1.57 *Let A, B, C be the outcomes of three distinguishable fair dice and consider the system*

$$Ax - By = C; \qquad x - 2y = 3.$$

Find the following probabilities

1. *that the system has no solution.*
2. *that the system has infinitely many solutions.*
3. *that the system has exactly one solution.*
4. *that the system has the unique solution $x = 3, y = 0$.*

4.2 Binomial Random Variables

173 Definition A random variable $\mathbf{X}$ has a binomial probability distribution if

$$\mathbf{P}(\mathbf{X} = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

where n is the number of independent trials, p is the probability of success in one trial, and k is the number of successes.

Thus a binomial random variable counts the number of successes in a sequence of independent trials.

Since

$$\sum_{k=0}^{n} \mathbf{P}(\mathbf{X} = k) = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1,$$

this is a bonafide random variable.

A few histograms for varying n and p follow.

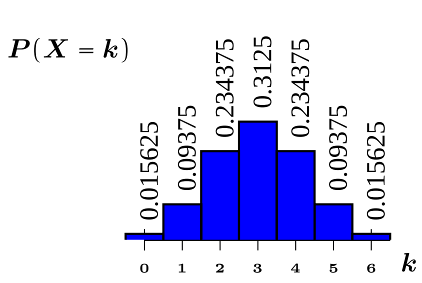

Figure 4.1: $n = 6, p = \frac{1}{2}$.

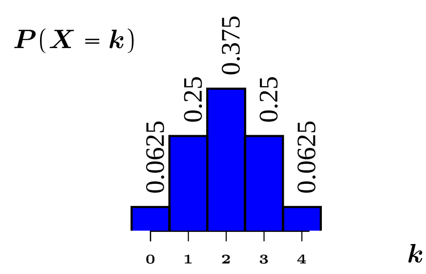

Figure 4.2: $n = 4, p = \frac{1}{2}$.

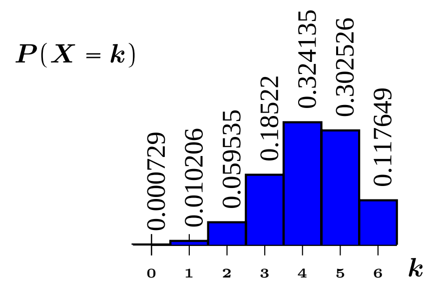

Figure 4.3: $n = 6, p = \frac{7}{10}$.

174 Example A fair coin is tossed 5 times.

❶ Find the probability of obtaining 3 heads.

❷ Find the probability of obtaining 3 tails.

❸ Find the probability of obtaining at most one head.

►**Solution:**

❶ *Let* X *be the random variables counting the number of heads. Here* $p = 1 - p = \dfrac{1}{2}$. *Hence*

$$P(X = 3) = \binom{5}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2 = \frac{5}{16}.$$

❷ *Obtaining* 3 *tails is equivalent to obtaining* 2 *heads, hence the probability sought is*

$$P(X = 2) = \binom{5}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 = \frac{5}{16}.$$

❸ *This is the probability of obtaining no heads or one head:*

$$\begin{aligned} P(X = 0) + P(X = 1) &= \binom{5}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5 + \binom{5}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4 \\ &= \frac{1}{32} + \frac{5}{32} \\ &= \frac{3}{16}. \end{aligned}$$

◄

175 Example A multiple-choice exam consists of 10 questions, and each question has four choices. You are clueless about this exam and hence, guessing the answer. It is assumed that for every question one, and only one of the choices is the correct answer.

❶ Find n, the number of trials, p, the probability of success, and $1 - p$, the probability of failure.

❷ Find the probability of answering exactly 7 questions right.

❸ Find the probability of answering 8 or more questions right.

❹ Find the probability of answering at most one question.

▶**Solution:**

❶ *Clearly* $n = 10$, $p = \frac{1}{4}$, *and also,* $1 - p = \frac{3}{4}$.

❷ *Let* $\mathbf{X}$ *be the random variables counting the number of right questions. Then*

$$\mathbf{P}(\mathbf{X} = 7) = \binom{10}{7}\left(\frac{1}{4}\right)^7\left(\frac{3}{4}\right)^3 = \frac{405}{131072}.$$

❸ *This is the probability of answering* 8 *or* 9 *or* 10 *questions right, so it is*

$$\begin{aligned}\mathbf{P}(\mathbf{X} = 8) + \mathbf{P}(\mathbf{X} = 9) + \mathbf{P}(\mathbf{X} = 10) &= \binom{10}{8}\left(\frac{1}{4}\right)^8\left(\frac{3}{4}\right)^2 \\ &\quad + \binom{10}{9}\left(\frac{1}{4}\right)^9\left(\frac{3}{4}\right)^1 \\ &\quad + \binom{10}{10}\left(\frac{1}{4}\right)^{10}\left(\frac{3}{4}\right)^0 \\ &= \frac{405}{1048576} + \frac{15}{524288} + \frac{1}{1048576} \\ &= \frac{109}{262144}.\end{aligned}$$

◀

176 Example A die is rolled repeatedly. Let $\mathbf{X}$ denote the number of the roll at which the third six occurs. Find $\mathbf{P}(\mathbf{X} > 1000)$.

▶**Solution:** *The desired probability is the same as the probability that there are at most two sixes within the first* 1000 *rolls:*

$$\mathbf{P}(\mathbf{X} > 1000) = \left(\frac{5}{6}\right)^{1000} + \binom{1000}{1}\left(\frac{5}{6}\right)^{999}\left(\frac{1}{6}\right)^1 + \binom{1000}{2}\left(\frac{5}{6}\right)^{998}\left(\frac{1}{6}\right)^2.$$

◀

Homework

Problem 4.2.1 *When two fair coins are tossed, what is the probability of getting no heads exactly four times in five tosses?*

Problem 4.2.2 *A coin is loaded so that* $\mathbf{P}(H) = \frac{3}{4}$ *and* $\mathbf{P}(T) = \frac{1}{4}$. *The coin is flipped* 5 *times and its outcome recorded. Find the probability that heads turns up at least once.*

Problem 4.2.3 *A fair coin is to be flipped* 1000 *times. What is the probability that the number of heads exceeds the number of tails?*

Problem 4.2.4 *In the world series of foosball, a five-game match is played, and the player who wins the most games is the champion. The probability of Player A winning any given game against player B is constant and equals* $\frac{1}{3}$. *What is the probability that Player A will be the champion? You may assume that all five games are played, even when a player wins three of the first five games.*

Problem 4.2.5 *In a certain game John's skill is to Peter's as* 3 *to* 2. *Find the chance of John winning* 3 *games at least out of* 5.

Problem 4.2.6 *A coin whose faces are marked* 2 *and* 3 *is thrown* 5 *times. What is the chance of obtaining a total* *of* 12?

Problem 4.2.7 *A calculator has a random number generator button which, when pushed displays a random digit* $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$*. The button is pushed four times. Assuming the numbers generated are independent, what is the probability of obtaining the digits of* 2007*, in any order?*

4.3 Geometric Random Variables

177 Definition (Geometric Random Variable) Let $0 < p < 1$. A random variable is said to have a *geometric or Pascal* distribution if

$$\mathbf{P}(X = k) = (1-p)^{k-1}p, \qquad k = 1, 2, 3, \ldots.$$

Thus the random variable X counts the number of trials necessary until success occurs.

Since

$$\sum_{k=1}^{\infty} \mathbf{P}(X = k) = \sum_{k=1}^{\infty}(1-p)^{k-1}p = \frac{p}{1-(1-p)} = 1,$$

this is a bonafide random variable.

Observe that

$$\mathbf{P}(X \geq k) = (1-p)^{k-1}, \qquad k = 1, 2, 3, \ldots, \tag{4.2}$$

since the probability that at least k trials are necessary for success is equal to the probability that the first $k-1$ trials are failures.

178 Example An urn contains 5 white, 4 black, and 1 red marble. Marbles are drawn, *with replacement*, until a red one is found. If X is the random variable counting the number of trials until a red marble appears, then

❶ $\mathbf{P}(X = 1) = \frac{1}{10}$ is the probability that the marble appears on the first trial.

❷ $\mathbf{P}(X = 2) = \frac{9}{10} \cdot \frac{1}{10} = \frac{9}{100}$ is the probability that the red marble appears on the second trial.

❸ $\mathbf{P}(X = k) = \frac{9^{k-1}}{10^k}$ is the probability that the marble appears on the k-th trial.

179 Example A drunk has five keys in his key-chain, and an only one will start the car [2] He tries each key until he finds the right one (he is so drunk that he may repeat the wrong key several times), then he starts his car and (by cheer luck), arrives home safely, where his wife is waiting for him, frying pan in hand. If X is the random variable counting the number of trials until he find the right key, then

❶ $\mathbf{P}(X = 1) = \frac{1}{5}$ is the probability that he finds the key on the first trial.

❷ $\mathbf{P}(X = 2) = \frac{4}{5} \cdot \frac{1}{5} = \frac{4}{25}$ is the probability that he finds the key on the second trial.

❸ $\mathbf{P}(X = 3) = \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} = \frac{16}{125}$ is the probability that he finds the key on the third trial.

❹ $\mathbf{P}(X = 4) = \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} = \frac{64}{625}$ is the probability that he finds the key on the fourth trial.

❺ $\mathbf{P}(X = 5) = \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{1}{5} = \frac{256}{3125}$ is the probability that he finds the key on the fifth trial.

[2] Caution: don't drink and drive!

❻ $\mathbf{P}(\mathbf{X}=6)=\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}=\frac{1024}{15625}$ is the probability that he finds the key on the sixth trial.

180 Example An urn contains 5 white, 4 black, and 1 red marble. Marbles are drawn, *with replacement*, until a red one is found. If $\mathbf{X}$ is the random variable counting the number of trials until the red marble appears.

❶ Find the probability that it takes at most 3 trials to obtain a red marble.

❷ Find the probability that it takes more than 3 trials to obtain a red marble.

►**Solution:**

❶ *This is asking for* $\mathbf{P}(\mathbf{X}=1)+\mathbf{P}(\mathbf{X}=2)+\mathbf{P}(\mathbf{X}=3)=\frac{1}{10}+\frac{9}{100}+\frac{81}{1000}=\frac{271}{1000}$.

❷ *This is asking for the infinite geometric sum*

$$\mathbf{P}(\mathbf{X}>3)=\sum_{k=4}^{\infty}\mathbf{P}(\mathbf{X}=k)=\sum_{k=4}^{\infty}\frac{9^{k-1}}{10^k}.$$

We can sum this directly, or we may resort to the fact that the event "more than 3 trials" is complementary to the event "at most 3 trials." Thus

$$\mathbf{P}(\mathbf{X}>3)=1-(\mathbf{P}(\mathbf{X}=1)+\mathbf{P}(\mathbf{X}=2)+\mathbf{P}(\mathbf{X}=3))=1-\frac{271}{1000}=\frac{729}{1000}.$$

We may also resort to (4.2) by noticing that

$$\mathbf{P}(\mathbf{X}>3)=\mathbf{P}(\mathbf{X}\geq 4)=\left(\frac{9}{10}\right)^{4-1}=\frac{729}{1000}.$$

◄

181 Example Three people, **X, Y, Z**, in order, roll a fair die. The first one to roll an even number wins and the game is ended. What is the probability that **X** will win?

►**Solution:** *We have*

$$\begin{aligned}\mathbf{P}(X\text{ wins}) &= \mathbf{P}(X\text{ wins on the first trial})\\ &\quad +\mathbf{P}(X\text{ wins on the fourth trial})\\ &\quad +\mathbf{P}(X\text{ wins on the seventh trial})+\cdots\\ &= \frac{1}{2}+\frac{1}{2}\left(\frac{1}{2}\right)^3+\frac{1}{2}\left(\frac{1}{2}\right)^6+\cdots\\ &= \frac{\frac{1}{2}}{1-\frac{1}{2^3}}\\ &= \frac{4}{7}.\end{aligned}$$

A different solution was given in example 155. ◄

182 Example A sequence of independent trials is performed by rolling a pair of fair dice. What is the probability that an 8 will be rolled before rolling a 7?

▶**Solution:** *The probability of rolling an* 8 *is* $\frac{5}{36}$ *and the probability of rolling a* 7 *is* $\frac{6}{36}$. *Let* A_n *be the event that no* 8 *or* 7 *appears on the first* $n-1$ *trials and that a* 8 *appears on the* n*th trial. Since the trials are independent,*

$$\mathbf{P}\left(A_n\right) = \left(1 - \frac{11}{36}\right)^{n-1}\frac{5}{36} = \left(\frac{25}{36}\right)^{n-1}\frac{5}{36}.$$

The probability sought is

$$\mathbf{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty}\mathbf{P}\left(A_n\right) = \sum_{n=1}^{\infty}\left(\frac{25}{36}\right)^{n-1}\frac{5}{36} = \frac{5}{11}.$$

A different solution to this problem will be given in example 154 ◀

Homework

Problem 4.3.1 *An urn has three red marbles and two white ones. Homer and Marge play alternately (Homer first, then Marge, then Homer, etc.) drawing marbles* with replacement *until one of them draws a white one, and then the game ends. What is the probability that Homer will eventually win?*

Problem 4.3.2 *Two people, X, Y, in order, roll a die. The first one to roll either a* 3 *or a* 6 *wins and the game is ended.*

❶ *What is the probability of throwing either a* 3 *or a* 6*?*

❷ *What is the probability that Y will win on the second throw?*

❸ *What is the probability that Y will win on the fourth throw?*

❹ *What is the probability that Y will win?*

Problem 4.3.3 *Six persons throw for a stake, which is to be won by the one who first throws head with a penny; if they throw in succession, find the chance of the fourth person.*

Problem 4.3.4 *Consider the following experiment: A fair coin is flipped until heads appear, and the number of flips is recorded. If this experiment is repeated three times, what is the probability that the result (number of flips) is the same all three times?*

Problem 4.3.5 *A game consists of looking for* 7*'s in rolls of a pair of dice. What is the probability that it takes ten rolls in order to observe eight* 7*'s?*

4.4 Negative Binomial Random Variables

183 Definition Consider a sequence of identical and independent trials, with individual probability of success p. A random variable $\mathbf{X}$ has a *negative binomial* distribution, if it measures the probability of obtaining the r success in the x trial:

$$\mathbf{P}\left(\mathbf{X} = x\right) = \binom{x-1}{r-1}p^r(1-p)^{x-r}, \qquad x \geq r.$$

This makes sense, since for the r-th success to occur on the x-th trial, the first $r-1$ successes must occur somewhere during the $x-1$ first trials, with probability $\binom{x-1}{r-1}p^{r-1}(1-p)^{x-r}$ and the x-th trial must be a success with probability p.

184 Example (The Problem of the Points) Consider a number of independent trials performed, each with probability p of success. What is the probability of having r successes occurring before m failures?

▶**Solution:** *Observe that* r *successes occur before* m *failure, if the last success occurs no later than on the* $r+m-1$*-th trial. The desired probability is thus*

$$\sum_{n=r}^{r+m-1}\binom{n-1}{r-1}p^r(1-p)^{n-r}.$$

◀

185 Example (Banach Matchbox Problem) A mathematician carries at all times two matchboxes, one in his left pocket, and the other in his right pocket, each having initially N matches. Each time he needs a match he reaches for either pocket with equal probability. At the moment when he first notices that one of the matchboxes is empty, what is the probability that there are k matches $(0 \leq k \leq N)$ matches in the other pocket?

▶**Solution:** *Let* $\mathbf{P}(\mathbf{L} = k)$ *be the probability that there are* k *matches in the left pocket when he first discovers that the right pocket is empty. Observe that this occurs on the* $N + 1 + N - k$ *trial. Thus*

$$\mathbf{P}(L = k) = \binom{2N - k}{N}\left(\frac{1}{2}\right)^{2N-k+1},$$

and the probability sought is hence

$$2\mathbf{P}(L = k) = \binom{2N - k}{N}\left(\frac{1}{2}\right)^{2N-k}.$$

◀

Homework

Problem 4.4.1 *A cholera patient lives in a building where his toilet stall has two dispensers (one on the left and another one on the right of the toilet). Initially each roll has* 100 *sheets of paper. Each time he visits the toilet (which is often, given that he has cholera), he chooses a dispenser at random and uses one sheet (OK, these sheets are very large, but let's continue with the problem. . .). At a certain moment, he first realises that one of the dispensers is empty. What is the probability that the other roll of paper has* 25 *sheets?*

5 Uniform Continuous Random Variables

186 Definition Let $\mathcal{C}$ be a body in one dimension (respectively, two, or three dimensions) having positive length $\text{meas}\left(\mathcal{C}\right)$ (respectively, positive area or positive volume). A *continuous random variable* $\mathbf{X}$ *defined on* $\mathcal{C}$ is a random variable with probability given by

$$\mathbf{P}\left(\mathbf{X} \in A\right) = \frac{\text{meas}\left(\mathcal{A}\right)}{\text{meas}\left(\mathcal{C}\right)}.$$

This means that the probability of of an event is proportional to the length (respectively, area or volume) that this body $\mathcal{A}$ occupies in $\mathcal{C}$.

187 Example A dartboard is made of three concentric circles of radii 3, 5, and 7, as in figure 5.1. A dart is thrown and it is assumed that it always lands on the dartboard. Here the inner circle is blue, the middle ring is white and the outer ring is red.

❶ The size of the sample space for this experiment is $\pi(7)^2 = 49\pi$.

❷ The probability of landing on blue is $\dfrac{\pi(3)^2}{49\pi} = \dfrac{9}{49}$.

❸ The probability of landing on white is $\dfrac{\pi(5)^2 - \pi(3)^2}{49\pi} = \dfrac{16}{49}$.

❹ The probability of landing on red is $\dfrac{\pi(7)^2 - \pi(5)^2}{49\pi} = \dfrac{24}{49}$.

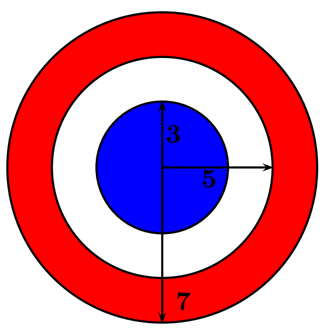

Figure 5.1: Example 187

188 Definition The *distribution function* F of a random variable $\mathbf{X}$ is $F(a) = \mathbf{P}\left(\mathbf{X} \leq a\right)$.

A distribution function satisfies

❶ If $a < b$ then $F(a) \leq F(b)$.

❷ $\lim_{a\to-\infty} F(a) = 0$,

❸ $\lim_{a\to+\infty} F(a) = 1$.

189 Example A random variable $\mathbf{X}$ has probability distribution

$$\mathbf{P}\left(\mathbf{X} \leq x\right) = \kappa \mathrm{meas}\left(x\right),$$

where $\mathrm{meas}\left(x\right)$ denotes the area of the polygon in figure 189 up to abscissa x. Assume that $\mathbf{P}\left(\mathbf{X} \leq 0\right) = 0$ and that $\mathbf{P}\left(\mathbf{X} \leq 6\right) = 1$.

❶ Find the value of κ.

❷ Find $\mathbf{P}\left(\mathbf{X} \leq 2\right)$.

❸ Find $\mathbf{P}\left(3 \leq \mathbf{X} \leq 4\right)$.

▶**Solution:**

❶ *The figure is composed of a rectangle and a triangle, and its total area is* $(4)(2)+\dfrac{1}{2}(4)(5) = 8 + 10 = 18$. *Since* $1 = \mathbf{P}\left(\mathbf{X} \leq 6\right) = \kappa \mathrm{meas}\left(6\right) = 18\kappa$ *we have* $\kappa = \dfrac{1}{18}$.

❷ $\mathbf{P}\left(\mathbf{X} \leq 2\right)$ *is the area of the rectangle between* $x = 0$ *and* $x = 2$ *and so* $\mathbf{P}\left(\mathbf{X} \leq 2\right) = \dfrac{1}{18}(8) = \dfrac{4}{9}$.

