Calculus Anyone?

As we all know, there is a calculus "crisis" and, as a result, a calculus "initiative". What the nature of the crisis is supposed to be, though, is by no means obvious, at least to me. In fact, while I attended the 1987 Calculus for a New Century conference in Washington, I didn't have the feeling that the audience wholeheartedly agreed that there was a crisis at all. Of course, we all have complaints. But the telltale here is the use of the plural. Your complaint is probably not my complaint. So, do all these complaints add up to a crisis? Yes, I think so, but only if we see all these complaints as symptoms of an illness. Then, we might have a chance to treat them.

In starting this column, I thought that we could examine the crisis and, eventually, arrive at least at a diagnostic. Then, maybe, we can discuss the remedy. So, I invite everyone to come up with their view of what the crisis is all about. This can range from a list of gripes, to use Steve Maurer's expression, to outraged letters to the editor to full fledged articles. And since not all of us are willing to write, I can usually be reached at (215)438 3820 in the evening, say until midnight. I shall try to put what you said to paper. By the way, I propose that we dispense with expressions such as "in my opinion", "I think that" etc. It goes without saying that this column is a column of opinions. Facts are welcome but they are so boring!

I should of course state my own vested interest. It is very simple: I have arrived at a diagnostic and I have the remedy. I just want to convince you. Seriously! Equally seriously, while I know that I have an existence theorem, I know that I don't have a uniqueness theorem. However, since, even if you don't write, I still shall, so that in this case the only things that will be written about will be *my* views. If you don't like what you read, you better write it yourself.

In the meantime and while I have no competition for this space, here I go with my diagnostic. I will argue that the root of all evil is in limits. I briefly adduce the following in support of my claim.

i. A lot of mathematics was written before limits even existed. I would say certainly most if not all of first year calculus. Moreover, some mathematical results were never reproven in terms of limits. This is the case, in particular, in classical differential geometry. As A. Robinson said, it just didn't seem worth the effort. But then what this means is that limits are not necessary for us to be convinced of the validity of a result.

ii. Up until at least the thirties, limits were not used in first year calculus and, at that time, most of the students who studied calculus were probably science majors. Contrast this with the current situation in which we teach such a higher grade calculus to ...

iii. There are no such things as limits. That they are not entities is evidenced by the fact that to say that "x is a limit" makes no more sense than to say "x is equal" Both 'limit' and 'equal' are just part of *binary* predicates. Given a function $\boldsymbol{f}$, there is only a relation, __ is the limit of $\boldsymbol{f}$ as x approaches __.

iv. In 1797, Lagrange published *Theorie des fonctions analytiques* in which he assumed that all functions were analytic, that is were the limit of a power series, and proceeded to study them as such. Of course, this wasn't really true but it certainly is true that all functions in first year calculus are, if not analytic, at least infinitely differentiable. In fact, analyticity is not necessary and all that is necessary is to notice that all functions in first year calculus have a Taylor expansion everywhere. For instance,

$$\boldsymbol{f}(x_0 + h) = A_0 + A_1 h + A_2 h^2 + \ldots A_n h^n + \boldsymbol{R}_{\mathrm{n}}(h), \qquad \text{with } \boldsymbol{R}_{\mathrm{n}}(h) = \mathbf{o}[h^n]$$

If, in addition, you realize that we can always find the Taylor expansion *from the definition* of the function, you find that you don't need limits at all. Thus, for instance, $f^{(n)}(x_0) = n!A_n$.

I hope to develop the above points in succeeding columns. For now, since , recently, I have spent a lot of time on Lagrange's approach, I will take up **iii.** for a change.

THERE ARE NO LIMITS.

Depending on what problems it was meant to solve or, perhaps, on the spirit of the time, Number Theory has been approached in either one of two ways. The Greeks considered that the whole numbers were **entities** in their own right, giving a platonic view of the world, granted by the gods, and, perhaps as a consequence, they believed that the whole numbers were the only possible numerical entities. Notice by the way how, to this day, whole numbers are the only ones deemed to be "natural". Indeed, even though he allowed for other numerical entities, Kronecker still said something to the effect that God gave us the whole numbers while all others were created by man.