❸ $\mathbf{P}\left(3 \leq \mathbf{X} \leq 4\right)$ *is the area of a trapezoid of bases of length* 2.5 *and* 5 *and height* 1, *thus*

$$\mathbf{P}\left(3 \leq \mathbf{X} \leq 4\right) = \frac{1}{18} \cdot \frac{1}{2}\left(\frac{5}{2} + 5\right) = \frac{5}{24}.$$

◀

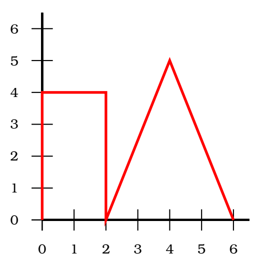

Figure 5.2: Example 189

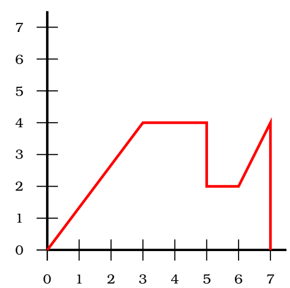

Figure 5.3: Example 190

190 Example A random variable $\mathbf{X}$ has probability distribution

$$\mathbf{P}\left(\mathbf{X} \leq x\right) = \kappa A(x),$$

where $A(x)$ denotes the area of the polygon in figure 190 up to abscissa x. Assume that $\mathbf{P}\left(\mathbf{X} \leq 0\right) = 0$ and that $\mathbf{P}\left(\mathbf{X} \leq 7\right) = 1$.

❶ Find the value of κ.

❷ Find $\mathbf{P}\left(\mathbf{X} \leq 3\right)$.

❸ Find $\mathbf{P}\left(\mathbf{X} \leq 5\right)$.

❹ Find $\mathbf{P}\left(\mathbf{X} \leq 6\right)$.

❺ Find $\mathbf{P}(1 \leq \mathbf{X} \leq 2)$.

❻ Find $\mathbf{P}(\mathbf{X} \geq 6)$.

❼ Find a median m of $\mathbf{X}$, that is, an abscissa that simultaneously satisfies $\mathbf{P}(\mathbf{X} \geq m) \geq \frac{1}{2}$ and $\mathbf{P}(\mathbf{X} \leq m) \geq \frac{1}{2}$.

▶**Solution:**

❶ *In* $[0;3]$ *the figure is a triangle with base* 3 *and height* 4*, and so its area is* 6*. In* $[3;5]$ *the figure is a rectangle, with base* 2 *and height* 4*, and so its area is* 8*. In* $[5;6]$ *the figure is a rectangle, with base* 1 *and height* 2*, and so its area is* 2*. In* $[6;7]$ *the figure is a trapezium, with bases* 2 *and* 4 *and height* 1*, and so its area is* 3*. Adding all these areas together we obtain* $6+8+2+3=19$*. Since*

$$1 = \mathbf{P}(\mathbf{X} \leq 7) = \kappa A(7) = \kappa(19),$$

we obtain $\kappa = \frac{1}{19}$.

❷ *This measures the proportion of the area enclosed by the triangle, and so* $\mathbf{P}(\mathbf{X} \leq 3) = \frac{6}{19}$.

❸ *This measures the proportion of the area enclosed by the triangle and the first rectangle, and so* $\mathbf{P}(\mathbf{X} \leq 5) = \frac{6+8}{19} = \frac{14}{19}$.

❹ *This measures the proportion of the area enclosed by the triangle, and the first and second rectangle, and so* $\mathbf{P}(\mathbf{X} \leq 6) = \frac{6+8+2}{19} = \frac{16}{19}$.

❺ *The area sought is that of a trapezium. One (of many possible ways to obtain this) is to observe that*

$$\mathbf{P}(1 \leq \mathbf{X} \leq 2) = \mathbf{P}(\mathbf{X} \leq 2) - \mathbf{P}(\mathbf{X} \leq 1).$$

To find $\mathbf{P}(\mathbf{X} \leq 2)$ *observe that the triangle with base on* $[0;4]$ *is similar to the one with base on* $[0;2]$*. If its height is* h_1 *then* $\frac{h_1}{4} = \frac{2}{3}$*, whence* $h_1 = \frac{8}{3}$*, and*

$$\mathbf{P}(\mathbf{X} \leq 2) = \frac{1}{19}\left(\frac{1}{2}\cdot 2 \cdot \frac{8}{3}\right) = \frac{8}{57}.$$

To find $\mathbf{P}(\mathbf{X} \leq 1)$ *observe that the triangle with base on* $[0;4]$ *is similar to the one with base on* $[0;1]$*. If its height is* h_2 *then* $\frac{h_2}{4} = \frac{1}{3}$*, whence* $h_2 = \frac{4}{3}$*, and*

$$\mathbf{P}(\mathbf{X} \leq 1) = \frac{1}{19}\left(\frac{1}{2}\cdot 1 \cdot \frac{4}{3}\right) = \frac{2}{57}.$$

Finally,

$$\mathbf{P}(1 \leq \mathbf{X} \leq 2) = \mathbf{P}(\mathbf{X} \leq 2) - \mathbf{P}(\mathbf{X} \leq 1) = \frac{8}{57} - \frac{2}{57} = \frac{2}{19}.$$

❻ *Since the curve does not extend from* $x = 7$*, we have*

$$\mathbf{P}(\mathbf{X} \geq 6) = \mathbf{P}(6 \leq \mathbf{X} \leq 7) = \frac{2}{19}.$$

❼ *From parts (2) and (3),* $3 < m < 5$*. For* m *in this range, a rectangle with base* $m-3$ *and height* 4 *has area* $4(m-3)$*. Thus we need to solve*

$$\frac{1}{2} = \mathbf{P}(\mathbf{X} \leq m) = \frac{6+4(m-3)}{19},$$

which implies

$$\frac{19}{2} = 6 + 4(m-3) \implies m = \frac{31}{8} = 3.875.$$

◀

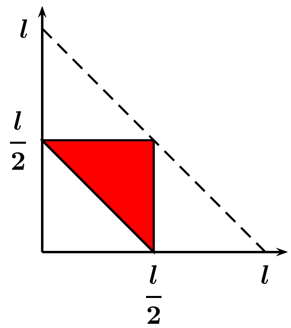

Figure 5.4: Example 191

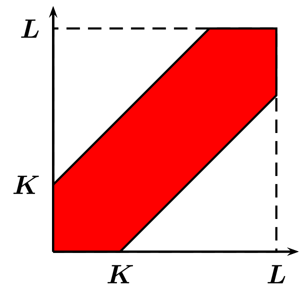

Figure 5.5: Example 192

191 Example A rod of length l is broken into three parts. What is the probability that these parts form a triangle?

▶**Solution:** *Let x, y, and $l-x-y$ be the lengths of the three parts of the rod. If these parts are to form a triangle, then the triangle inequality must be satisfied, that is, the sum of any two sides of the triangle must be greater than the third. So we simultaneously must have*

$$x + y > l - x - y \implies x + y > \frac{l}{2},$$

$$x + l - x - y > y \implies y < \frac{l}{2},$$

$$y + l - x - y > x \implies x < \frac{l}{2}.$$

Since trivially $0 \leq x + y \leq l$, what we are asking is for the ratio of the area of the region

$$\mathcal{A} = \{(x, y) : 0 < x < \frac{l}{2}, 0 < y < \frac{l}{2}, x + y > \frac{l}{2}\}$$

to that of the triangle with vertices at $(0,0)$, $(l,0)$ and $(0,l)$. This is depicted in figure 5.4. The desired probability is thus

$$\frac{\frac{l^2}{8}}{\frac{l^2}{2}} = \frac{1}{4}.$$

◀

192 Example Two points are chosen at random on a segment of length L. Find the probability that the distance between the points is at most K, where $0 < K < L$.

▶**Solution:** *Let the points chosen be X and Y with $0 \leq \mathrm{X} \leq L$, $0 \leq \mathrm{Y} \leq L$, as in figure 5.5. The distance of the points is at most K if $|\mathrm{X} - \mathrm{Y}| \leq K$, that is*

$$\mathrm{X} - K \leq \mathrm{Y} \leq \mathrm{X} + K.$$

The required probability is the ratio of the area shaded inside the square to the area of the square:

$$\frac{L^2 - 2\frac{(K-L)^2}{2}}{L^2} = \frac{K(2L-K)}{L^2}.$$

◀

Homework

Problem 5.0.2 *A point (x, y) are chosen at random on a rectangle 5 feet by 3 feet. What is the probability that their coordinates are within one foot of each other?*

Problem 5.0.3 *The amount 2.5 is split into two nonnegative real numbers uniformly at random, for instance, into 2.03 and 0.47 or into $2.5 - \sqrt{3}$ and $\sqrt{3}$. Then each of the parts is rounded to the nearest integer, for instance 2 and 0 in the first case above and 1 and 2 in the second. What is the probability that the two numbers so obtained will add up to 3?*

6 Expectation and Variance

6.1 Expectation and Variance

193 Definition Let X be a discrete random variable taking on the values $x_1, x_2, \ldots, x_k, \ldots$. The *mean value or expectation of* X, denoted by $E(X)$ is defined by

$$E(X) = \sum_{k=1}^{\infty} x_k P(X = x_k).$$

194 Example A player is paid \$1 for getting heads when flipping a fair coin and he loses \$0.50 if he gets tails.

❶ Let G denote the random variables measuring his gain. What is the image of G?

❷ Find the distribution of G.

❸ What is his expected gain in the long run?

►**Solution:**

❶ G *can either be* 1 *or* -0.50.

❷ $P(G = 1) = \frac{1}{2}$, *and* $P(G = -0.5) = \frac{1}{2}$,

❸
$$E(G) = 1P(G = 1) - 0.5P(G = 0.5) = \frac{3}{4}.$$

◄

195 Example A player is playing with a fair die. He gets \$2 if the die lands on a prime, he gets nothing if the die lands on 1, and he loses \$1 if the die lands on a composite number.

❶ Let G denote the random variables measuring his gain. What is the image of G?

❷ Find the distribution of G.

❸ What is his expected gain in the long run?

►**Solution:**

❶ G *can either be* 2, 0 *or* -1.

❷ $P(G = 2) = \frac{3}{6}$, $P(G = 0) = \frac{1}{6}$, *and* $P(G = -1) = \frac{2}{6}$.

❸
$$E(G) = 2P(G = 2) + 0P(G = 0) - 1P(G = -1) = \frac{6}{6} + 0 - \frac{2}{6} = \frac{2}{3}.$$

◄

196 Example A player chooses, without replacement, two cards from a standard deck of cards. He gets \$2 for each heart suit card.

❶ Let $\mathbf{G}$ denote the random variables measuring his gain. What is the image of $\mathbf{G}$?

❷ Find the distribution of $\mathbf{G}$.

❸ What is his expected gain in the long run?

►**Solution:**

❶ $\mathbf{G}$ *can either be* $\mathbf{0}$, $\mathbf{1}$ *or* $\mathbf{2}$.

❷

$$\mathbf{P}(\mathbf{G}=\mathbf{0}) = \frac{\binom{13}{0}\binom{39}{2}}{\binom{52}{2}} = \frac{19}{34},$$

$$\mathbf{P}(\mathbf{G}=\mathbf{1}) = \frac{\binom{13}{1}\binom{39}{1}}{\binom{52}{2}} = \frac{13}{34},$$

and

$$\mathbf{P}(\mathbf{G}=\mathbf{2}) = \frac{\binom{13}{2}\binom{39}{0}}{\binom{52}{2}} = \frac{1}{17}.$$

❸

$$\mathbf{E}(\mathbf{G}) = \mathbf{0P}(\mathbf{G}=\mathbf{0}) + \mathbf{1P}(\mathbf{G}=\mathbf{1}) + \mathbf{2P}(\mathbf{G}=\mathbf{2}) = \mathbf{0} + \frac{13}{34} + \frac{2}{17} = \frac{1}{2}.$$

◄

197 Definition Let $\mathbf{X}$ be a discrete random variable taking on the values $x_1, x_2, \ldots, x_k, \ldots$. Then $\mathbf{E}(\mathbf{X}^2)$ is defined by

$$\mathbf{E}(\mathbf{X}^2) = \sum_{k=1}^{\infty} x_k^2 \mathbf{P}(\mathbf{X} = x_k).$$

198 Definition Let $\mathbf{X}$ be a random variable. The *variance* $\mathbf{var}(\mathbf{X})$ of $\mathbf{X}$ is defined by

$$\mathbf{var}(\mathbf{X}) = \mathbf{E}(\mathbf{X}^2) - (\mathbf{E}(\mathbf{X}))^2.$$

199 Example A random variable has distribution function as shewn below.

$\mathbf{X}$	$\mathbf{P}(\mathbf{X})$
-1	$2k$
1	$3k$
2	$4k$

❶ Find the value of k.

❷ Determine the actual values of $\mathbf{P}(\mathbf{X}=-1)$, $\mathbf{P}(\mathbf{X}=1)$, and $\mathbf{P}(\mathbf{X}=2)$.

❸ Find $\mathbf{E}(\mathbf{X})$.

❹ Find $\mathbf{E}(\mathbf{X}^2)$.

❺ Find var (X).

▶**Solution:**

❶ *The probabilities must add up to* 1*:*

$$2k + 3k + 4k = 1 \implies k = \frac{1}{9}.$$

❷

$$P(X = -1) = 2k = \frac{2}{9},$$

$$P(X = 1) = 3k = \frac{3}{9},$$

$$P(X = 2) = 4k = \frac{4}{9}.$$

❸

$$E(X) = -1P(X = -1) + 1P(X = 1) + 2P(X = 2) = -1 \cdot \frac{2}{9} + 1 \cdot \frac{3}{9} + 2 \cdot \frac{4}{9} = 1.$$

❹

$$E\left(X^2\right) = (-1)^2 P(X = -1) + 1^2 P(X = 1) + 2^2 P(X = 2) = 1 \cdot \frac{2}{9} + 1 \cdot \frac{3}{9} + 4 \cdot \frac{4}{9} = \frac{21}{9}.$$

❺

$$\mathrm{var}(X) = E\left(X^2\right) - (E(X))^2 = \frac{21}{9} - 1^2 = \frac{4}{3}.$$

◀

200 Example John and Peter play the following game with three fair coins: John plays a stake of \$10 and tosses the three coins in turn. If he obtains three heads, his stake is returned together with a prize of \$30. For two consecutive heads, his stake money is returned, together with a prize of \$10. In all other cases, Peter wins the stake money. Is the game fair?

▶**Solution:** *The game is fair if the expected gain of both players is the same. Let* J *be the random variable measuring John's gain and let* P *be the random variable measuring Peter's gain. John wins when the coins shew* HHH, HHT, THH*. Thus*

$$\begin{aligned} E(J) &= 30P(HHH) + 10P(HHT) + 10P(THH) \\ &= 30 \cdot \frac{1}{8} + 10 \cdot \frac{1}{8} + 10 \cdot \frac{1}{8} \\ &= \frac{25}{4}. \end{aligned}$$

Peter wins when the coins shew HTH, HTT, THT, TTH, TTT*. Thus*

$$\begin{aligned} E(P) &= 10P(HTH) + 10P(HTT) + 10P(THT) + 10P(TTH) + 10P(TTT) \\ &= 10 \cdot \frac{1}{8} + 10 \cdot \frac{1}{8} + 10 \cdot \frac{1}{8} + 10 \cdot \frac{1}{8} + 10 \cdot \frac{1}{8} \\ &= \frac{25}{4}, \end{aligned}$$

whence the game is fair. ◀

201 Example There are eight socks in a box, of which four are white and four are black. Socks are drawn one at a time (without replacement) until a pair is produced. What is the expected value of drawings? (Clearly, this number should be between 2 and 3.)

▶**Solution:** *Let X be the random variable counting the number of drawings. Now, $X = 2$ means that matching socks are obtained when 2 socks are drawn. Hence*

$$\mathbf{P}\left(X = 2\right) = \frac{\binom{2}{1}\binom{4}{2}}{\binom{8}{2}} = \frac{3}{7},$$

and thus $\mathbf{P}\left(X = 3\right) = \frac{4}{7}$. *Therefore*

$$EX = 2\mathbf{P}\left(X = 2\right) + 3\mathbf{P}\left(X = 3\right) = 2 \cdot \frac{3}{7} + 3 \cdot \frac{4}{7} = \frac{18}{7}.$$

◀

202 Example Suppose that a player starts with a fortune of \$8. A fair coin is tossed three times. If the coin comes up heads, the player's fortune is doubled, otherwise it is halved. What is the player's expected fortune?

▶**Solution:** *The player may have:*

- *three wins, with probability* $\binom{3}{3}\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ *and his fortune increases eightfold.*
- *two wins, and one loss, with probability* $\binom{3}{2}\left(\frac{1}{2}\right)^3 = \frac{3}{8}$ *and his fortune doubles.*
- *one win, and two losses, with probability* $\binom{3}{1}\left(\frac{1}{2}\right)^3 = \frac{3}{8}$, *and his fortune halves.*
- *three losses, with probability* $\binom{3}{0}\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ *and his fortune reduces by a factor of* 8.

His expected fortune is thus

$$8\left(8 \cdot \frac{1}{8} + 2 \cdot \frac{3}{8} + \frac{1}{2} \cdot \frac{3}{8} + \frac{1}{8} \cdot \frac{1}{8}\right) = \frac{125}{8}.$$

◀

Homework

Problem 6.1.1 *A fair die is tossed. If the resulting number is even, you multiply your score by 2 and get that many dollars. If the resulting number is odd, you add 1 to your score and get that many dollars. Let $\mathbf{X}$ be the random variable counting your gain, in dollars.*

❶ *Give the range of* $\mathbf{X}$.

❷ *Give the distribution of* $\mathbf{X}$.

❸ *Find* $\mathbf{E}\left(\mathbf{X}\right)$.

❹ *Find* $\mathrm{var}\left(\mathbf{X}\right)$.

Problem 6.1.2 *A casino game consists of a single toss of a fair die and pays off as follows: if the die comes up with an odd number, the player is paid that number of dollars (i.e., \$1 for rolling a 1, \$3 for rolling a 3, and \$5 for rolling a 5), and if an even number comes up the player is paid nothing. What fee should the casino charge to play the game to make it exactly fair?*

Problem 6.1.3 *At a local carnival, Osa pays \$1 to play a game in which she chooses a card at random from a standard deck of 52 cards. If she chooses a heart, then she* *receives \$2 (that is, \$1 plus her initial bet of \$1). If she chooses the Queen of Spades she receives \$13. Which of the following is closest to Osa's expected net profit from playing the game?*

Problem 6.1.4 *Consider the random variable* $\mathbf{X}$ *with distribution table as follows.*

$\mathbf{X}$	$\mathbf{P}(\mathbf{X})$
-2	0.3
-1	k
0	$5k$
1	$2k$

❶ *Find the value of* k.

❷ *Find* $\mathbf{E}(\mathbf{X})$.

❸ *Find* $\mathbf{E}(\mathbf{X}^2)$.

❹ *Find* $\mathbf{var}(\mathbf{X})$.

Problem 6.1.5 *A fair coin is to be tossed thrice. The player receives \$10 if all three tosses turn up heads, and pays \$3 if there is one or no heads. No gain or loss is incurred otherwise. If* $\mathbf{Y}$ *is the gain of the player, find* $E\mathbf{Y}$.

Problem 6.1.6 *A die is loaded so that if* $\mathbf{D}$ *is the random variable giving the score on the die, then* $\mathbf{P}(\mathbf{D} = k) = \dfrac{k}{21}$, *where* $k = 1, 2, 3, 4, 5, 6$. *Another die is loaded differently, so that if* $\mathbf{X}$ *is the random variable giving the score on the die, then* $\mathbf{P}(\mathbf{X} = k) = \dfrac{k^2}{91}$.

❶ *Find the expectation* $\mathbf{E}(\mathbf{D} + \mathbf{X})$.

❷ *Find the variance* $\mathbf{var}(\mathbf{D} + \mathbf{X})$.

Problem 6.1.7 *John and Peter each put \$1 into a pot. They then decide to throw a pair of dice alternately (John plays first, Peter second, then John again, etc.). The first one who throws a* 5 *wins the pot. How much money should John add to the pot in order to make the game fair?*

Problem 6.1.8 *A man pays \$1 to throw three fair dice. If at least one* 6 *appears, he receives back his stake together with a prize consisting of the number of dollars equal to the number of sixes shewn. Does he expect to win or lose?*

Problem 6.1.9 (AHSME 1989) *Suppose that* k *boys and* $n - k$ *girls line up in a row. Let* S *be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row*

$$GBBGGGBGBGGGBGBGGBGG,$$

with $k = 7, n = 20$ *we have* $S = 12$. *Shew that the average value of* S *is* $\dfrac{2k(n-k)}{n}$.

6.2 Indicator Random Variables

203 Example Six different pairs of socks are put in the laundry (12 socks in all, and each sock has only one mate), but only 7 socks come back. What is the expected number of pairs of socks that come back?

▶**Solution:** *Let* $\mathbf{X}_i = 0$ *if the* i*-th pair does not come back, and* $\mathbf{X}_i = 1$ *if it does. We want*

$$E\mathbf{X}_1 + \cdots + E\mathbf{X}_6 = 6E\mathbf{X}_1 = 6\mathbf{P}(\mathbf{X}_1 = 1),$$

since the $\mathbf{X}_i$ *have the same distribution. Now*

$$\mathbf{P}(\mathbf{X}_1 = 1) = \frac{\binom{2}{2}\cdot\binom{10}{5}}{\binom{12}{7}} = \frac{7}{22},$$

and the required expectation is $\dfrac{21}{11}$. ◀

204 Example A standard deck of cards is turned face up one card at a time. What is the expected number of cards turned up in order to obtain a king?

▸**Solution:** *(1) Consider the 48 cards which are not kings and for $1 \leq i \leq 48$ put*

$$X_i = \begin{cases} 1 & \text{if the } i\text{—th non — king appears before a king.} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$X = 1 + \sum_{i=1}^{48} X_i$$

is the number of cards turned up in order to obtain a king. Let us prove that $\mathbf{P}(X_i = 1) = \frac{1}{5}$. *To this end, paint card i blue, then we have 47 cards which are not kings, card i, and 4 kings. The experiment consists in permuting all these cards, which can be done in* $\frac{52!}{47!4!}$ *ways. A favourable arrangement has the form*

$$x_1 B x_2 K x_3 K x_4 K x_5 K x_6,$$

where the B is the blue card, K is a king, and x_n can be any of the of the 47 other non-Kings. The number of favourable arrangements is thus the number of non-negative integral solutions to $x_1 + \cdots + x_6 = 47$, *which is* $\binom{47+6-1}{5} = \frac{52!}{5!47!}$. *Hence*

$$\mathbf{P}(X_i = 1) = \frac{\frac{52!}{5!47!}}{\frac{52!}{4!47!}} = \frac{1}{5}.$$

Notice that

$$\mathbf{P}(X_i = 1) = \frac{1}{5} \implies EX = 1 + \frac{48}{5} = \frac{53}{5}.$$

◂

205 Example An urn contains 30 cards: two numbered **1**, two numbered **2**, . . . , two numbered **15**. Ten cards are drawn at random from the urn. What is the expected number of pairs remaining in the urn?

▸**Solution:** *For $1 \leq i \leq 15$ put put*

$$X_i = \begin{cases} 1 & \text{if the } i\text{—th pair remains in the urn.} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\mathbf{P}(X_i = 1) = \frac{\binom{28}{10}\binom{2}{2}}{\binom{30}{10}} = \frac{\frac{28!}{18!10!}}{\frac{30!}{20!18!}} = \frac{38}{87},$$

and the desired expectation is $\frac{15 \cdot 38}{87} = \frac{190}{29}$. ◂

Homework

Problem 6.2.1 *A standard deck of cards is turned face up one card at a time. What is the expected number of cards turned up in order to obtain a heart?*

Problem 6.2.2 *If* X *denotes the number of* 1*'s when* 72 *dice are thrown, find* $E\mathrm{X}^2$.

Problem 6.2.3 *There are* 10 *boys and* 15 *girls in a class, and* 8 *students are to be selected at random from the class without replacement. Let* X *denote the number of boys that are selected and let* Y *denote the number of girls that are selected. Find* $E(X - Y)$.

Problem 6.2.4 *Seven married couples, the Adams, the Browns, the Castros, the Friedmans, the Lignowskis, the Santos, and the Jias , go to a desert island. Unbeknownst to them, a group of savages and cannibals awaits them. After an agonic week, five of the fourteen people survive. What is the average number of last names which are represented? (A last name is represented if either spouse, or possibly, both spouses, survived.)*

6.3 Conditional Expectation

206 Example A fair coin is tossed. If a head occurs, one fair die is rolled, else, two fair dice are rolled. Let X be the total on the die or dice. Find EX.

▶**Solution:**

$$E\mathrm{X} = \mathbf{P}(\mathrm{H})\,\mathbf{P}(\mathrm{X}|\mathrm{H}) + \mathbf{P}(\mathrm{T})\,\mathbf{P}(\mathrm{X}|\mathrm{T}) = \frac{1}{2}\cdot\frac{7}{2} + \frac{1}{2}\cdot 7 = \frac{21}{4}.$$

◀

207 Example In the city of Jerez de la Frontera, in Cádiz, Spain, true sherry is made according to a multistage system called *Solera*. Assume that a winemaker has three barrels, A, B, and C. Every year, a third of the wine from barrel C is bottled and replaced by wine from B; then B is topped off with a third of the wine from A; finally A is topped off with new wine. Find the mean of the age of the wine in each barrel, under the assumption that the operation has been going on since time immemorial.

▶**Solution:** *We start with barrel A. Abusing notation, we will let* A *the random variable indicating the number of years of wine in barrel A, etc. After the transfer has been made, the mean age of the new wine is* 0 *years and the mean age of the old wine is a year older than what it was. Hence*

$$A = \frac{1}{3}A_{\mathrm{new}} + \frac{2}{3}A_{\mathrm{old}} \implies EA = \frac{1}{3}EA_{\mathrm{new}} + \frac{2}{3}EA_{\mathrm{old}} \implies EA = \frac{1}{3}\cdot 0 + \frac{2}{3}(1+EA) \implies EA = 2.$$

Thus $EA_{\mathrm{old}} = 3$. *Now,*

$$B = \frac{1}{3}B_{\mathrm{new}} + \frac{2}{3}B_{\mathrm{old}} = \frac{1}{3}A_{\mathrm{old}} + \frac{2}{3}B_{\mathrm{old}} \implies EB = \frac{1}{3}\cdot 3 + \frac{2}{3}EB_{\mathrm{old}} \implies EB = \frac{3}{3} + \frac{2}{3}(1+EB) \implies EB = 5.$$

Hence, $EB_{\mathrm{old}} = 6$. *Similarly,*

$$C = \frac{1}{3}C_{\mathrm{new}} + \frac{2}{3}C_{\mathrm{old}} = \frac{1}{3}B_{\mathrm{old}} + \frac{2}{3}C_{\mathrm{old}} \implies EC = \frac{1}{3}\cdot 6 + \frac{2}{3}EC_{\mathrm{old}} \implies EC = \frac{6}{3} + \frac{2}{3}(1+EC) \implies EC = 8.$$

◀

Homework

Problem 6.3.1 *A fair coin is tossed repeatedly until heads is produced. If it is known that the coin produces heads within the first flip, what is the expected number of flips to produce the first heads?*

A Answers

1.1.1 $2^A = \{\varnothing, \{a\}, \{b\}, A\}$ so $\mathbf{card}\left(2^A\right) = 4$.

1.1.2 $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, and $\{2, 3, 4\}$, whence $\mathbf{card}\,(A) = 4$.

1.1.3 $A = \{-2, -1, 0, 1, 2\}$. Yes.

1.1.4 $2^0 = 1$, namely itself. $2^{10} = 1024$.

1.1.5 Yes. The first is the empty set, and has 0 elements. The second is a set containing the empty set, and hence it has 1 element.

1.1.6 Observe that

$$1 = 1 + 6 \cdot 0, \quad 7 = 1 + 6 \cdot 1, \quad 13 = 1 + 6 \cdot 2, \quad \ldots, \quad 397 = 1 + 6 \cdot 66,$$

and hence, there are $66 + 1 = 67$ elements, where we add the 1 because our count started at 0. Notice that every element has the form $1 + 6k$. If $295 = 1 + 6k$ then $k = 49$, and hence 295 is in this set.

Let the sum of the elements be S. Observe that we obtain S by also adding backwards, Adding,

$$\begin{array}{rclcccccccc} S & = & 1 & + & 7 & + & \cdots & + & 391 & + & 397 \\ S & = & 397 & + & 391 & + & \cdots & + & 7 & + & 1 \\ \hline 2S & = & 398 & + & 398 & + & \cdots & + & 398 & + & 398 \\ & = & 67 \cdot 398, \end{array}$$

whence

$$S = \frac{67 \cdot 398}{2} = 13333.$$

1.2.1 $\{HH, HT, TH, TT\}$

1.2.2 We have

$$\begin{aligned} X &= \{(1, 6), (2, 3), (3, 2), (6, 1)\} \\ T &= \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (3, 1)\} \\ U &= \varnothing \end{aligned}$$

1.2.3 R denotes a red marble and B denotes a blue one. The sample space is $\Omega = \{BRR, RBR, RRB, BBR, BRB, RBB, RRR\}$.

1.2.4 Let S_1, S_2 represent the Spanish novels, I the Italian novel, and G the German book. Then the sample space has 24 elements:

$$\begin{aligned} \Omega = \{ & S_1S_2IG, S_1S_2GI, IGS_1S_2, GIS_1S_2, S_1IGS_2, S_1GIS_2, S_1IS_2G, S_1GS_2I, IS_1S_2G, GS_1S_2I, S_1GS_2I, S_1IS_2G, \\ & S_2S_1IG, S_2S_1GI, IGS_2S_1, GIS_2S_1, S_2IGS_1, S_2GIS_1, S_2IS_1G, S_2GS_1I, IS_2S_1G, GS_2S_1I, S_2GS_1I, S_2IS_1G\}. \end{aligned}$$

The event that the Spanish books remain together is

$$E = \{S_1S_2IG, S_1S_2GI, IGS_1S_2, GIS_1S_2, IS_1S_2G, GS_1S_2I, S_2S_1IG, S_2S_1GI, IGS_2S_1, GIS_2S_1, IS_2S_1G, GS_2S_1I\}.$$

1.2.5 Let P, N, D, Q represent a penny, a nickel, a dime, and a quarter, respectively. Then

1. this is $\{QP, PQ\}$,
2. this is the null event $\varnothing$,
3. this is $\{PD, DP, NN, ND, DN\}$.

1.3.1 There are four ways:

❶ $\{1\} \cup \{2, 3\}$

❷ $\{2\} \cup \{1, 3\}$

❸ $\{3\} \cup \{1, 2\}$

❹ $\{1\} \cup \{2\} \cup \{3\}$

1.3.2 A

1.3.3 B

1.3.4 $A \cap B$

1.3.5 One possible answer is $A \cup (B \setminus A)$. Another one is $B \cup (A \setminus B)$.

1.3.6 $(A \setminus B) \cup (B \setminus A) \cup (A \cap B)$

1.3.7 $(A \cap B^c) \cup (A^c \cap B) = (A \cup B) \setminus (A \cap B)$.

1.3.8 By the De Morgan Laws,

$$(A^c \cup B^c)^c = (A^c)^c \cap (B^c)^c = A \cap B = \varnothing.$$

1.3.9 We have

❶ $(A \cap B \cap C^c) \cup (A \cap B^c \cap C) \cup (A^c \cap B \cap C) \cup (A \cap B \cap C)$.

❷ $(A \cap B^c \cap C^c) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C) \cup (A^c \cap B^c \cap C^c)$.

1.3.10 We have

❶ $\{1, 4, 6, 8, 9, 10, 12, 14, 15\}$

❷ $\{4, 6, 8, 10, 12, 14, 16\}$

❸ $\{4, 6, 8, 10, 12, 14\}$

1.3.11 The progression in A has common difference 10 and the one in B has common difference 12. Observe that the smallest element they share is 13, and hence, they will share every $\mathrm{lcm}[10, 12] = 60$ elements, starting with 13. We now want the largest k so that

$$13 + 60k \leq 361,$$

where we have chosen 361 since it is the minimum of 361 and 456. Solving,

$$k \leq \llbracket \frac{361 - 13}{60} \rrbracket = 5.$$

Hence there are $5 + 1 = 6$ elements in the intersection. They are

$$A \cap B = \{13, 73, 133, 193, 253, 313\}.$$

1.3.12 $B \subsetneq A$.

1.3.13 $A \subsetneq B$.

1.3.14 $B \cup C \subsetneq A$.

1.4.1 There are $2^3 = 8$ such functions:

❶ f_1 given by $f_1(0) = f_1(1) = f_1(2) = -1$

❷ f_2 given by $f_2(0) = 1, f_2(1) = f_2(2) = -1$

❸ f_3 given by $f_3(0) = f_3(1) = -1, f_3(2) = 1$

❹ f_4 given by $f_4(0) = -1, f_4(1) = 1, f_4(2) = -1$

❺ f_5 given by $f_5(0) = f_5(1) = f_5(2) = 1$

❻ f_6 given by $f_6(0) = -1, f_6(1) = f_6(2) = 1$

❼ f_7 given by $f_7(0) = f_7(1) = 1, f_7(2) = -1$

❽ f_8 given by $f_8(0) = 1, f_8(1) = -1, f_8(2) = 1$

Of these, 0 are injective, and 6, f_2, f_3, f_4, f_6, f_7 and f_8 are surjective.

1.4.2 There are $3^2 = 9$ such functions:

❶ f_1 given by $f_1(-1) = f_1(1) = 0$

❷ f_2 given by $f_2(-1) = f_2(1) = 1$

❸ f_3 given by $f_3(-1) = f_3(1) = 2$

❹ f_4 given by $f_4(-1) = 0, f_4(1) = 1$

❺ f_5 given by $f_5(-1) = 1, f_5(1) = 0$

❻ f_6 given by $f_6(-1) = 0, f_6(1) = 2$

❼ f_7 given by $f_7(-1) = 2$, $f_7(1) = 0$

❽ f_8 given by $f_8(-1) = 1$, $f_8(1) = 2$

❾ f_9 given by $f_9(-1) = 2$, $f_6(1) = 1$

Of these, 6, f_4, f_5, f_6, f_7, f_8 and f_9 are injective, and 0 are surjective.

1.4.3 There are two.

$$f_1 : \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \qquad f_2 : \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

1.4.4 There are six.

$$f_1 : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad f_2 : \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad f_3 : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$f_4 : \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad f_5 : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad f_6 : \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

2.1.1 Let $A_k \subseteq A$ be the set of those integers divisible by k.

❶ Notice that the elements are $2 = 2(1)$, $4 = 2(2)$, $\ldots$, $114 = 2(57)$. Thus $\mathbf{card}\left(A\right) = 57$.

❷ There are $\lfloor \frac{57}{3} \rfloor = 19$ integers in A divisible by 3. They are

$$\{6, 12, 18, \ldots, 114\}.$$

Notice that $114 = 6(19)$. Thus $\mathbf{card}\left(A_3\right) = 19$.

❸ There are $\lfloor \frac{57}{5} \rfloor = 11$ integers in A divisible by 5. They are

$$\{10, 20, 30, \ldots, 110\}.$$

Notice that $110 = 10(11)$. Thus $\mathbf{card}\left(A_5\right) = 11$

❹ There are $\lfloor \frac{57}{15} \rfloor = 3$ integers in A divisible by 15. They are $\{30, 60, 90\}$. Notice that $90 = 30(3)$. Thus $\mathbf{card}\left(A_{15}\right) = 3$, and observe that by Theorem **??** we have $\mathbf{card}\left(A_{15}\right) = \mathbf{card}\left(A_3 \cap A_5\right)$.

❺ We want $\mathbf{card}\left(A_3 \cup A_5\right) = 19 + 11 - 3 = 27$.

❻ We want

$$\begin{aligned} \mathbf{card}\left(A \setminus (A_3 \cup A_5)\right) &= \mathbf{card}\left(A\right) - \mathbf{card}\left(A_3 \cup A_5\right) \\ &= 57 - 27 \\ &= 30. \end{aligned}$$

❼ We want

$$\begin{aligned} \mathbf{card}\left((A_3 \cup A_5) \setminus (A_3 \cap A_5)\right) &= \mathbf{card}\left((A_3 \cup A_5)\right) \\ &\quad -\mathbf{card}\left(A_3 \cap A_5\right) \\ &= 27 - 3 \\ &= 24. \end{aligned}$$

2.1.2 We have

❶ $\lfloor \frac{100}{2} \rfloor = 50$

❷ $\lfloor \frac{100}{3} \rfloor = 33$

❸ $\lfloor \frac{100}{7} \rfloor = 14$

❹ $\lfloor \frac{100}{6} \rfloor = 16$

❺ $\lfloor \frac{100}{14} \rfloor = 7$

❻ $\lfloor \frac{100}{21} \rfloor = 4$

❼ $\llbracket \frac{100}{42} \rrbracket = 2$

❽ $100 - 50 - 33 - 14 + 16 + 7 + 4 - 2 = 28$

❾ $16 - 2 = 14$

❿ 52

2.1.3 52%

2.1.4 Let A be the set of students liking Mathematics, B the set of students liking theology, and C be the set of students liking alchemy. We are given that

$$\mathbf{card}\,(A) = 14, \mathbf{card}\,(B) = 16,$$
$$\mathbf{card}\,(C) = 11, \mathbf{card}\,(A \cap B) = 7, \mathbf{card}\,(B \cap C) = 8, \mathbf{card}\,(A \cap C) = 5,$$

and

$$\mathbf{card}\,(A \cap B \cap C) = 4.$$

By the Principle of Inclusion-Exclusion,

$$\begin{aligned}\mathbf{card}\,(A^c \cap B^c \cap C^c) &= 40 - \mathbf{card}\,(A) - \mathbf{card}\,(B) - \mathbf{card}\,(C) \\ &\quad + \mathbf{card}\,(A \cap B) + \mathbf{card}\,(A \cap C) + \mathbf{card}\,(B \cap C) \\ &\quad - \mathbf{card}\,(A \cap B \cap C).\end{aligned}$$

Substituting the numerical values of these cardinalities

$$40 - 14 - 16 - 11 + 7 + 5 + 8 - 4 = 15.$$

2.1.5 We have

❶ 31

❷ 10

❸ 3

❹ 3

❺ 1

❻ 1

❼ 1

❽ 960

2.1.6 Let Y, F, S, M stand for young, female, single, male, respectively, and let H stand for married.[1] We have

$$\begin{aligned}\mathbf{card}\,(Y \cap F \cap S) &= \mathbf{card}\,(Y \cap F) - \mathbf{card}\,(Y \cap F \cap H) \\ &= \mathbf{card}\,(Y) - \mathbf{card}\,(Y \cap M) \\ &\qquad -(\mathbf{card}\,(Y \cap H) - \mathbf{card}\,(Y \cap H \cap M)) \\ &= 3000 - 1320 - (1400 - 600) \\ &= 880.\end{aligned}$$

2.1.7 34

2.1.8 $30; 7; 5; 18$

2.1.9 4

2.1.10 Let C denote the set of people who like candy, I the set of people who like ice cream, and K denote the set of people who like cake. We are given that $\mathbf{card}\,(C) = 816$, $\mathbf{card}\,(I) = 723$, $\mathbf{card}\,(K) = 645$, $\mathbf{card}\,(C \cap I) = 562$, $\mathbf{card}\,(C \cap K) = 463$, $\mathbf{card}\,(I \cap K) = 470$, and $\mathbf{card}\,(C \cap I \cap K) = 310$. By Inclusion-Exclusion we have

$$\begin{aligned}\mathbf{card}\,(C \cup I \cup K) &= \mathbf{card}\,(C) + \mathbf{card}\,(I) + \mathbf{card}\,(K) \\ &\quad - \mathbf{card}\,(C \cap I) - \mathbf{card}\,(C \cap K) - \mathbf{card}\,(I \cap C) \\ &\quad + \mathbf{card}\,(C \cap I \cap K) \\ &= 816 + 723 + 645 - 562 - 463 - 470 + 310 \\ &= 999.\end{aligned}$$

The investigator miscounted, or probably did not report one person who may not have liked any of the three things.