To return to the Greeks, when they eventually were faced with the need to represent points between integral points, and rather than to invent fractions, i.e. numerical entities, they prefered to invent *proportionality*, i.e. a **relation**. That is, instead of saying "2/3 of A" they said "That which is to A as $2A$ is to $3A$". Of course, they also invented *ratios* as equivalence classes of pairs of things which were to one another as a given natural number is to another but they never accepted them as numerical entities; they failed to invent rational numbers. We would say today that they never went to the quotient space.

But, later, through a desire to solve an ever larger number of equations, new numerical entities were repeatedly introduced. For instance, rational numbers were invented to solve $ax = b$ when b was *not* a multiple of a, irrational numbers to solve $x^2 = a$ when a was *not* a square, negative numbers to solve $x + a = b$ when b was *not* larger than a. The process eventually came to an end, the system was said to be complete and $\mathbb{R}$ was declared to represent the real world. However, each time, this was done with a show of great reluctance and many oratorical precautions. In fact, and perhaps because the Greeks had accepted rationality as a platonic representation of reality, so that the word *rational* had come to be identified with *that which makes sense*, the rationals were accepted as numbers when the religious imperative on numbers disappeared but the names given to the numbers introduced subsequently remain quite significant of the resistance they elicited: irrational and negative still aren't desirable qualities, negative numbers were first called absurd numbers (at least by Descartes) and there was a lot of nonsense written about imaginary numbers..

In dealing with "evanescent" phenomena, both Newton and Leibniz had continued the *numerical* tradition by introducing new entities, fluxions and infinitesimals respectively, in order to solve $ax = b$ when a was *not* non-zero. Just as, before anyone had had a clear idea of what they were, imaginary numbers had first been a way to find the real root of a cubic, so were infinitesimals and infinities used, just as successfully, as means towards ends which were often real numbers. Leibniz was perfectly aware that his calculus implied the introduction of "ideal numbers". Here again, that we call the Real Numbers *real* is significant but is really just a reflection of our *preconceived* notion of what the fine structure of a line is. What is rational, real, ideal or, for that matter, hyperreal is mostly a cultural prejudice.

But when, eventually, mathematicians started to need precise operational rules to manipulate infinitesimals, they soon realized that they couldn't even produce a definition. Bolzano and, later, Weierstrass then returned to the Greek approach: instead of continuing to use the new numbers which, they had decided, made no sense, they invented a new relation, the *limit relation*, among the old numbers. For example, instead of saying that the value of $1/x$ is infinite when x is infinitesimal, they prefered to say that $1/x$ is larger than any preassigned number M as soon as x is smaller than some number δ. By a wonderful piece of apparent sophistry, this is all right because, since, as one does not set x actually equal to 0, $1/x$ need not equal ∞ but the limit relation between 0 and ∞ holds as soon as one has a *Skolem function* $\delta = \delta(M)$, that is of a function giving the necessary condition on x (to be smaller than δ) in terms of the prespecified tolerance on $1/x$ (to be larger than ε).

Yet, the very success of infinitesimals should have been a cue not to give up on entities. Even if it took almost three centuries, A. Robinson finally rehabilitated Leibniz's ideal numbers by defining what came to be called Hyperreal Numbers which do for the calculus what rational numbers did for algebra. An obvious question, though, is the completion issue but observe that the completeness of the Reals is defined relatively to the Natural Numbers, a subset of the Reals, so that the paradox is resolved by defining the completeness of the Hyperreals in terms of their own subset of"natural" numbers, the Hypernatural Numbers.

Finally, note that history does not always repeat itself. The Dirac δ "function" and the derivative of the Heaviside function made no sense as functions but were so indispensable that, in Dirac's lifetime, they were given status by Laurent Schwartz as *distributions*.

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