[1] Or H for *hanged*, if you prefer.

2.1.11 A set with k elements has 2^k different subsets. We are given

$$2^{100} + 2^{100} + 2^{\mathbf{card}(C)} = 2^{\mathbf{card}(A\cup B\cup C)}.$$

This forces $\mathbf{card}(C) = 101$, as $1 + 2^{\mathbf{card}(C)-101}$ is larger than 1 and a power of 2. Hence $\mathbf{card}(A\cup B\cup C) = 102$. Using the Principle Inclusion-Exclusion, since $\mathbf{card}(A) + \mathbf{card}(B) + \mathbf{card}(C) - \mathbf{card}(A\cup B\cup C) = 199$,

$$\begin{aligned}\mathbf{card}(A\cap B\cap C) &= \mathbf{card}(A\cap B) + \mathbf{card}(A\cap C) + \mathbf{card}(B\cap C) - 199\\ &= (\mathbf{card}(A) + \mathbf{card}(B) - \mathbf{card}(A\cup B))\\ &\quad +(\mathbf{card}(A) + \mathbf{card}(C)\\ &\quad -\mathbf{card}(A\cup C)) + \mathbf{card}(B) + \mathbf{card}(C)\\ &\quad -\mathbf{card}(B\cup C) - 199\\ &= 403 - \mathbf{card}(A\cup B) - \mathbf{card}(A\cup C) - \mathbf{card}(B\cup C).\end{aligned}$$

As $A\cup B, A\cup C, B\cup C \subseteq A\cup B\cup C$, the cardinalities of all these sets are ≤ 102. Thus

$$\begin{aligned}\mathbf{card}(A\cap B\cap C) &= 403 - \mathbf{card}(A\cup B) - \mathbf{card}(A\cup C)\\ &\quad -\mathbf{card}(B\cup C) \geq 403 - 3\cdot 102\\ &= 97.\end{aligned}$$

By letting

$$A = \{1, 2, \ldots, 100\}, B = \{3, 4, \ldots, 102\},$$

and

$$C = \{1, 2, 3, 4, 5, 6, \ldots, 101, 102\}$$

we see that the bound $\mathbf{card}(A\cap B\cap C) = \mathbf{card}(\{4, 5, 6, \ldots, 100\}) = 97$ is achievable.

2.1.12 One computes the sum of all integers from 1 to 1000 and weeds out the sum of the multiples of 3 and the sum of the multiples of 5, but puts back the multiples of 15, which one has counted twice. The desired sum is

$$\begin{aligned}&(1 + 2 + 3 + \cdots + 1000) - (3 + 6 + 9 + \cdots + 999)\\ &\quad -(5 + 10 + 15 + \cdots + 1000)\\ &\quad +(15 + 30 + 45 + \cdots + 990)\\ &= (1 + 2 + 3 + \cdots + 1000) - 3(1 + 2 + 3 + \cdots + 333)\\ &\quad -5(1 + 2 + 3 + \cdots + 200)\\ &\quad +15(1 + 2 + 3 + \cdots + 66)\\ &= 500500 - 3\cdot 55611\\ &\quad -5\cdot 20100 + 15\cdot 2211\\ &= 266332\end{aligned}$$

2.1.13 Let A denote the set of those who lost an eye, B denote those who lost an ear, C denote those who lost an arm and D denote those losing a leg. Suppose there are n combatants. Then

$$\begin{aligned}n &\geq \mathbf{card}(A\cup B)\\ &= \mathbf{card}(A) + \mathbf{card}(B) - \mathbf{card}(A\cap B)\\ &= .7n + .75n - \mathbf{card}(A\cap B),\end{aligned}$$

$$\begin{aligned}n &\geq \mathbf{card}(C\cup D)\\ &= \mathbf{card}(C) + \mathbf{card}(D) - \mathbf{card}(C\cap D)\\ &= .8n + .85n - \mathbf{card}(C\cap D).\end{aligned}$$

This gives

$$\mathbf{card}(A\cap B) \geq .45n,$$

$$\mathbf{card}\,(C\cap D)\geq .65n.$$

This means that

$$\begin{aligned} n &\geq \mathbf{card}\,((A\cap B)\cup(C\cap D)) \\ &= \mathbf{card}\,(A\cap B)+\mathbf{card}\,(C\cap D)-\mathbf{card}\,(A\cap B\cap C\cap D) \\ &\geq .45n+.65n-\mathbf{card}\,(A\cap B\cap C\cap D)\,, \end{aligned}$$

whence

$$\mathbf{card}\,(A\cap B\cap C\cap D)\geq .45+.65n-n=.1n.$$

This means that at least 10% of the combatants lost all four members.

2.2.1 $2^{10}=1024$

2.2.2 I can choose a right shoe in any of nine ways, once this has been done, I can choose a non-matching left shoe in eight ways, and thus I have 72 choices.
Aliter: I can choose any pair in $9\times 9=81$ ways. Of these, 9 are matching pairs, so the number of non-matching pairs is $81-9=72$.

2.2.3 $=(20)(19)(20)(19)(20)(20)=57760000$

2.2.4 $10^3 5^3-10^2 5^2=122500$

2.2.5 The number of different license plates is the number of different four-tuples (Letter $_1$, Letter $_2$, Digit $_1$, Digit $_2$). The first letter can be chosen in 26 ways, and so we have

$$\begin{array}{|c|c|c|c|}\hline 26 & \quad & \quad & \quad \\ \hline\end{array}.$$

The second letter can be chosen in any of 26 ways:

$$\begin{array}{|c|c|c|c|}\hline 26 & 26 & \quad & \quad \\ \hline\end{array}.$$

The first digit can be chosen in 10 ways:

$$\begin{array}{|c|c|c|c|}\hline 26 & 26 & 10 & \quad \\ \hline\end{array}.$$

Finally, the last digit can be chosen in 10 ways:

$$\begin{array}{|c|c|c|c|}\hline 26 & 26 & 10 & 10 \\ \hline\end{array}.$$

By the multiplication principle, the number of different four-tuples is $26\cdot 26\cdot 10\cdot 10=67600.$

2.2.6 (i) In this case we have a grid like

$$\begin{array}{|c|c|c|c|}\hline 26 & 26 & 10 & 9 \\ \hline\end{array},$$

since after a digit has been used for the third position, it cannot be used again. Thus this can be done in $26\cdot 26\cdot 10\cdot 9=60840$ ways.
(ii) In this case we have a grid like

$$\begin{array}{|c|c|c|c|}\hline 26 & 25 & 10 & 10 \\ \hline\end{array},$$

since after a letter has been used for the first position, it cannot be used again. Thus this can be done in $26\cdot 25\cdot 10\cdot 10=65000$ ways.
(iii) After a similar reasoning, we obtain a grid like

$$\begin{array}{|c|c|c|c|}\hline 26 & 25 & 10 & 9 \\ \hline\end{array}.$$

Thus this can be done in $26\cdot 25\cdot 10\cdot 9=58500$ ways.

2.2.7 [1] 8, [2] $5^2 3^2=225$, [3] $5^2\cdot 3\cdot 2=150$, [4] $5\cdot 4\cdot 3^2=180$, [5] $8\cdot 7\cdot 6\cdot 5=1680$.

2.2.8 432

2.2.9 Solution:

❶ The first letter can be one of any 4 ways. After choosing the first letter, we have 3 choices for the second letter, etc.. The total number of words is thus $4\cdot 3\cdot 2\cdot 1=24$.

❷ The first letter can be one of any 4. Since we are allowed repetitions, the second letter can also be one of any 4, etc.. The total number of words so formed is thus $4^4=256$.

2.2.10 The last digit must perforce be 5. The other five digits can be filled with any of the six digits on the list: the total number is thus 6^5.

2.2.11 We have

❶ This is $5 \cdot 8^6 = 1310720$.

❷ This is $5 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 25200$.

❸ This is $5 \cdot 8^5 \cdot 4 = 655360$.

❹ This is $5 \cdot 8^5 \cdot 4 = 655360$.

❺ We condition on the last digit. If the last digit were 1 or 5 then we would have 5 choices for the first digit, and so we would have

$$5 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 = 7200$$

phone numbers. If the last digit were either 3 or 7, then we would have 4 choices for the last digit and so we would have

$$4 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 = 5760$$

phone numbers. Thus the total number of phone numbers is

$$7200 + 5760 = 12960.$$

2.2.12 $26 \cdot 25^4 = 10156250$

2.2.13 For the leftmost digit cannot be 0 and so we have only the nine choices

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

for this digit. The other $n-1$ digits can be filled out in 10 ways, and so there are

$$9 \cdot \underbrace{10 \cdots 10}_{n-1\ 10'\mathrm{s}} = 9 \cdot 10^{n-1}.$$

2.2.14 The leftmost digit cannot be 0 and so we have only the nine choices

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

for this digit. If the integer is going to be even, the last digit can be only one of the five $\{0, 2, 4, 6, 8\}$. The other $n-2$ digits can be filled out in 10 ways, and so there are

$$9 \cdot \underbrace{10 \cdots 10}_{n-2\ 10'\mathrm{s}} \cdot 5 = 45 \cdot 10^{n-2}.$$

2.2.15 9 1-digit numbers and $8 \cdot 9^{n-1}$ n-digit numbers $n \geq 2$.

2.2.16 One can choose the last digit in 9 ways, one can choose the penultimate digit in 9 ways, etc. and one can choose the second digit in 9 ways, and finally one can choose the first digit in 9 ways. The total number of ways is thus 9^n.

2.2.17 m^2, $m(m-1)$

2.2.18 We will assume that the positive integers may be factorised in a unique manner as the product of primes. Expanding the product

$$(1 + 2 + 2^2 + \cdots + 2^8)(1 + 3 + 3^2 + \cdots + 3^9)(1 + 5 + 5^2)$$

each factor of $2^8 3^9 5^2$ appears and only the factors of this number appear. There are then, as many factors as terms in this product. This means that there are $(1 + 8)(1 + 9)(1 + 3) = 320$ factors.

The sum of the divisors of this number may be obtained by adding up each geometric series in parentheses. The desired sum is then

$$\frac{2^9 - 1}{2 - 1} \cdot \frac{3^{10} - 1}{3 - 1} \cdot \frac{5^3 - 1}{5 - 1} = 467689684.$$

☞ *A similar argument gives the following. Let $p_1, p_2, \ldots, p_k$ be different primes. Then the integer*

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$$

has

$$\mathrm{d}(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$$

positive divisors. Also, if $\sigma(n)$ denotes the sum of all positive divisors of n, then

$$\sigma(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}.$$

2.2.19 The 96 factors of 2^{95} are $1, 2, 2^2, \ldots, 2^{95}$. Observe that $2^{10} = 1024$ and so $2^{20} = 1048576$. Hence

$$2^{19} = 524288 < 1000000 < 1048576 = 2^{20}.$$

The factors greater than $1,000,000$ are thus $2^{20}, 2^{21}, \ldots 2^{95}$. This makes for $96 - 20 = 76$ factors.

2.2.20 $(1 + 3)(1 + 2)(1 + 1) = 24$; 18; 6; 4.

2.2.21 16

2.2.22 $n = \underbrace{1+1+\cdots+1}_{n-1\ +'\text{s}}$. One either erases or keeps a plus sign.

2.2.23 There are **589** such values. The easiest way to see this is to observe that there is a bijection between the divisors of n^2 which are $> n$ and those $< n$. For if $n^2 = ab$, with $a > n$, then $b < n$, because otherwise $n^2 = ab > n \cdot n = n^2$, a contradiction. Also, there is exactly one decomposition $n^2 = n \cdot n$. Thus the desired number is

$$\lfloor \frac{d(n^2)}{2} \rfloor + 1 - d(n) = \lfloor \frac{(63)(39)}{2} \rfloor + 1 - (32)(20) = 589.$$

2.2.24 The total number of sequences is 3^n. There are 2^n sequences that contain no **0**, **1** or **2**. There is only one sequence that contains only **1**'s, one that contains only **2**'s, and one that contains only **0**'s. Obviously, there is no ternary sequence that contains no **0**'s or **1**'s or **2**'s. By the Principle of Inclusion-Exclusion, the number required is

$$3^n - (2^n + 2^n + 2^n) + (1 + 1 + 1) = 3^n - 3 \cdot 2^n + 3.$$

2.2.25 The conditions of the problem stipulate that both the region outside the circles in diagram 2.3 and R_3 will be empty. We are thus left with **6** regions to distribute **100** numbers. To each of the **100** numbers we may thus assign one of **6** labels. The number of sets thus required is 6^{100}.

2.3.1 21

2.3.2 56

2.3.3 There are $26^2 - 25^2 = 51$ using two letters with at least one of the letters a D, since from the total of 26^2 we delete the 25^2 that do not have a D. Similarly, there are $26^3 - 25^3$ with three letters, with at least one of the letters a D. Thus there is a total of $(26^2 - 25^2) + (26^3 - 25^3) = 2002$.

Aliter: The ones with two initials are of the form ■D, D■, or DD, where ■ is any of the **25** letters not D. Hence there are $25 + 25 + 1 = 51$ with two letters. The ones with three letters are of the form DDD, DD■, D■D, ■DD, ■■D, ■D■, ■■D, and hence there are

$$1 + 3 \cdot 25 + 3 \cdot 25^2 = 1951.$$

Altogether there are

$$51 + 1951 = 2002,$$

like before.

2.3.4

$$\begin{aligned} & 9 + 9 \cdot 9 \\ & \quad + 9 \cdot 9 \cdot 8 + 9 \cdot 9 \cdot 8 \cdot 7 \\ & \quad + 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 + 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \\ & \quad + 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 + 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \\ & \quad + 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \\ & \quad + 9 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \\ & \quad = 8877690 \end{aligned}$$

2.3.5 $2 + 4 + 8 + 16 = 30$.

2.3.6 8; $12(n-2)$; $6(n-2)^2$; $(n-2)^3$
Comment: This proves that $n^3 = (n-2)^3 + 6(n-2)^2 + 12(n-2) + 8$.

2.3.7 We condition on the first digit, which can be $4, 5$, or 6. If the number starts with **4**, in order to satisfy the conditions of the problem, we must choose the last digit from the set $\{0, 2, 6, 8\}$. Thus we have four choices for the last digit. Once this last digit is chosen, we have **8** choices for the penultimate digit and **7** choices for the antepenultimate digit. There are thus $4 \times 8 \times 7 = 224$ even numbers which have their digits distinct and start with a **4**. Similarly, there are **224** even numbers will all digits distinct and starting with a **6**. When they start with a 5, we have **5** choices for the last digit, **8** for the penultimate and **7** for the antepenultimate. This gives $5 \times 8 \times 7 = 280$ ways. The total number is thus $224 + 224 + 280 = 728$.

2.3.8 When the number **99** is written down, we have used

$$1 \cdot 9 + 2 \cdot 90 = 189$$

digits. If we were able to write 999, we would have used

$$1 \cdot 9 + 2 \cdot 90 + 3 \cdot 900 = 2889$$

digits, which is more than **1002** digits. The **1002**nd digit must be among the three-digit positive integers. We have $1002 - 189 = 813$ digits at our disposal, from which we can make $\lfloor \frac{813}{3} \rfloor = 271$ three-digit integers, from 100 to **370**. When the **0** in **370** is written, we have used $189 + 3 \cdot 271 = 1002$ digits. The **1002**nd digit is the **0** in **370**.

2.3.9 4

2.3.10 There is 1 such number with 1 digit, 10 such numbers with 2 digits, 100 with three digits, 1000 with four digits, etc. Starting with 2 and finishing with 299 we have used $1 \cdot 1 + 2 \cdot 10 + 3 \cdot 100 = 321$ digits. We need $1978 - 321 = 1657$ more digits from among the 4-digit integers starting with 2. Now $\llbracket \frac{1657}{4} \rrbracket = 414$, so we look at the 414th 4-digit integer starting with 2, namely, at 2413. Since the 3 in 2413 constitutes the $321 + 4 \cdot 414 = 1977$-th digit used, the 1978-th digit must be the 2 starting 2414.

2.3.11 19990

2.3.12 [1] 125, [2] 25, [3] 25, [4] $5 + 2 \cdot 3 + 3 \cdot 6 = 29$.

2.3.13 8

2.3.14 4095

2.3.15 144

2.3.16 Observe that there are $9 \times 10^{n-1}$ palindromes of n digits. The number 1003001 has seven digits. After writing 999999, the last palindrome with six digits, one has written

$$9 + 9 + 90 + 90 + 900 + 900 = 1998$$

palindromes. The 1999-th, 2000-th, 2001-st and 2002-nd are thus

$$1000001, 1001001, 1002001, 1003001,$$

and so 1003001 occupies the 2002-nd position.

2.3.17 There are none used when writing the numbers from 1 through 9.

When writing the numbers from 10 to 99, there are 9 zeroes used, when writing $\{10, 20, \ldots, 90\}$.

When writing a three-digit integer ABC (numbers in the 100-999 range), one can use either one or two zeroes. If ABC has exactly one zero, then it is either B or C. If one of B or C is 0, then there are 9 choices for the other and 9 for A. Thus of the numbers ABC there are $9 \cdot 9 \cdot 2 = 162$ that use exactly one 0. If ABC has exactly two 0's then B and C must be 0 and there are 9 choices for A. Those 9 numbers use $2 \cdot 9 = 18$ zeroes. Thus in this range we have used $162 + 18 = 180$ zeroes.

A number in the 1000-1999 range has the form 1ABC. When writing them, one may use exactly one, two, or three zeroes. If there is only exactly one zero, then exactly one of A, B, or C, is 0, and since there are 9 choices for each of the other two letters, one has used $9 \cdot 9 \cdot 3 = 243$ zeroes this way. If there are exactly two zeroes, then either A and B, or A and C, or B and C, are zero, and there are 9 for the remaining letter. Therefore there are $9 \cdot 3 = 27$ numbers with 2 zeroes and $27 \cdot 2 = 54$ zeroes are used. Also, there is exactly one number in the 1000-1999 range using 3 zeroes. Altogether in this range we have used $243 + 54 + 3 = 300$ zeroes in this range.

Finally, in the range 2000-2007, there is one number using 3 zeroes, and 7 numbers using 2 zeroes. Hence in this range we have used $3 + 2 \times 7 = 17$ zeroes.

In summary, we have used

$$9 + 180 + 300 + 17 = 506$$

zeroes.

2.3.18 Observe that we need $x > y$. Since $x^2 - y^2 = 81 \iff (x + y)(x - y) = 81$, we examine the positive divisors of 81. We need

$$x + y = 81, x - y = 1, \qquad x + y = 27, x - y = 3, \qquad x + y = 9, x - y = 9.$$

Hence, by inspection, the following solutions lie on the first quadrant,

$$(41, 40), (15, 12),$$

and the solution $(9, 0)$ lies on the x-axis. Thus on each quadrant there are two solutions, and a solution each on the positive and the negative portion of the x-axis, giving a total of

$$4 \cdot 2 + 2 = 10$$

solutions.

2.3.19 Choose a specific colour for the upper face of the cube, say A. Then we have five choices for colouring the lower face of the cube, say with colour B. Rotate the cube so that some colour C is facing us. Now the remaining sides are fixed with respect to these three. We can distribute the three colours in $3 \times 2 \times 1 = 6$ ways, giving $5 \times 6 = 30$ possibilities.

2.3.20 Put the 6 in any of the 6 faces, leaving five faces. You have only one face to put the 1 (opposite of the 6), leaving 4 faces. Put the 4 in any of the 4 remaining faces, leaving 3 faces. You must put the 3 in the opposite face, leaving 2 faces. You can now put the 2 in any of the two remaining faces, and in the last face you put the 5. In total you have $6 \cdot 4 \cdot 2 = 48$ different dice.

2.3.21 First observe that $1 + 7 = 3 + 5 = 8$. The numbers formed have either one, two, three or four digits. The sum of the numbers of 1 digit is clearly $1 + 7 + 3 + 5 = 16$.

There are $4 \times 3 = 12$ numbers formed using 2 digits, and hence 6 pairs adding to 8 in the units and the tens. The sum of the 2 digits formed is $6((8)(10) + 8) = 6 \times 88 = 528$.

There are $4 \times 3 \times 2 = 24$ numbers formed using 3 digits, and hence 12 pairs adding to 8 in the units, the tens, and the hundreds. The sum of the 3 digits formed is $12(8(100) + (8)(10) + 8) = 12 \times 888 = 10656$.

There are $4 \times 3 \times 2 \cdot 1 = 24$ numbers formed using 4 digits, and hence 12 pairs adding to 8 in the units, the tens the hundreds, and the thousands. The sum of the 4 digits formed is $12(8(1000) + 8(100) + (8)(10) + 8) = 12 \times 8888 = 106656$.

The desired sum is finally

$$16 + 528 + 10656 + 106656 = 117856.$$

2.3.22 Observe that

❶ We find the pairs

$$\{1, 6\}, \{2, 7\}, \{3, 8\}, \ldots, \{45, 50\},$$

so there are 45 in total. (Note: the pair $\{a, b\}$ is indistinguishable from the pair $\{b, a\}$.

❷ If $|a - b| = 1$, then we have

$$\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{49, 50\},$$

or 49 pairs. If $|a - b| = 2$, then we have

$$\{1, 3\}, \{2, 4\}, \{3, 5\}, \ldots, \{48, 50\},$$

or 48 pairs. If $|a - b| = 3$, then we have

$$\{1, 4\}, \{2, 5\}, \{3, 6\}, \ldots, \{47, 50\},$$

or 47 pairs. If $|a - b| = 4$, then we have

$$\{1, 5\}, \{2, 6\}, \{3, 7\}, \ldots, \{46, 50\},$$

or 46 pairs. If $|a - b| = 5$, then we have

$$\{1, 6\}, \{2, 7\}, \{3, 8\}, \ldots, \{45, 50\},$$

or 45 pairs.
The total required is thus

$$49 + 48 + 47 + 46 + 45 = 235.$$

2.3.23 If $x = 0$, put $m(x) = 1$, otherwise put $m(x) = x$. We use three digits to label all the integers, from 000 to 999 If a, b, c are digits, then clearly $p(100a + 10b + c) = m(a)m(b)m(c)$. Thus

$$p(000) + \cdots + p(999) = m(0)m(0)m(0) + \cdots + m(9)m(9)m(9),$$

which in turn

$$\begin{aligned} &= (m(0) + m(1) + \cdots + m(9))^3 \\ &= (1 + 1 + 2 + \cdots + 9)^3 \\ &= 46^3 \\ &= 97336. \end{aligned}$$

Hence

$$\begin{aligned} S &= p(001) + p(002) + \cdots + p(999) \\ &= 97336 - p(000) \\ &= 97336 - m(0)m(0)m(0) \\ &= 97335. \end{aligned}$$

2.3.24 Points $16, 17, \ldots, 48$ are 33 in total and are on the same side of the diameter joining 15 to 49. For each of these points there is a corresponding diametrically opposite point. There are thus a total of $2 \cdot 33 + 2 = 68$ points.

2.3.25 There are 27 different sums. The sums 1 and 27 only appear once (in 100 and 999), each of the other 25 sums appears thrice. Thus if $27 + 25 + 1 = 53$ are drawn, at least 3 chips will have the same sum.

2.3.26 The shortest equality under the stated conditions must involve 3 numbers, and hence a maximum of 33 equalities can be achieved. The 33 equalities below shew that this maximum can be achieved.

$1+75=76$	$23+64=87$	$45+53=98$
$3+74=77$	$25+63=88$	$47+52=99$
$5+73=78$	$27+62=89$	$49+51=100$
$7+72=79$	$29+61=90$	$24+26=50$
$9+71=80$	$31+60=91$	$20+28=48$
$11+70=81$	$33+59=92$	$16+30=46$
$13+69=82$	$35+58=93$	$12+32=44$
$15+68=83$	$37+57=94$	$8+34=42$
$17+67=84$	$39+56=95$	$2+38=40$
$19+66=85$	$41+55=96$	$4+6=10$
$21+65=86$	$43+54=97$	$14+22=36$

2.3.27 Since $a+d=b+c$, we can write the four-tuple (a,b,c,d) as $(a,b,c,d)=(a,a+x,a+y,a+x+y)$, with integers $x,y,0<x<y$. Now, $93=bc-ad=(a+x)(a+y)-a(a+x+y)=xy$. Thus either $(x,y)=(1,93)$ or $(x,y)=(3,31)$. In the first case

$$(a,b,c,d)=(a,a+1,a+93,a+94)$$

is in the desired range for $1\leq a\leq 405$. In the second case,

$$(a,b,c,d)=(a,a+3,a+31,a+34)$$

is in the desired range for $1\leq a\leq 465$. These two sets of four-tuples are disjoint, and so the sought number of four-tuples is 870.

2.3.28 Let m be the largest member of the set and let n be its smallest member. Then $m\geq n+99$ since there are 100 members in the set. If the triangle with sides n,n,m is non-obtuse then $m^2\leq 2n^2$ from where

$$(n+99)^2\leq 2n^2 \iff n^2-198n-99^2\geq 0 \iff n\geq 99(1+\sqrt{2}) \iff n\geq 240.$$

If $n<240$ the stated condition is not met since $m^2\geq(n+99)^2\geq 2n^2$ and the triangle with sides of length n,n,m is not obtuse. Thus the set

$$\mathscr{A}=\{240,241,242,\ldots,339\}$$

achieves the required minimum. There are $100^3=1000000$ triangles that can be formed with length in $\mathscr{A}$ and so 3000000 sides to be added. Of these $3000000/100=30000$ are 240, 30000 are 241, etc. Thus the value required is

$$30000(240+241+\cdots+339)=(30000)\left(\frac{100(240+339)}{2}\right)=868500000.$$

2.3.29 Pair a with $(10^n-1-a.)$

2.3.30 Observe that person d changes the status of door n if and only if d divides n. Each divisor d of n can be paired off with $\frac{n}{d}$, and unless $d=\frac{n}{d}$, n would have an even number of divisors. Thus the doors closed are those for which n has an odd number of divisors, i.e. $d^2=n$, or n is a square. Hence doors $1,4,9,16,25,36,49,64,81$, and 100 are closed.

2.3.31 Assume $x^2+x-n=(x+a)(x-b)$, with $1\leq ab\leq 100$, and $a-b=1$. This means that $b=a-1$, and so we need integers a such that $1\leq a(a-1)\leq 100$. If $a>0$, then $2\leq a\leq 10$, and there are 9 possibilities for n. If $a<0$, then $-9\leq a\leq -1$, give the same 9 possibilities for n. Conclusion: there are 9 possibilities for n.

2.3.32 We condition on a, which can take any of the values $a=1,2,\ldots,100$. Given a, b can be any of the $101-a$ values in $\{a+1,a+2,\ldots,101\}$. Similarly, c can be any of the $101-a$ values in $\{a+1,a+2,\ldots,101\}$. Given a then, b and c may be chosen in $(101-a)(101-a)=(101-a)^2$ ways. The number of triplets is therefore

$$\begin{aligned}\sum_{a=1}^{100}(101-a)^2 &= 100^2+99^2+98^2+\cdots+1^2\\ &= \frac{100(100+1)(2(100)+1)}{6}\\ &= 338350.\end{aligned}$$

2.3.33 There are $9 \cdot 10^{j-1}$ j-digit positive integers. The total number of digits in numbers with at most r digits is the arithmetic-geometric sum

$$g(r) = \sum_{j=1}^{j} j \cdot 9 \cdot 10^{j-1} = r10^r - \frac{10^r - 1}{9}.$$

As $0 < \frac{10^r - 1}{9} < 10^r$, we get

$$(r-1)10^r < g(r) < r10^r.$$

Thus $g(1983) < 1983 \cdot 10^{1983} < 10^4 10^{1983} = 10^{1987}$ and $g(1984) > 1983 \cdot 10^{1984} > 10^3 10^{1984} = 10^{1987}$. Therefore $f(1987) = 1984$.

2.4.1 120

2.4.2 479001600; 4838400; 33868800

2.4.3 720; 24; 120; 144

2.4.4 1440

2.4.5 128

2.4.6 81729648000

2.4.7 249

2.4.8 We have

❶ This is $8!$.

❷ Permute XY in $2!$ and put them in any of the 7 spaces created by the remaining 6 people. Permute the remaining 6 people. This is $2! \cdot 7 \cdot 6!$.

❸ In this case, we alternate between sexes. Either we start with a man or a woman (giving 2 ways), and then we permute the men and the women. This is $2 \cdot 4!4!$.

❹ Glue the couples into 4 separate blocks. Permute the blocks in $4!$ ways. Then permute each of the 4 blocks in $2!$. This is $4!(2!)^4$.

❺ Sit the women first, creating 5 spaces in between. Glue the men together and put them in any of the 5 spaces. Permute the men in $4!$ ways and the women in $4!$. This is $5 \cdot 4!4!$.

2.5.1 1816214400

2.5.2 548

2.5.3 18

2.5.4 A. [1] 10000, [2] 5040, B. [1] 12 , [2] 10

2.5.5

1. $\frac{10!}{2!3!5!}$
2. $\frac{5!}{2!3!}$
3. $\frac{8!}{3!5!}$

2.5.6 We have

❶ This is

$$\frac{10!}{4!3!2!}$$

❷ This is

$$\frac{9!}{4!3!2!}$$

❸ This is

$$\frac{8!}{2!3!2!}$$

2.5.7 36

2.5.8 25

2.5.9 126126; 756756

2.6.2 $\binom{7}{2} = 21$

2.6.3 $\binom{7}{1}\binom{5}{3} = (7)(10) = 70$

2.6.4 $\binom{N}{2}$

2.6.5 $\binom{8}{4}4! = 1680$

2.6.6 $\binom{25}{2} = 300$

2.6.7 Let the subsets be A and B. We have either $\mathbf{card}\,(A) = 1$ or $\mathbf{card}\,(A) = 2$. If $\mathbf{card}\,(A) = 1$ then there are $\binom{4}{1} = 4$ ways of choosing its elements and $\binom{3}{3} = 1$ ways of choosing the elements of B. If $\mathbf{card}\,(A) = 2$ then there are $\binom{4}{2} = 6$ ways of choosing its elements and $\binom{2}{2} = 1$ ways of choosing the elements of B. Altogether there are $4 + 6 = 10$ ways.

2.6.8 $\binom{6}{3} = 20$

2.6.9 We count those numbers that have exactly once, twice and three times. There is only one number that has it thrice (namely 333). Suppose the number xyz is to have the digit 3 exactly twice. We can choose these two positions in $\binom{3}{2}$ ways. The third position can be filled with any of the remaining nine digits (the digit 3 has already been used). Thus there are $9\binom{3}{2}$ numbers that the digit 3 exactly twice. Similarly, there are $9^2\binom{3}{2}$ numbers that have 3 exactly once. The total required is hence $3 \cdot 1 + 2 \cdot 9 \cdot \binom{3}{2} + 9^2\binom{3}{1} = 300.$

2.6.10 $\binom{5}{3} = 10$

2.6.11 $\binom{5}{1} + \binom{5}{3} + \binom{5}{5} = 5 + 10 + 1 = 16.$

2.6.12 $10 \times 3! = 60$

2.6.13 We have

❶ $(E + F + S + I)!$

❷ $4! \cdot E!F!S!I!$

❸ $\binom{E+F+I+1}{1}S!(E+F+I)!$

❹ $\binom{E+F+I+1}{S}S!(E+F+I)!$

❺ $2!\binom{F+I+1}{2}S!E!(F+I)!$

2.6.14 We can choose the seven people in $\binom{20}{7}$ ways. Of the seven, the chairman can be chosen in seven ways. The answer is thus

$$7\binom{20}{7} = 542640.$$

Aliter: Choose the chairman first. This can be done in twenty ways. Out of the nineteen remaining people, we just have to choose six, this can be done in $\binom{19}{6}$ ways. The total number of ways is hence $20\binom{19}{6} = 542640$.

2.6.15 We can choose the seven people in $\binom{20}{7}$ ways. Of these seven people chosen, we can choose the chairman in seven ways and the secretary in six ways. The answer is thus $7 \cdot 6\binom{20}{7} = 3255840.$

Aliter: If one chooses the chairman first, then the secretary and finally the remaining five people of the committee, this can be done in $20 \cdot 19 \cdot \binom{18}{5} = 3255840$ ways.

2.6.16 For a string of three-digit numbers to be decreasing, the digits must come from $\{0, 1, \ldots, 9\}$ and so there are $\binom{10}{3} = 120$ three-digit numbers with all its digits in decreasing order. If the string of three-digit numbers is increasing, the digits have to come from $\{1, 2, \ldots, 9\}$, thus there are $\binom{9}{3} = 84$ three-digit numbers with all the digits increasing. The total asked is hence $120 + 84 = 204.$

2.6.17 We can choose the four students who are going to take the first test in $\binom{20}{4}$ ways. From the remaining ones, we can choose students in $\binom{16}{4}$ ways to take the second test. The third test can be taken in $\binom{12}{4}$ ways. The fourth in $\binom{8}{4}$ ways and the fifth in $\binom{4}{4}$ ways. The total number is thus

$$\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}.$$

2.6.18 We align the thirty-nine cards which are not hearts first. There are thirty-eight spaces between them and one at the beginning and one at the end making a total of forty spaces where the hearts can go. Thus there are $\binom{40}{13}$ ways of choosing the *places* where the hearts can go. Now, since we are interested in arrangements, there are $39!$ different configurations of the non-hearts and $13!$ different configurations of the hearts. The total number of arrangements is thus $\binom{40}{13}39!13!$.

2.6.19 The equality signs cause us trouble, since allowing them would entail allowing repetitions in our choices. To overcome that we establish a one-to-one correspondence between the vectors $(a, b, c, d), 0 \leq a \leq b \leq c \leq d \leq n$ and the vectors $(a', b', c', d'), 0 \leq a' < b' < c' < d' \leq n+3$. Let $(a', b', c', d') = (a, b+1, c+2, d+3)$. Now we just have to pick four different numbers from the set $\{0, 1, 2, 3, \ldots, n, n+1, n+2, n+3\}$. This can be done in $\binom{n+4}{4}$ ways.

2.6.20 We have

❶ $(T+L+W)!$

❷ $3!T!L!W! = 6T!L!W!$

❸ $\binom{T+L+1}{W}(T+L)!W!$

❹ $\binom{T+L+1}{1}(T+L)!W!$

2.6.21 The required number is

$$\binom{20}{1}+\binom{20}{2}+\cdots+\binom{20}{20} = 2^{20}-\binom{20}{0} = 1048576-1 = 1048575.$$

2.6.22 The required number is

$$\binom{20}{4}+\binom{20}{6}+\cdots+\binom{20}{20} = 2^{19}-\binom{20}{0}-\binom{20}{2} = 524288-1-190 = 524097.$$

2.6.23 We have

❶ $\frac{13!}{2!3!3!} = 86486400$

❷ $\frac{11!}{2!3!} = 3326400$

❸ $\frac{11!}{2!2!2!} = 4989600$

❹ $\binom{12}{1}\frac{11!}{3!3!} = 13305600$

❺ $\binom{12}{2}\frac{11!}{3!3!} = 73180800$

❻ $\binom{10}{1}\frac{9!}{3!3!2!} = 50400$

2.6.24 We have

❶ $\binom{M+W}{C}$

❷ $\binom{M}{C-T}\binom{W}{T}$

❸ $\binom{M+W-2}{C-2}$

❹ $\binom{M+W-2}{C}$

2.6.25

$$\binom{M+W}{C}-\binom{M+W-2}{C-2} = 2\binom{M+W-2}{C-1}+\binom{M+W-2}{C}.$$

2.6.26 2030

2.6.27 $2\binom{50}{2}$

2.6.28 $\binom{n+k-1}{k}$

2.6.29 [1] For the first column one can put any of 4 checkers, for the second one, any of 3, etc. hence there are $4\cdot 3\cdot 2\cdot 1 = 24$. [2] If there is a column without a checker then there must be a column with 2 checkers. There are 3 choices for this column. In this column we can put the two checkers in $\binom{4}{2} = 6$ ways. Thus there are $4\cdot 3\binom{4}{2}4\cdot 4 = 1152$ ways of putting the checkers. [3] The number of ways of filling the board with no restrictions is $\binom{16}{4}$. The number of ways of of of filling the board so that there is one checker per column is 4^4. Hence the total is $\binom{16}{4} - 4^4 = 1564$.

2.6.30 7560.

2.6.31 $\frac{1}{4!}\binom{8}{2}\binom{6}{2}\binom{4}{2}$.

2.6.32 $\binom{15}{7}\binom{8}{4}$.

2.6.32 There are 6513215600 of former and 3486784400 of the latter.

2.6.33 $\binom{17}{5}\binom{12}{5}\binom{7}{4}\binom{3}{3}$; $\binom{17}{3}\binom{14}{4}2^{10}$.

2.6.34 $\sum_{k=3}^{7}\binom{7}{k} = 99$

2.6.35 $2^{10} - 1 - 1 - \binom{10}{5} = 1024 - 2 - 252 = 770$

2.6.36 $\binom{n}{2}$; $n-1$; $\binom{n}{3}$; $\binom{n-1}{2}$

2.6.37 $\binom{12}{1}\binom{11}{5}\binom{6}{2}\binom{4}{4}$

2.6.38 $\binom{6}{3}^{20} = 104857600000000000000000000$

2.6.39 $\binom{9}{3}\binom{5}{3} = 840$

2.6.40 $\binom{b}{c}\binom{g}{c}c!$

2.6.41 $(2^3 - 1)(2^4 - 1)(2^2 - 1) = 315$

2.6.44 $\binom{10}{2}2^8$

2.6.45 We have

$$\begin{aligned}
\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-k-1)!} \\
&= \frac{(n-1)!}{(n-k-1)!(k-1)!}\left(\frac{1}{n-k} + \frac{1}{k}\right) \\
&= \frac{(n-1)!}{(n-k-1)!(k-1)!}\frac{n}{(n-k)k} \\
&= \frac{n!}{(n-k)!k!}. \\
&= \binom{n}{k}.
\end{aligned}$$

A combinatorial interpretation can be given as follows. Suppose we have a bag with n red balls. The number of ways of choosing k balls is n. If we now paint one of these balls blue, the number of ways of choosing k balls is the number of ways of choosing balls if we always *include* the blue ball (and this can be done in $\binom{n-1}{k-1}$) ways, plus the number of ways of choosing k balls if we always *exclude* the blue ball (and this can be done in $\binom{n-1}{k}$ ways).

2.6.46 The sinistral side counts the number of ways of selecting r elements from a set of n, then selecting k elements from those r. The dextral side counts how many ways to select the k elements first, then select the remaining $r-k$ elements to be chosen from the remaining $n-k$ elements.

2.6.47 The dextral side sums
$$\binom{n}{0}\binom{n}{0}+\binom{n}{1}\binom{n}{1}+\binom{n}{2}\binom{n}{2}+\cdots+\binom{n}{n}\binom{n}{n}.$$
By the symmetry identity, this is equivalent to summing
$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n}\binom{n}{0}.$$
Now consider a bag with $2n$ balls, n of them red and n of them blue. The above sum is counting the number of ways of choosing 0 red balls and n blue balls, 1 red ball and $n-1$ blue balls, 2 red balls and $n-2$ blue balls, etc.. This is clearly the number of ways of choosing n balls of either colour from the bag, which is $\binom{2n}{n}$.

2.6.48 Consider choosing n balls from a bag of r yellow balls and s white balls.

2.6.49 The numbers belong to the following categories: (I) all six digits are identical; (II) there are exactly two different digits used, three of one kind, three of the other; (III) there are exactly two different digits used, two of one kind, four of the other; (IV) there are exactly three different digits used, two of each kind.

There are clearly 9 numbers belonging to category (I). To count the numbers in the remaining categories, we must consider the cases when the digit 0 is used or not. If 0 is not used, then there are $\binom{9}{2}\cdot\frac{6!}{3!3!}=720$ integers in category (II); $\binom{9}{1}\binom{8}{1}\cdot\frac{6!}{2!4!}=1080$ integers in category (III); and $\binom{9}{3}\cdot\frac{6!}{2!2!2!}=7560$ integers in category (IV). If 0 is used, then the integers may not start with 0. There are $\binom{9}{1}\cdot\frac{5!}{2!3!}=90$ in category (II) ; $\binom{9}{1}\cdot(\frac{5!}{1!4!}+\frac{5!}{3!2!})=135$ in category (III) ; and $\binom{9}{2}\cdot 2\cdot\frac{5!}{1!2!2!}=3240$ in category (IV). Thus there are altogether
$$9+720+1080+7560+90+135+3240=12834$$
such integers.

2.6.50 The numbers belong to the following categories: (I) all seven digits are identical; (II) there are exactly two different digits used, three of one kind, four of the other.

There are clearly 9 numbers belonging to category (I). To count the numbers in the remaining category (II), we must consider the cases when the digit 0 is used or not. If 0 is not used, then there are $\binom{9}{1}\binom{8}{1}\cdot\frac{7!}{3!4!}=2520$ integers in category (II). If 0 is used, then the integers may not start with 0. There are $\binom{9}{1}\cdot\frac{6!}{2!4!}+\binom{9}{1}\cdot\frac{6!}{3!3!}=315$ in category (II). Thus there are altogether $2520+315+9=2844$ such integers.

2.6.51 432

2.6.52 $\binom{15}{9}$; $15!/6!$

2.6.53 29.

2.6.54 2^4

2.6.55 $\binom{8}{5}5!$

2.6.56 175308642

2.6.57 Hint: There are k occupied boxes and $n-k$ empty boxes. Align the balls first! $\binom{k+1}{n-k}$.

2.6.58 There are $n-k$ empty seats. Sit the people in between those seats. $\binom{n-k+1}{k}$.

2.6.59 Let A_i be the property that the i-th letter is put back into the i-th envelope. We want
$$\mathbf{card}\left(A_1^c\cap A_2^c\cap\cdots\cap A_{10}^c\right).$$
Now, if we accommodate the i-th letter in its envelope, the remaining nine letters can be put in 9! different ways in the nine remaining envelopes, thus $\mathbf{card}\left(A_i\right)=9!$. Similarly $\mathbf{card}\left(A_i\cap A_j\right)=8!$, $\mathbf{card}\left(A_i\cap A_j\cap A_k\right)=7!$, etc. for unequal $i,j,k,\ldots$. Now, there are $\binom{10}{1}$ ways of choosing i, $\binom{10}{2}$ ways of choosing different pairs i,j, etc.. Since
$$\mathbf{card}\left(A_1\cup A_2\cup\cdots\cup A_{10}\right)+\mathbf{card}\left(A_1^c\cap A_2^c\cap\cdots+\cap A_{10}^c\right)=10!,$$
by the Inclusion-Exclusion Principle we gather that
$$\mathbf{card}\left(A_1^c\cap A_2^c\ \cap\cdots\cap A_{10}^c\right)=10!-\left(\binom{10}{1}9!+\binom{10}{2}8!-\binom{10}{3}7!\ +\cdots-\binom{10}{9}1!+\binom{10}{10}0!\right).$$

2.7.1 36

2.7.2 From the preceding problem subtract those sums with $1+2+7$ ($3! = 6$ of them) and those with $1+1+8$ ($\frac{3!}{2!} = 3$ of them). The required total is $36 - 9 = 27$.

2.7.3 $\binom{14}{4}$

2.7.4 Put $x_k = y_k + k - 1$ with $y_k \geq 1$. Then

$$(y_1 + 0) + (y_2 + 1) + \cdots + (y_{100} + 99) = n$$

implies that

$$y_1 + y_2 + \cdots + y_{100} = n - 4950.$$

Hence there are $\binom{n-4951}{99}$ solutions.

2.7.5 Put $a = 2a' - 1$ with $a' \geq 1$, etc. Then

$$(2a' - 1) + \cdots + (2d' - 1) = 98 \implies a' + \cdots + d' = 51.$$

Thus there are $\binom{50}{3} = 19600$ solutions.

2.7.6 Consider the following categorization:

1. with (x, y) in the first quadrant
2. $(0, 0)$, the origin,
3. $(x, 0)$ with $1 \leq x \leq 99$.

Thus the number of lattice points sought is four times the number in (1), plus 1, plus four times the number in (3).

Clearly, the number of lattice points in (3) is 99.

The number of (1) is the number of strictly positive solutions to $x + y < 100$. Let $z = 100 - x - y$, the discrepancy of $x + y$ from 100. Then we are counting the number of strictly positive solutions to $x + y + z = 100$. To count these, write 100 as a sum of 100 ones:

$$\underbrace{1 + 1 + \cdots + 1}_{100 \text{ ones}}.$$

Observe that there are 99 plus signs. Of these, we must choose two, because the equation $x + y + z = 100$ has two. Thus there are $\binom{99}{2} = 4851$ such points.

The required number of points is thus

$$4 \cdot 4851 + 4 \cdot 99 + 1 = 19801.$$

3.2.1 We are given that $\mathbf{P}(a) = 2\mathbf{P}(b)$, $\mathbf{P}(b) = 4\mathbf{P}(c)$, $\mathbf{P}(c) = 2\mathbf{P}(d)$. The trick is to express three of the probabilities in terms of one of the four. We will express all probabilities in terms of outcome d. Thus

$$\mathbf{P}(b) = 4\mathbf{P}(c) = 4(2\mathbf{P}(d)) = 8\mathbf{P}(d),$$

and

$$\mathbf{P}(a) = 2\mathbf{P}(b) = 2(8\mathbf{P}(d)) = 16\mathbf{P}(d).$$

Now

$$\begin{aligned} \mathbf{P}(a) + \mathbf{P}(b) + \mathbf{P}(c) + \mathbf{P}(d) = 1 \quad &\implies \quad 16\mathbf{P}(d) + 8\mathbf{P}(d) + 2\mathbf{P}(d) + \mathbf{P}(d) = 1 \\ &\implies \quad 27\mathbf{P}(d) = 1, \end{aligned}$$

whence $\mathbf{P}(d) = \frac{1}{27}$. This yields

$$\mathbf{P}(a) = 16\mathbf{P}(d) = \frac{16}{27},$$

$$\mathbf{P}(b) = 8\mathbf{P}(d) = \frac{8}{27},$$

and

$$\mathbf{P}(c) = 2\mathbf{P}(d) = \frac{2}{27}.$$

Observe that all probabilities are between 0 and 1 and that they add up to 1.

3.2.2 The trick is to express all probabilities in terms of a single one. We will express $\mathbf{P}(a)$, $\mathbf{P}(b)$, $\mathbf{P}(c)$, in terms of $\mathbf{P}(d)$. We have

$$\mathbf{P}(b) = 3\mathbf{P}(c) = 3(3\mathbf{P}(d)) = 9\mathbf{P}(d),$$
$$\mathbf{P}(a) = 3\mathbf{P}(b) = 3(9\mathbf{P}(d)) = 27\mathbf{P}(d).$$

Now

$$\mathbf{P}(a)+\mathbf{P}(b)+\mathbf{P}(c)+\mathbf{P}(d)=1 \implies 27\mathbf{P}(d)+9\mathbf{P}(d)+3\mathbf{P}(d)+\mathbf{P}(d)=1$$
$$\implies \mathbf{P}(d)=\frac{1}{40}.$$

Whence

$$\mathbf{P}(a)=27\mathbf{P}(d)=\frac{27}{40},$$
$$\mathbf{P}(b)=9\mathbf{P}(d)=\frac{9}{40},$$
$$\mathbf{P}(c)=3\mathbf{P}(d)=\frac{3}{40}.$$

3.2.3 By Theorem 121,

$$\mathbf{P}(A\cup B)=0.8+0.5-0.4=0.9.$$

By Corollary 119 and the De Morgan Law's,

$$\mathbf{P}(A^c\cap B^c)=\mathbf{P}((A\cup B)^c)=1-\mathbf{P}(A\cup B)=1-0.9=0.1,$$
$$\mathbf{P}(A^c\cup B^c)=\mathbf{P}((A\cap B)^c)=1-\mathbf{P}(A\cap B)=1-0.4=0.5.$$

3.2.4 The maximum is 0.6, it occurs when $B\subseteq A$. Now by Theorem 121 and using the fact that $\mathbf{P}(A\cup B)\leq 1$, we have

$$\mathbf{P}(A\cap B)=\mathbf{P}(A)+\mathbf{P}(B)-\mathbf{P}(A\cup B)\geq 1.5-1=0.5,$$

whence the minimum value is 0.5.

3.2.5 Let C be the event that patient visits a chiropractor, and T be the event that patient visits a physical therapist. The data stipulates that

$$\mathbf{P}(C)=\mathbf{P}(T)+0.14, \qquad \mathbf{P}(C\cap T)=0.22, \qquad \mathbf{P}(C^c\cap T^c)=0.12.$$

Now,

$$0.88=1-\mathbf{P}(C^c\cap T^c)=\mathbf{P}(C\cup T)=\mathbf{P}(C)+\mathbf{P}(T)-\mathbf{P}(C\cap T)=2\mathbf{P}(T)-0.08,$$

whence $\mathbf{P}(T)=0.48$.

3.2.6 0.8

3.2.7 Let R and B are the events that Rocinante wins, and that Babieca wins, respectively, in 8-horse race. As R and B are mutually exclusive, we deduce that

$$\begin{aligned}\mathbf{P}(R\cup B) &= \mathbf{P}(A)+\mathbf{P}(B)\\ &= \frac{2}{2+5}+(1-\mathbf{P}(B^c))\\ &= \frac{2}{7}+\left(1-\frac{7}{7+3}\right)\\ &= \frac{2}{7}+\frac{3}{10}\\ &= \frac{41}{70}.\end{aligned}$$

3.3.1 Let $\mathbf{P}(\mathrm{X}=k)=\alpha k$. Then

$$1=\mathbf{P}(\mathrm{X}=1)+\cdots+\mathbf{P}(\mathrm{X}=6)=\alpha(1^2+\cdots+6^2)=91\alpha$$

giving $\alpha=\dfrac{1}{91}$ and $\mathbf{P}(\mathrm{X}=k)=\dfrac{k}{91}$.

3.3.5 Since the probability of obtaining the sum 1994 is strictly positive, there are $n\geq \llbracket\frac{1994}{6}\rrbracket=333$ dice. Let $x_1+x_2+\cdots+x_n=1994$ be the sum of the faces of the n dice adding to 1994. We are given that

$$(7-x_1)+(7-x_2)+\cdots+(7-x_n)=\mathrm{S}$$

or $7n-1994=\mathrm{S}$. The minimal sum will be achieved with the minimum dice, so putting $n=333$ we obtain the minimal $\mathrm{S}=7(333)-1994=337$.

3.4.1 $\left(\dfrac{1}{27}\right)^{48}$

3.4.2 $\dfrac{5}{144}$

3.4.3 $\dfrac{3}{8}$

3.4.4 $\frac{1}{52}$

3.4.5 $\frac{41}{81}$

3.4.6 Use Theorem 3.2.8. The desired probability is $\frac{23}{32}$.

3.4.7 We have

$$\begin{aligned}
\mathbf{P}\left(|X-Y|=1\right) &= \mathbf{P}\left(X-Y=1\right)+\mathbf{P}\left(Y-X=1\right)\\
&= 2\mathbf{P}\left(X-Y=1\right)\\
&= 2(\mathbf{P}\left(X=1\cap Y=0\right)+\mathbf{P}\left(X=2\cap Y=1\right))\\
&= 2(\mathbf{P}\left(X=1\right)\mathbf{P}\left(Y=0\right)+\mathbf{P}\left(X=2\right)\mathbf{P}\left(Y=1\right))\\
&= 2((.4)(.2)+(.4)(.4))\\
&= .48,
\end{aligned}$$

since the sampling with replacement gives independence.

3.4.8 Suppose there are n re-reading necessary in order that there be no errors. At each re-reading, the probability that a typo is not corrected is $\frac{2}{3}$. Thus the probability that a particular typo is never corrected is $(\frac{2}{3})^n$. Hence the probability that a particular typo is corrected in the n re-readings is $1-(\frac{2}{3})^n$. Thus the probability that all typos are corrected is

$$\left(1-\left(\frac{2}{3}\right)^n\right)^4.$$

We need

$$\left(1-\left(\frac{2}{3}\right)^n\right)^4 \geq 0.9$$

and with a calculator we may verify that this happens for $n \geq 10$.

3.4.9 The probability of not obtaining a six in a single trial is $\frac{5}{6}$. The probability of not obtaining a single six in the three trials is $(\frac{5}{6})^3=\frac{125}{216}$. Hence the probability of obtaining at least one six in three rolls is $1-\frac{125}{216}=\frac{91}{216}$.

3.4.10 By inclusion-exclusion and by independence,

$$\begin{aligned}
\mathbf{P}\left(A\cup B\cup C\right) &= \mathbf{P}\left(A\right)+\mathbf{P}\left(B\right)+\mathbf{P}\left(C\right)-\mathbf{P}\left(A\cap B\right)-\mathbf{P}\left(A\cap C\right)\\
&\quad -\mathbf{P}\left(B\cap C\right)+\mathbf{P}\left(A\cap B\cap C\right)\\
&= \frac{1}{3}+\frac{1}{3}+\frac{1}{3}-\frac{1}{9}-\frac{1}{9}-\frac{1}{9}+\frac{1}{27}\\
&= \frac{19}{27}.
\end{aligned}$$

3.4.11 $(\frac{7}{9})^{10}$

3.5.1 $\frac{13}{51}$; $\frac{25}{102}$; $\frac{4}{663}$; $\frac{1}{221}$

3.5.2 $\frac{1}{116}$

3.5.3 $\frac{473}{16215}$

3.5.4 We have

$$\mathbf{P}\left(A\cap B\right)=\mathbf{P}\left(A|B\right)\mathbf{P}\left(B\right)=\frac{1}{6},\implies \mathbf{P}\left(A\cup B\right)=\mathbf{P}\left(A\right)+\mathbf{P}\left(B\right)-\mathbf{P}\left(A\cap B\right)=1$$

whence

$$\mathbf{P}\left(A^c\cap B^c\right)=\mathbf{P}\left((A\cup B)^c\right)=1-\mathbf{P}\left(A\cup B\right)=0.$$

3.5.5 $\frac{3}{8}$

3.5.6 $\dfrac{3}{16}$

3.5.7 Observe that there are 10 ways of getting a sum of six in three dice: the 3 permutations of $(1,1,4)$, the 6 permutations of $(1,2,3)$, and the 1 permutation of $(2,2,2)$. Of these, only $(2,2,2)$ does not require a 1. Let S be the event that the sum of the dice is 6 and let N be the event that no die landed on a 1. We need

$$\mathbf{P}\left(N|S\right) = \frac{\mathbf{P}\left(N\cap S\right)}{\mathbf{P}\left(S\right)} = \frac{\frac{1}{216}}{\frac{10}{216}} = \frac{1}{10}.$$

3.5.9 $\dfrac{1}{4}$

3.5.10 $\dfrac{30}{31}$

3.5.11 $\dfrac{7}{18}$

3.5.12 Let A be the event that Peter's letter is received by Paul and B be the event that Paul's letter is received by Peter. Then we want $\mathbf{P}\left(A|B^c\right)$. Then

$$\begin{aligned}\mathbf{P}\left(A|B^c\right) &= \frac{A\cap B^c}{\mathbf{P}\left(B^c\right)} \\ &= \frac{\mathbf{P}\left(B^c|A\right)\cdot\mathbf{P}\left(A\right)}{\mathbf{P}\left(B^c|A\right)\cdot\mathbf{P}\left(A\right)+\mathbf{P}\left(B^c|A^c\right)\cdot\mathbf{P}\left(A^c\right)} \\ &= \frac{\frac{1}{n}\cdot\frac{n-1}{n}}{\frac{1}{n}\cdot\frac{n-1}{n}+1\cdot\frac{1}{n}} \\ &= \frac{n-1}{2n-1}.\end{aligned}$$

3.5.13 We condition on whether the interchanged card is the one selected on the second half. Let A be the event that the selected on the second half card was an ace, and let I be the event that the card selected was the interchanged one. Then

$$\mathbf{P}\left(A\right) = \mathbf{P}\left(A|I\right)\mathbf{P}\left(I\right) + \mathbf{P}\left(A|I^c\right)\mathbf{P}\left(I^c\right) = 1\cdot\frac{1}{27} + \frac{3}{51}\cdot\frac{26}{27} = \frac{43}{459}.$$

3.5.14 Let I be the event that a customer insures more than one car. Let S be the event that a customer insures a sports car. We are given that

$$\mathbf{P}\left(I\right) = 0.7, \quad \mathbf{P}\left(S\right) = 0.2, \quad \mathbf{P}\left(S|I\right) = 0.15.$$

This gives

$$\mathbf{P}\left(S\cap I\right) = \mathbf{P}\left(S|I\right)\mathbf{P}\left(I\right) = (0.15)(0.7) = 0.105.$$

We want $\mathbf{P}\left(I^c\cap S^c\right)$. By the De Morgan Laws and Inclusion-Exclusion

$$\begin{aligned}\mathbf{P}\left(I^c\cap S^c\right) &= \mathbf{P}\left((I\cup S)^c\right) \\ &= 1-\mathbf{P}\left(I\cup S\right) \\ &= 1-\left(\mathbf{P}\left(I\right)+\mathbf{P}\left(S\right)-\mathbf{P}\left(I\cap S\right)\right) \\ &= 1-(0.7+0.2-0.105) \\ &= 0.205.\end{aligned}$$

3.5.15

1. We may write

$$D = (A\cup B)\setminus(A\cap B) = (A\setminus B)\cup(B\setminus A).$$

Thus by Inclusion-Exclusion,

$$\mathbf{P}\left(D\right) = \mathbf{P}\left(A\cup B\right) - \mathbf{P}\left(A\cap B\right) = \mathbf{P}\left(A\right) + \mathbf{P}\left(B\right) - 2\mathbf{P}\left(A\cap B\right).$$

By independence, $\mathbf{P}\left(A\cap B\right) = \mathbf{P}\left(A\right)\mathbf{P}\left(B\right)$ and so,

$$\mathbf{P}\left(D\right) = \mathbf{P}\left(A\right) + \mathbf{P}\left(B\right) - 2\mathbf{P}\left(A\right)\mathbf{P}\left(B\right) = 0.2 + 0.3 - 2(0.2)(0.3) = 0.38.$$

2. First observe that

$$D = (A\setminus B)\cup(B\setminus A) \implies A\cap D = (A\setminus B)\cap A = A\setminus A\cap B,$$

and so

$$\mathbf{P}\left(A\cap D\right) = \mathbf{P}\left(A\right) - \mathbf{P}\left(A\cap B\right) = 0.2-(0.2)(0.3) = 0.14.$$

Hence,

$$\mathbf{P}\left(A|D\right) = \frac{\mathbf{P}\left(A\cap D\right)}{\mathbf{P}\left(D\right)} = \frac{0.14}{0.38} = \frac{7}{19}.$$

3. The events C and D are disjoint, hence $\mathbf{P}(A \cap D) = \mathbf{P}(\varnothing) = 0$. On the other hand, $\mathbf{P}(C)\mathbf{P}(D) = (0.06)(0.38) \neq 0$, and therefore the events are not independent.

3.6.1 Let Y, F, E denote the events of choosing the 30% heads, the 50% heads, and the 80% heads, respectively. Now,

$$\begin{aligned}\mathbf{P}(HHT) &= \mathbf{P}(HHT|Y)\cdot\mathbf{P}(Y) + \mathbf{P}(HHT|F)\cdot\mathbf{P}(F) + \mathbf{P}(HHT|E)\cdot\mathbf{P}(E) \\ &= \frac{3\times 3\times 7}{1000}\cdot\frac{1}{3} + \frac{5\times 5\times 5}{1000}\cdot\frac{1}{3} + \frac{8\times 8\times 2}{1000}\cdot\frac{1}{3} \\ &= \frac{79}{750},\end{aligned}$$

whence

$$\begin{aligned}\mathbf{P}(F|HHT) &= \frac{\mathbf{P}(F\cap HHT)}{\mathbf{P}(HHT)} \\ &= \frac{\mathbf{P}(HHT|F)\cdot\mathbf{P}(F)}{\mathbf{P}(HHT)} \\ &= \frac{\frac{5\times 5\times 5}{1000}\cdot\frac{1}{3}}{\frac{79}{750}} \\ &= \frac{125}{316}\end{aligned}$$

3.6.2 Let T denote the event that Tom operates the machinery, S the event that Sally operates the machinery and H that two out of three pieces of the output be of high quality. Then

$$\begin{aligned}\mathbf{P}(H) &= \mathbf{P}(H|T)\cdot\mathbf{P}(T) + \mathbf{P}(H|S)\cdot\mathbf{P}(S) \\ &= \binom{3}{2}\left(\frac{70}{100}\right)^2\left(\frac{30}{100}\right)\cdot\frac{3}{5} + \binom{3}{2}\left(\frac{90}{100}\right)^2\left(\frac{10}{100}\right)\cdot\frac{2}{5} \\ &= \frac{1809}{5000},\end{aligned}$$

whence

$$\begin{aligned}\mathbf{P}(T|H) &= \frac{\mathbf{P}(H|T)\cdot\mathbf{P}(T)}{\mathbf{P}(H)} \\ &= \frac{\binom{3}{2}\left(\frac{70}{100}\right)^2\left(\frac{30}{100}\right)\cdot\frac{3}{5}}{\frac{1809}{5000}} \\ &= \frac{49}{67}.\end{aligned}$$

3.6.3

❶
$$\mathbf{P}(6) = \mathbf{P}(6\cap I) + \mathbf{P}(6\cap II) = \frac{1}{2}\cdot p + \frac{1}{2}\cdot 1 = \frac{p+1}{2}$$

❷ $\mathbf{P}(6\cap I) = \frac{1}{2}\cdot p = \frac{p}{2}$

❸
$$\mathbf{P}(I|6) = \frac{\mathbf{P}(6\cap I)}{\mathbf{P}(6)} = \frac{p}{p+1}.$$

3.6.4

❶
$$\mathbf{P}(Q) = \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2}.$$

❷
$$\mathbf{P}(Q\cap III) = \frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6}$$

❸
$$\mathbf{P}(III|Q) = \frac{\mathbf{P}(III\cap Q)}{\mathbf{P}(Q)} = \frac{1}{3}.$$

3.6.5 ❶ Conditioning on the urn chosen,

$$\begin{aligned}\mathbf{P}(G) &= \mathbf{P}(G|\mathrm{A})\,\mathbf{P}(\mathrm{A}) + \mathbf{P}(G|\mathrm{B})\,\mathbf{P}(\mathrm{B}) + \mathbf{P}(G|\mathrm{C})\,\mathbf{P}(\mathrm{C}) \\ &= \frac{b}{a+b}\cdot\frac{1}{3} + \frac{d}{c+d}\cdot\frac{1}{3} + \frac{c}{a+c}\cdot\frac{1}{3}.\end{aligned}$$

❷ This is clearly $\dfrac{c}{a+c}$.

❸ We use Bayes' Rule

$$\begin{aligned}\mathbf{P}\left(\mathrm{C}|G\right) &= \frac{\mathbf{P}\left(\mathrm{C}\cap G\right)}{\mathbf{P}\left(G\right)} \\ &= \frac{\mathbf{P}\left(G|\mathrm{C}\right)\mathbf{P}\left(\mathrm{C}\right)}{\mathbf{P}\left(G\right)} \\ &= \frac{\frac{c}{a+c}\cdot\frac{1}{3}}{\frac{b}{a+b}\cdot\frac{1}{3}+\frac{d}{c+d}\cdot\frac{1}{3}+\frac{c}{a+c}\cdot\frac{1}{3}} \\ &= \frac{\frac{c}{a+c}}{\frac{b}{a+b}+\frac{d}{c+d}+\frac{c}{a+c}}\end{aligned}$$

❹ Conditioning on the urn chosen,

$$\begin{aligned}\mathbf{P}\left(R\right) &= \mathbf{P}\left(R|\mathrm{A}\right)\mathbf{P}\left(\mathrm{A}\right)+\mathbf{P}\left(R|\mathrm{B}\right)\mathbf{P}\left(\mathrm{B}\right)+\mathbf{P}\left(R|\mathrm{C}\right)\mathbf{P}\left(\mathrm{C}\right) \\ &= \frac{a}{a+b}\cdot\frac{1}{3}+\frac{c}{c+d}\cdot\frac{1}{3}+\frac{a}{a+c}\cdot\frac{1}{3}.\end{aligned}$$

❺ This is clearly $\dfrac{a}{a+b}$.

❻ We use Bayes' Rule

$$\begin{aligned}\mathbf{P}\left(\mathrm{A}|R\right) &= \frac{\mathbf{P}\left(\mathrm{A}\cap R\right)}{\mathbf{P}\left(R\right)} \\ &= \frac{\mathbf{P}\left(R|\mathrm{C}\right)\mathbf{P}\left(\mathrm{C}\right)}{\mathbf{P}\left(R\right)} \\ &= \frac{\frac{a}{a+b}\cdot\frac{1}{3}}{\frac{a}{a+b}\cdot\frac{1}{3}+\frac{c}{c+d}\cdot\frac{1}{3}+\frac{a}{a+c}\cdot\frac{1}{3}} \\ &= \frac{\frac{a}{a+b}}{\frac{a}{a+b}+\frac{c}{c+d}+\frac{a}{a+c}}\end{aligned}$$

3.6.6 $\dfrac{p}{p+q+r}$

3.6.8 $\dfrac{10}{17}$

3.6.9 $\dfrac{91}{371}$

3.6.10 $\dfrac{15}{43}$

3.6.11 $\dfrac{2}{5}$

3.6.12 $\dfrac{1}{35}$

3.6.13 $\dfrac{1}{3}$

3.6.14

1. 0.76
2. 0.91

3.6.15 Let H be the event that Hugh was infected and let C_1, C_2, C_3 be the events that child $i = 1, 2, 3$, respectively, is a cyclops. For the first question we want $\mathbf{P}\left(C_1^c \cap C_2^c\right)$. We will condition on Hugh getting infected, and thus

$$\mathbf{P}\left(C_1^c \cap C_2^c\right) = \mathbf{P}\left(C_1^c \cap C_2^c|H\right)\mathbf{P}\left(H\right) + \mathbf{P}\left(C_1^c \cap C_2^c|H^c\right)\mathbf{P}\left(H^c\right).$$

Since C_1 and C_2 are independent, this becomes

$$\mathbf{P}\left(C_1^c \cap C_2^c\right) = \mathbf{P}\left(C_1^c|H\right)\mathbf{P}\left(C_2^c|H\right)\mathbf{P}\left(H\right) + \mathbf{P}\left(C_1^c|H^c\right)\mathbf{P}\left(C_2^c|H^c\right)\mathbf{P}\left(H^c\right) = \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{3}+1\cdot1\cdot\frac{2}{3} = \frac{41}{48}.$$

The answer to the second question is

$$\mathbf{P}\left(C_3|C_1^c \cap C_2^c\right) = \frac{\mathbf{P}\left(C_1^c \cap C_2^c \cap C_3\right)}{\mathbf{P}\left(C_1^c \cap C_2^c\right)} = \frac{\mathbf{P}\left(C_1^c \cap C_2^c|C_3\right)\mathbf{P}\left(C_3\right)}{\mathbf{P}\left(C_1^c \cap C_2^c\right)}.$$

Now,

$$\mathbf{P}\left(C_3\right) = \mathbf{P}\left(C_3|H\right)\mathbf{P}\left(H\right) + \mathbf{P}\left(C_3|H^c\right)\mathbf{P}\left(H^c\right) = \frac{1}{4}\cdot\frac{1}{3} + 0\cdot\frac{2}{3} = \frac{1}{12},$$

and

$$\mathbf{P}\left(C_1^c \cap C_2^c|C_3\right) = \frac{9}{16}.$$

Assembling all the pieces,

$$\mathbf{P}\left(C_3|C_1^c \cap C_2^c\right) = \frac{\frac{9}{16}\cdot\frac{1}{12}}{\frac{41}{48}} = \frac{9}{164}.$$

3.6.16 Let C_1, C_2, C_3 be the event David's family has one, two, or three children, respectively. Let A be the event that David's family has only one boy. We are operating on the assumption that $\mathbf{P}\left(C_1\right) = \mathbf{P}\left(C_2\right) = \mathbf{P}\left(C_3\right) = \frac{1}{3}$. Observe that

$$\mathbf{P}\left(A|C_1\right) = \mathbf{P}\left(A|C_2\right) = \frac{1}{2}, \qquad \mathbf{P}\left(A|C_3\right) = \frac{3}{8}.$$

This gives,

$$\mathbf{P}\left(C_1|A\right) = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{3}{8}} = \frac{4}{11}.$$

For the second question let B be the event that David's family has no girls. Then

$$\mathbf{P}\left(B|C_1\right) = \frac{1}{2}, \qquad \mathbf{P}\left(B|C_2\right) = \frac{1}{4}, \qquad \mathbf{P}\left(B|C_3\right) = \frac{1}{8}.$$

This gives,

$$\mathbf{P}\left(C_1|B\right) = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{8}} = \frac{4}{7}.$$

Somewhat counterintuitive!

3.6.17 We have

$$\mathbf{P}\left(HHT\right) = \frac{1}{4}\cdot\frac{4}{5^3} + \frac{1}{4}\cdot\frac{12}{5^3} + \frac{1}{4}\cdot\frac{18}{5^3} + \frac{1}{4}\cdot\frac{16}{5^3} = \frac{1}{10}.$$

Hence

$$\mathbf{P}\left(A|HHT\right) = \frac{\frac{1}{4}\cdot\frac{4}{5^3}}{\mathbf{P}\left(HHT\right)} = \frac{2}{25},$$

$$\mathbf{P}\left(B|HHT\right) = \frac{\frac{1}{4}\cdot\frac{12}{5^3}}{\mathbf{P}\left(HHT\right)} = \frac{6}{25},$$

$$\mathbf{P}\left(C|HHT\right) = \frac{\frac{1}{4}\cdot\frac{18}{5^3}}{\mathbf{P}\left(HHT\right)} = \frac{9}{25},$$

$$\mathbf{P}\left(D|HHT\right) = \frac{\frac{1}{4}\cdot\frac{16}{5^3}}{\mathbf{P}\left(HHT\right)} = \frac{8}{25},$$

so it is more likely to be coin C.

4.1.1 $\frac{1}{10}$; $\frac{3}{10}$; $\frac{3}{100}$; $\frac{2}{5}$; 0; $\frac{1}{100}$; $\frac{13}{100}$; $\frac{8}{25}$; $\frac{1}{50}$; $\frac{29}{100}$

4.1.2 $\frac{11}{36}$

4.1.3 $1 - \frac{6\cdot 5\cdots(6-n+1)}{6^n}$. This is 1 for $n \geq 7$.

4.1.4 The sample space consists of all possible phone numbers in this town: $7\cdot 10^6$. A phone number will be divisible by 5 if it ends in 0 or 5 and so there are $7\cdot 10^5\cdot 2$ phone numbers that are divisible by 5. The probability sought is

$$\frac{7\cdot 10^5\cdot 2}{7\cdot 10^6} = \frac{2}{10} = \frac{1}{5}.$$

4.1.5 For this to happen, we choose the ticket numbered 9, the one numbered 15 and the other two tickets must be chosen from amongst the five tickets numbered $10, 11, 12, 13, 14$. The probability sought is thus

$$\frac{\binom{5}{2}}{\binom{20}{4}} = \frac{10}{4845} = \frac{2}{969}.$$

4.1.6 There are $4 + 6 + 2 = 12$ bills. The experiment can be performed in $\binom{12}{2} = 66$ ways. To be successful we must choose either 2 tens (in $\binom{4}{2} = 6$ ways), **or** 2 fives (in $\binom{6}{2} = 15$ ways), **or** 2 ones (in $\binom{2}{2} = 1$ way). The probability sought is thus

$$\frac{\binom{4}{2} + \binom{6}{2} + \binom{2}{2}}{\binom{12}{2}} = \frac{6 + 15 + 1}{66} = \frac{1}{3}.$$

4.1.7 $N^2 + 1$ is divisible by 10 if it ends in 0. For that N^2 must end in 9. This happens when $N \in \{3, 7, 13, 17, 23\}$. Thus the probability sought is $\frac{5}{25} = \frac{1}{5}$.

4.1.8 $\frac{\binom{3}{1}\binom{4}{1}\binom{5}{1}}{\binom{12}{3}} = \frac{3}{11}$

4.1.9 $\frac{1}{221}$

4.1.10 $\frac{6}{20825}, \frac{6327}{20825}$

4.1.11 There are only 5 numbers in the set that leave remainder 1 upon division by 6, namely $\{1, 7, 13, 19, 25\}$. The probability sought is thus $\frac{5}{25} = \frac{1}{5}$.

4.1.12 $\frac{4}{9}$

4.1.13 (a) $\frac{90 \cdot 89 \cdot 88 \cdot 87 \cdot 10}{100 \cdot 99 \cdot 98 \cdot 97 \cdot 96}$, (b) $\frac{90^4 \cdot 10}{100^5}$

4.1.14 $\frac{\binom{2}{1}^{12}}{\binom{24}{12}} = \frac{2^{12}}{2,704,156} = \frac{4096}{2,704,156} \approx 0.00151.$

4.1.15 Let A_k denote the event that there are exactly k married couples, $0 \le k \le 3$, among the escapees. First choose the couples that will escape, among the ten couples, this can be done in $\binom{10}{k}$ ways. Then select from the $10 - k$ couples remaining the $6 - 2k$ couples that will have only one partner. For each of these $6 - 2k$ couples, there are two ways to select one partner. Therefore

$$\mathbf{P}(A_k) = \frac{\binom{10}{k} \cdot \binom{10-k}{6-2k} \cdot 2^{6-2k}}{\binom{20}{6}}.$$

4.1.16 Each of the dice may land in 6 ways and hence the size of the sample space for this experiment is $6^3 = 216$. Notice that there is a one to one correspondence between vectors

$$(R, W, B), \quad 1 \le R \le W \le B \le 6$$

and vectors

$$(R', W', B'), \quad 1 \le R' < W' < B' \le 8.$$

This can be seen by putting $R' = R, W' = W + 1$, and $B' = B + 2$. Thus the number of vectors (R', W', B') with $1 \le R' < W' < B' \le 8$ is $\binom{8}{3} = 56$. The probability sought is thus

$$\frac{56}{216} = \frac{7}{27}.$$

4.1.17 We have

❶ First observe that this experiment has a sample space of size $\binom{A+B}{C}$. There are $\binom{B}{T}$ ways of choosing the females. The remaining $C-T$ members of the committee must be male, hence the desired probability is

$$\frac{\binom{B}{T}\binom{A}{C-T}}{\binom{A+B}{C}}.$$

❷ Either $C-2$ or $C-1$ or C males will be chosen. Corresponding to each case, we must choose either 2 or 1 or 0 women, whence the desired probability is

$$\frac{\binom{B}{C-2}\binom{A}{2}+\binom{B}{C-1}\binom{A}{1}+\binom{B}{C}\binom{A}{0}}{\binom{A+B}{C}}.$$

❸ Either 3 or 2 or 1 or 0 women will be chosen. In each case, either $C-3$ or $C-2$ or $C-1$ or C men will be chosen. Thus the desired probability is

$$\frac{\binom{A}{C-3}\binom{B}{3}+\binom{A}{C-2}\binom{B}{2}+\binom{A}{C-1}\binom{B}{1}+\binom{A}{C}\binom{B}{0}}{\binom{A+B}{C}}.$$

❹ We must assume that Peter and Mary belong to the original set of people, otherwise the probability will be 0. Since Peter and Mary must belong to the committee, we must choose $C-2$ other people from the pool of the $A+B-2$ people remaining. The desired probability is thus

$$\frac{\binom{A+B-2}{C-2}}{\binom{A+B}{C}}.$$

❺ Again, we must assume that Peter and Mary belong to the original set of people, otherwise the probability will be 1. Observe that one of the following three situations may arise: (1) Peter is in a committee, Mary is not, (ii) Mary is in a committee, Peter is not, (iii) Neither Peter nor Mary are in a committee. Perhaps the easiest way to count these options (there are many ways of doing this) is to take the total number of committees and subtract those including (simultaneously) Peter and Mary. The desired probability is thus

$$\frac{\binom{A+B}{C}-\binom{A+B-2}{C-2}}{\binom{A+B}{C}}.$$

Aliter: The number of committees that include Peter but exclude Mary is $\binom{A+B-2}{C-1}$, the number of committees that include Mary but exclude Peter is $\binom{A+B-2}{C-1}$, and the number of committees that exclude both Peter and Mary is $\binom{A+B-2}{C}$. Thus the desired probability is seen to be

$$\frac{\binom{A+B-2}{C-1}+\binom{A+B-2}{C-1}+\binom{A+B-2}{C}}{\binom{A+B}{C}}$$

That this agrees with the preceding derivation is a simple algebraic exercise.

4.1.18 The experiment is choosing five people from amongst 12, and so the sample space has size $\binom{12}{5}=792$. The women will outnumber the men if there are (a) 3 women and 2 men; (b) 4 women and 1 man; or (c) 5 women. The numbers of successes is thus

$$\binom{5}{3}\binom{7}{2}+\binom{5}{4}\binom{7}{1}+\binom{5}{5}\binom{7}{0}=246.$$

The probability sought is thus $\frac{246}{792}=\frac{41}{132}$.

4.1.19 Let A be the event that all the camels are together, and let B be the event that all the goats are together. The answer to the first question is

$$\mathbf{P}\left(A\cap B\right)=\frac{2\cdot 5!\cdot 5!}{10!}\approx 0.00794.$$

The answer to the second question is

$$\mathbf{P}\left(A\cup B\right)=\mathbf{P}\left(A\right)+\mathbf{P}\left(B\right)-\mathbf{P}\left(A\cap B\right)=\frac{5!\cdot 5!\cdot 6}{10!}+\frac{5!\cdot 5!\cdot 6}{10!}-\frac{2\cdot 5!\cdot 5!}{10!}=\frac{10\cdot 5!\cdot 5!}{10!}=\frac{115,200}{3,628,800}\approx 0.0317.$$

4.1.20 We use inclusion-exclusion, where C, F, S, respectively, denote the sets of Chinese, French and Spanish speakers. We have

$$\begin{aligned}\text{card}\,(C \cup F \cup S) &= \text{card}\,(C) + \text{card}\,(F) + \text{card}\,(S) \\ &\quad -\text{card}\,(C \cap F) - \text{card}\,(F \cap S) - \text{card}\,(S \cap C) \\ &\quad +\text{card}\,(C \cap F \cap S) \\ &= 30 + 50 + 75 - 15 - 30 - 12 + 7 \\ &= 105,\end{aligned}$$

students speak at least one language, hence $120 - 105 = 15$ students speak none of the languages. The probability sought is $\frac{15}{120} = \frac{1}{8}$.

4.1.21 The experiment consists in permuting the letters $RRRWWWWBBB$ and hence the sample space size is $\frac{10!}{3!4!3!}$. In order to obtain success, we must have an arrangement of the form

$$x_1Rx_2Rx_3Rx_4Wx_5Wx_6Wx_7Wx_8,$$

where the x_i may have from 0 to 3 blue balls. The number of such arrangements is the number of non-negative integral solutions to $x_1 + x_2 + \cdots + x_8 = 3$, namely $\binom{8+3-1}{8-1} = \binom{10}{7} = \frac{10!}{7!3!}$. Hence the probability sought is

$$\frac{\frac{10!}{7!3!}}{\frac{10!}{3!4!3!}} = \frac{3!4!}{7!} = \frac{1}{35}.$$

Aliter: Observe that the position of the red balls is irrelevant for success. Thus we only worry about permutations of of $RRRWWWW$ and only one of this is successful. The desired probability is $\frac{1}{\frac{7!}{4!3!}} = \frac{4!3!}{7!} = \frac{1}{35}$.

4.1.22 $\frac{125}{216}$; $\frac{91}{216}$; $\frac{4}{9}$; $\frac{1}{2}$

4.1.23 $\frac{\binom{26}{3}^2}{\binom{52}{6}}$; $\frac{\binom{4}{2}^3}{\binom{52}{6}}$; $\frac{\binom{13}{1}\binom{4}{4}\binom{48}{2}}{\binom{52}{6}}$; $\frac{\binom{4}{1}\binom{13}{4}\binom{39}{2}}{\binom{52}{6}}$; $\frac{\binom{48}{6}}{\binom{52}{6}}$

4.1.24 $\frac{7}{18}$

4.1.26 The sample space consists of all vectors $D_1D_2D_3$ where D_i is a day of the week, hence the sample space size is $7^3 = 343$. Success consists in getting a vector with all the D_i different, and there are $7 \cdot 6 \cdot 5 = 210$ of these. The desired probability is thus $\frac{210}{343} = \frac{30}{49}$.

4.1.27 $\frac{\llbracket \frac{N}{k} \rrbracket}{N}$

4.1.28 $\frac{2}{5}$

4.1.29 $\frac{1}{5}$

4.1.30 In the numbers $\{1, 2, \ldots, 20\}$ there are 6 which are multiples of 3, 7 which leave remainder 1 upon division by 3, and 7 that leave remainder 2 upon division by 3. The sum of three numbers will be divisible by 3 when (a) the three numbers are divisible by 3; (b) one of the numbers is divisible by 3, one leaves remainder 1 and the third leaves remainder 2 upon division by 3; (c) all three leave remainder 1 upon division by 3; (d) all three leave remainder 2 upon division by 3. The required probability is thus

$$\frac{\binom{6}{3} + \binom{6}{1}\binom{7}{1}\binom{7}{1} + \binom{7}{3} + \binom{7}{3}}{\binom{20}{3}} = \frac{32}{95}.$$

4.1.31 The person will have to try exactly 17 guns if either the third firing gun occurs on the seventeenth place or the firing guns occur on the last three places. Hence the probability sought is $\dfrac{\binom{16}{2}+1}{\binom{20}{3}} = \dfrac{121}{1140}$.

4.1.32 The possible pairs with $X < Y$ are $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(2,3)$, $(2,4)$, $(2,5)$, and $(3,4)$ for a total of 8 pairs. There are also eight corresponding pairs with $Y < X$. The probability sought is $\dfrac{64}{\binom{27}{2}} = \dfrac{64}{351}$.

4.1.34 Notice that the sample space of this experiment has size $10 \cdot 10$ since X and Y are chosen with replacement. Observe that if $N = 3k$ then $N^2 = 9k^2$, leaves remainder 0 upon division by 3. If $N = 3k+1$ then $N^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1$ leaves remainder 1 upon division by 3. Also, if $N = 3k+2$ then $N^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1$ leaves remainder 1 upon division by 3. Observe that there are 3 numbers—$3, 6, 9$—divisible by 3 in the set, 4 numbers—$1, 4, 7, 10$—of the form $3k+1$, and 3 numbers—$2, 5, 8$—of the form $3k+2$ in the set. Now, $X^2 - Y^2$ is divisible by 3 in the following cases: (i) both X and Y are divisible by 3, (ii) both X and Y are of the form $3k+1$, (iii) both X and Y are of the form $3k+2$, (iv) X is of the form $3k+1$ and Y of the form $3k+2$, (v) X is of the form $3k+2$ and Y of the form $3k+1$. Case (i) occurs $3 \cdot 3 = 9$ instances, case (ii) occurs in $4 \cdot 4 = 16$ instances, case (iii) occurs in $3 \cdot 3 = 9$ instances, case (iv) occurs in $4 \cdot 3 = 12$ instances and case (v) occurs in $3 \cdot 4 = 12$ instances. The favourable cases are thus $9 + 16 + 9 + 12 + 12 = 58$ in number and the desired probability is $\dfrac{58}{100} = \dfrac{29}{50}$.

4.1.35 $\dfrac{\binom{15}{3}\binom{5}{2}}{\binom{20}{5}}$

4.1.36 The TA chooses 3 problems in $\binom{20}{3} = 1140$ ways. Success means $\binom{15}{2}\binom{5}{1} = 525$ ways of choosing exactly two correct answers. The probability sought is thus $\dfrac{525}{1140} = \dfrac{35}{76}$.

4.1.37 The experiment consists of choosing 3 people out of 10, and so the sample space size is $\binom{10}{3} = 120$. Success occurs when one man and two women chosen, which can be done in $\binom{6}{1}\binom{4}{2} = 36$ ways. The probability sought is $\dfrac{36}{120} = \dfrac{3}{10}$.

4.1.38 This is plainly

$$\frac{\binom{5}{3}\binom{7}{2} + \binom{5}{4}\binom{7}{1} + \binom{5}{5}\binom{7}{0}}{\binom{12}{5}} = \frac{41}{132}.$$

4.1.39 The $r-1$ integers before i must be taken from the set $\{1, 2, \ldots, i-1\}$ and the $k-r$ after i must be taken from the set $\{i+1, i+2, \ldots, n\}$. Hence $\mathbf{P}(i, r, k, n) = \dfrac{\binom{i-1}{r-1}\binom{n-i}{k-r}}{\binom{n}{k}}$.

4.1.40 $\dfrac{5}{9}$; $\dfrac{2}{9}$

4.1.41 $\dfrac{5}{108}$

4.1.42 $\dfrac{1}{2}$

4.1.43 The sample space has size $6^3 = 216$. A simple count yields 25 ways of obtaining a 9 and 27 of getting a 10. Hence $\mathbf{P}(S = 9) = \dfrac{25}{216} \approx 0.1157$, and $\mathbf{P}(S = 10) = \dfrac{27}{216} = \dfrac{1}{8} = 0.125$.

4.1.44 $\dfrac{360}{2401}$

4.1.45 $\dfrac{118}{231}$

4.1.46 $\dfrac{\binom{4}{3}}{\binom{7}{3}} = \dfrac{4}{35}$

4.1.47 $\frac{1}{3}$

4.1.48 We have

$$\frac{\binom{n}{3}}{\binom{2n}{3}} = \frac{1}{12} \implies \frac{n(n-1)(n-2)}{2n(2n-1)(2n-2)} = \frac{1}{12}$$
$$\implies \frac{n-2}{4(2n-1)} = \frac{1}{12}$$
$$\implies 3(n-2) = 2n-1$$
$$\implies n = 5.$$

4.1.49 $\frac{\binom{13}{1}^4}{\binom{52}{4}}$

4.1.50 The experiment consists in choosing three positions to be occupied by the three cards, this can be done in $\binom{12}{3}$ ways. Success is accomplished by selecting one of the players, in $\binom{3}{1}$ and three of his cards, (in $\binom{4}{3}$) ways, to be the three lowest cards. The probability required is thus $\frac{\binom{3}{1}\binom{4}{3}}{\binom{12}{3}} = \frac{3}{55}$.

4.1.51 To have 2 distinct roots we need the discriminant $\mathbf{A}^2 - 4\mathbf{B} > 0$. Since $1 \leq \mathbf{A} \leq 6$ and $1 \leq \mathbf{B} \leq 6$ this occurs for the 17 ordered pairs $(\mathbf{A}, \mathbf{B})$: $(3,1)$, $(3,2)$, $(4,1)$, $(4,2)$, $(4,3)$, $(5,1)$, $(5,2)$, $(5,3)$, $(5,4)$, $(5,5)$, $(5,6)$, $(6,1)$, $(6,2)$, $(6,3)$, $(6,4)$, $(6,5)$, $(6,6)$, so the desired probability is $\frac{17}{36}$.

To have a double root we need $\mathbf{A}^2 - 4\mathbf{B} = 0$. This occurs when for the 2 ordered pairs $(\mathbf{A}, \mathbf{B})$: $(2,1)$ and $(4,4)$. Hence the desired probability is $\frac{2}{36} = \frac{1}{18}$.

If $x = -3$ is a root, then $(-3)^2 - 3\mathbf{A} + \mathbf{B} = 0$, that is $9 + \mathbf{B} = 3\mathbf{A}$. This occurs for the 2 ordered pairs $(\mathbf{A}, \mathbf{B})$: $(4,3)$ and $(5,6)$. Hence the desired probability is $\frac{2}{36} = \frac{1}{18}$.

If $x = 3$ were a root, then $3^2 + 3\mathbf{A} + \mathbf{B} = 0$, which is impossible since the sum on the sinistral side is strictly positive and hence never 0. The desired probability is thus 0.

4.1.52 This is plainly

$$\frac{\binom{3}{1}\binom{n}{2} + \binom{3}{2}\binom{n}{1}}{\binom{3n}{2}} = \frac{3n(n-1)+6n}{3n(3n-1)} = \frac{n+1}{3n-1}.$$

4.1.53 106

4.1.54 $\frac{25}{648}$

4.1.55 This is plainly $\frac{\binom{4}{3}\binom{48}{10}}{\binom{52}{13}} = \frac{858}{20825}$.

4.1.56 The sample space is the number of permutations of 10 objects of two types: 8 of type W (for *white*) and 2 of type R (for *red*). There are $\frac{10!}{8!2!} = 45$ such permutations. Now, to count the successful permutations, observe that we need a configuration of the form

$$X_1RX_2RX_3.$$

If one of the $X_i = 7W$ then another one must be $1W$ and the third must be $0W$, so there are $3! = 6$ configurations of this type. Similarly, if one of the $X_i = 8W$, the other two must be $0W$ and again there are $\frac{3!}{2!} = 3$ configurations of this type. The desired probability is hence $\frac{9}{45} = \frac{1}{5}$.

4.1.57 By subtracting A times the second equation from the first, the system becomes

$$(2A - B)x = (C - 3A)y; \quad x - 2y = 3.$$

For infinitely many solutions, we need $2A = B; 3A = C$, hence B is even and C is a multiple of 3, giving $(A, B, C) = (1, 2, 3)$ or $(2, 4, 6)$. The probability of infinitely many solutions is thus $\frac{2}{216} = \frac{1}{108}$.

If the system will have no solutions, then $2A = B$ and $3A \neq C$. For $(A, B) = (1, 2)$ we have 5 choices of C; for $(A, B) = (2, 4)$ we have 5 choices of C; and for $(A, B) = (3, 6)$ we have 6 choices of C. Hence there are $5 + 5 + 6 = 16$ successes, and the probability sought is $\frac{16}{216} = \frac{2}{27}$.

For the system to have exactly one solution we need $2A \neq B$. If $A = 1, 2$ or 3, then B cannot $B = 2, 4$ or 6, giving $5+5+5 = 15$ choices of B in these cases. If $A = 4, 5$ or 6, then B can be any of the 6 choices, giving $6 + 6 + 6 = 18$ in these cases. These $15 + 18 = 33$ choices of B can be combined with any 6 choices of C, giving $33 \cdot 6 = 198$ choices. The probability in this case is thus $\frac{198}{216} = \frac{11}{12}$.

For the system to have $x = 3$, $y = 0$ as its unique solution, we need $2A \neq B$ and $3A = C$. If $A = 1$ then $C = 3$ and we have 5 choices for B. If $A = 2$ then $C = 6$ and again, we have 5 choices for B. Hence there are 10 successes and the probability sought is $\frac{10}{216} = \frac{5}{108}$.

4.2.1 $\frac{15}{1024}$

4.2.2 Let A denote the event whose probability we seek. Then A^c is the event that no heads turns up. Thus

$$\mathbf{P}(A^c) = \binom{5}{5}\left(\frac{3}{4}\right)^0\left(\frac{1}{4}\right)^5 = \frac{1}{1024}.$$

Hence

$$\mathbf{P}(A) = 1 - \mathbf{P}(A^c) = 1 - \frac{1}{1024} = \frac{1023}{1024}.$$

Notice that if we wanted to find this probability directly, we would have to add the five terms

$$\begin{aligned}\mathbf{P}(A) &= \binom{5}{1}\left(\frac{3}{4}\right)^1\left(\frac{1}{4}\right)^4 + \binom{5}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^3 + \binom{5}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^2 \\ &\quad + \binom{5}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^1 + \binom{5}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)^0. \\ &= \frac{15}{1024} + \frac{90}{1024} + \frac{270}{1024} + \frac{405}{1024} + \frac{243}{1024} \\ &= \frac{1023}{1024}.\end{aligned}$$

4.2.3 $\frac{1}{2} - \frac{\binom{1000}{500}}{2^{1001}}$

4.2.4 This is plainly

$$\binom{5}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 + \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^1 + \binom{5}{5}\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)^0 = \frac{17}{81}.$$

4.2.5 $\frac{2133}{3125}$

4.2.6 $\frac{5}{16}$

4.2.7 A particular configuration with one '2', one '7', and two '0's has probability $(\frac{1}{10})^1(\frac{1}{10})^1(\frac{1}{10})^2 = \frac{1}{10000}$ of occurring. Since there are $\frac{4!}{2!} = 12$ such configurations, the desired probability is thus $\frac{12}{10000} = \frac{3}{2500}$.

4.3.1 $\frac{5}{8}$

4.3.2 $\frac{1}{3}$; $\frac{2}{9}$; $\frac{8}{81}$; $\frac{2}{5}$

4.3.3 $\frac{4}{63}$

4.3.4 Let $\mathbf{X}_i$ be the random variable counting the number of times until heads appears for times $i = 1, 2, 3$. Observe that $\mathbf{P}(\mathbf{X}_i = n) = \frac{1}{2^n}$ (in fact, $\mathbf{X}_i$ is geometric with $p = \frac{1}{2}$). Hence the desired probability is

$$\sum_{n=1}^{\infty} \mathbf{P}(\mathbf{X}_1 = n)\,\mathbf{P}(\mathbf{X}_2 = n)\,\mathbf{P}(\mathbf{X}_3 = n) = \sum_{n=1}^{\infty} \frac{1}{8^n} = \frac{\frac{1}{8}}{1 - \frac{1}{8}} = \frac{1}{7}.$$

4.3.5 $\frac{5^2}{6^8}$

4.4.1 For the patient to notice for the first time that the left dispenser is empty, he must have pulled out 100 sheets from the left, 75 from the right, and on the 101st attempt on the left he finds that there is no sheet. So we have a configuration like

$$\underbrace{\ldots L \ldots R \ldots}_{100\ L\text{'s and }75\ R\text{'s}}\quad L,$$

where all the L's, except for the one on the last position, can be in any order, and all the R's can be in any order. This happens with probability $\binom{175}{75}(\frac{1}{2})^{75}(\frac{1}{2})^{100} \cdot \frac{1}{2} = \binom{175}{75}(\frac{1}{2})^{176}$. The same probability can be obtained for the right dispenser and hence the probability sought is $2\binom{175}{75}(\frac{1}{2})^{176} = \binom{175}{75}(\frac{1}{2})^{175}$.

5.0.2 We want $\mathbf{P}(|x - y| < 1) = \mathbf{P}(-1 + x < y < 1 + x)$. This is the area shaded in figure A.1. The area of the rectangle is $3 \cdot 5 = 15$, of the white triangle $\frac{1}{2} \cdot (2)(2) = 2$, and of the white trapezoid $\frac{1}{2} \cdot (1 + 4)(3) = \frac{15}{2}$. The desired probability is thus

$$\frac{15 - 2 - \frac{15}{2}}{15} = \frac{11}{30}.$$

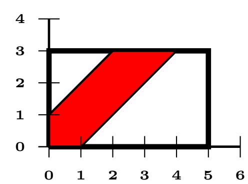

Figure A.1: Problem 5.0.2.

5.0.3 Consider x and y with $0 \leq x \leq 2.5$ and $x + y = 2.5$ Observe that the sample space has size 2.5. We have a successful pair (x, y) if it happens that $(x, y) \in [0.5; 1] \times [1.5; 2]$ or $(x, y) \in [1.5; 2] \times [0.5; 1]$ The measure of all successful x is thus $0.5 + 0.5 = 1$. The probability sought is thus $\frac{1}{2.5} = \frac{2}{5}$.

6.1.2 Let $\mathbf{G}$ be the random variable denoting the gain of the player. Then $\mathbf{G}$ has image $\{0, 1, 3, 5\}$ and

$$\mathbf{P}(\mathbf{G} = 0) = \frac{1}{2}, \qquad \mathbf{P}(\mathbf{G} = 1) = \mathbf{P}(\mathbf{G} = 3) = \mathbf{P}(\mathbf{G} = 5) = \frac{1}{6}.$$

Thus

$$E\mathbf{G} = 0\mathbf{P}(\mathbf{G} = 0) + 1\mathbf{P}(\mathbf{G} = 1) + 3\mathbf{P}(\mathbf{G} = 3) + 5\mathbf{P}(\mathbf{G} = 5) = \frac{1 + 3 + 5}{6} = \frac{3}{2},$$

meaning that the fee should be $\$1.50$.

6.1.3 Let $\mathbf{G}$ be the random variable denoting Osa's net gain. Then $\mathbf{G}$ has image $\{-1, 1, 12\}$ and

$$\mathbf{P}(\mathbf{G} = -1) = \frac{38}{52}, \quad \mathbf{P}(\mathbf{G} = 1) = \frac{13}{52}, \quad \mathbf{P}(\mathbf{G} = 12) = \frac{1}{52}.$$

Thus

$$\begin{aligned} E\mathbf{G} &= -1\mathbf{P}(\mathbf{G} = -1) + 1\mathbf{P}(\mathbf{G} = 1) + 12\mathbf{P}(\mathbf{G} = 13) \\ &= \frac{-38 + 13 + 12}{52} \\ &= -\frac{13}{52} \\ &= -0.25, \end{aligned}$$

and so the net gain is $-\$0.25$.

6.1.4 0.0875; -0.5125 ; 1.4625 ; 1.19984375

6.1.5 -0.25

6.1.7 $\$\frac{1}{8}$

6.1.8 Lose.

6.2.1 $1+\frac{39}{14}=\frac{53}{14}$

6.2.2 X is a binomial random variable with $EX=np=\frac{72}{6}=12$ and $\text{var}X=np(1-p)=72\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)=10$. But $EX^2=\text{var}(X)+(EX)^2=10+12^2=154$.

6.2.3 The fastest way to do this is perhaps the following. Let $X_i=1$ if the i-th boy is selected, $X_i=0$ otherwise. Then $\mathbf{P}(X_i=1)=\frac{\binom{24}{7}}{\binom{25}{8}}=\frac{8}{25}$ and $EX=\frac{10\cdot 8}{25}=\frac{16}{5}$. Similarly, let $Y_i=1$ if the i-th girl is selected, $Y_i=0$ otherwise. Then $\mathbf{P}(Y_i=1)=\frac{\binom{24}{7}}{\binom{25}{8}}=\frac{8}{25}$ and $EY=\frac{15\cdot 8}{25}=\frac{24}{5}$. Thus $E(X-Y)=EX-EY=-\frac{8}{5}$.

6.2.4 $7\left(\frac{\binom{2}{1}\binom{12}{4}+\binom{2}{2}\binom{12}{3}}{\binom{14}{5}}\right)=\frac{55}{13}$

6.3.1 Let F be the random variable counting the number of flips till the first heads. Then $\text{Im}(F)=\{1,2,3\}$. Let A be the event that heads is produced within the first three flips. Then

$$\mathbf{P}(A)=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}.$$

Hence

$$\mathbf{P}(F=1|A)=\frac{\mathbf{P}((F=1)\cap A)}{\mathbf{P}(A)}=\frac{\frac{1}{2}}{\frac{7}{8}}=\frac{4}{7};$$

$$\mathbf{P}(F=2|A)=\frac{\mathbf{P}((F=2)\cap A)}{\mathbf{P}(A)}=\frac{\frac{1}{4}}{\frac{7}{8}}=\frac{2}{7};$$

$$\mathbf{P}(F=3)=\frac{\mathbf{P}((F=3|A)\cap A)}{\mathbf{P}(A)}=\frac{\frac{1}{8}}{\frac{7}{8}}=\frac{1}{7}.$$

Thus

$$E(F|A)=1\cdot\frac{4}{7}+2\cdot\frac{2}{7}+3\cdot\frac{1}{7}=\frac{11}{7}.$$