Notes from the Mathematical Underground

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Posted: This column seeks to advocate (provoke?) an analysis and a discussion of the *mathematics* underlying the courses we teach, Basic Arithmetic, Calculus, Linear Algebra, … , and of exactly *what* the students get out of it and of *why* they should get it.

The reason we in the mathematical profession do not like to do this is that, to begin with, we don't understand much mathematics and, then, usually haven't given much thought to what little we assume we know. The late I. N. Herstein used to advocate the creation of a Ph.D. in mathematical *knowledge* alongside the conventional Ph.D. in mathematical *research*. Briefly, he argued that the research Ph.D. mostly resulted in infinite expertise in infinitesimal areas and that this was quite incompatible with the training of college students. As I recall, he said that the research Ph.D. produced people with no idea of how to present anything outside their area of expertise, if that. (Hence the need for the "fat text".) On the other hand, the Ed.D. produces people who claim to know all about "how to teach" with no idea of what it is they are talking about. (Hence the need for … .) Predictably, his adjurations had no effect whatsoever. The only example I know of a thesis written along such lines is *A Proposed Sophomore-Level Experimental Course in Geometric Algebra Based Primarily on the Work of Emil Artin.* (Judd, 1969) and, in fact, it was more a paraphrase of (Artin, 1957) than the construction of a real course.

But even distinguished mathematicians often don't seem to see that lesser beings might want to approach the subject in alternate ways. Not too long ago, a very well known mathematician wrote that "(t)*he central notion of calculus is that of a limit*" and consequently complained about the lack of precision in the definition of limit in some text: How, under such circumstances, could the students be expected to learn what a limit is? I replied to him that

"I would rather put it like (Gleason, 1967):

> The course we teach in college which is usually called Calculus frequently hurries into such questions as differentiation and integration, and often fails to put the proper emphasis on what the subject is all about, namely functions of a real variable, or of several variables. The differential and integral calculus are, after all, techniques used to find out certain properties of functions, and should not be considered as ends in themselves.

"Thus, I would say that limits are just a tool. A powerful tool but not one indispensable to beginners. There was once an exchange among, I believe, Thom, Serre and Dieudonné about the latter's "*A bas Euclide*": The question was the extent to which linear algebra and axiomatic affine geometry ought to replace Euclid's twelve books. One of Dieudonné's adversaries maintained that one could enter geometry at many different points and that what most mattered was that the point of entry be clearly delineated.

"I would argue that it is precisely the identification of "the" calculus with its Bolzano-Cauchy-Weirstrass avatar that has doomed the "calculus reform". For instance, one can easily enter the calculus by way of little ohs given with their operating rules (rather than defined from limits). To be sure, even if, for a Freshman (Freshperson?), enough functions admit asymptotic expansions, not all do. For those who will continue, that very fact will be the "raison d'être" for limits. Of course, at the level of *my* students, all little ohs are replaced by (…), read "a little bit" and justified only by considerations of orders of magnitude for powers of 10: When they see h, they are to think of 0.1.

"Where I most disagree is in what constitutes "le mot juste". I think it was Church who pointed out that a perfectly correct definition of the human animal is that it is a featherless biped. Is a draughtman's drawing of an apple more specific, more evocative than a rendering by Cézanne? I can well imagine that a thousand counter examples must have gone through your mind as you read the Harvard text but, much as I dislike the book, I must point out that no such counter examples are likely to come to the mind of most of its readers, instructors included, so that what portion of the message *can* be transmitted will have been effectively transmitted. How precise the language should depend on the audience. The criteria for rigor imposed by research mathematicians on mathematics beginners have resulted, for instance, in basic algebra texts that, while they nowhere mention complex numbers, make sure that the equation $x^2 + 1 = 0$ is said to have no *real* solution. On the other hand, these same texts never discuss issues such as how $\mathbb{Z}$ arises out of the manner in which, for instance, $3 + x = 2$ has no solution in $\mathbb{N}$ or how, with the equation $3x = 2$, we can either construct $\mathbb{Q}$ and say the solution is $\frac{2}{3}$, which is exact but says nothing about its order of magnitude, or use approximate decimal solutions. In a different vein, that, for instance, constant functions are degenerate affine functions needs to be pointed out but taking it constantly into consideration complicates the language to the point where it becomes completely opaque. I would rather warn the students that my language is not foolproof and invite them to find the loopholes I left, wittingly or not.

"I also think that the current efforts to make calculus "*user friendly*" are misplaced: First Year Calculus has essentially no application to the real world; Differential Equations is the first course that does. If one insists to the contrary, then I am afraid that calculus will go the way of Greek and Latin, to be replaced by some kind of data analysis. For the most part, and certainly for beginners, mathematics can only be a labor of love, even when unrequited.

"As to proof versus justification or even plausibility, I think, again, that the issue is more the consistency of the level of rigor with the audience. In my experience, "just plain folks" ("*weak students*"?) will enjoy mathematics in general and calculus in particular when the idea of proof is presented as what attorneys do in front of a jury, namely give convincing arguments with the amount of supporting evidence depending largely on the challenges of the other side. Which is perhaps why I have a nostalgia for the oral examinations of my youth in which what we said might be challenged at any moment."

The above is slightly edited but the extremely influential and very redoubtable mathematician's full answer was:

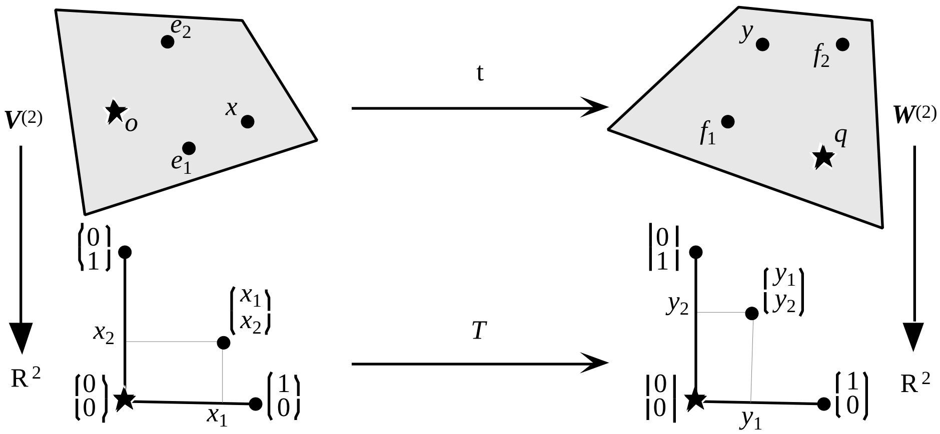
Professor Alain Schremmer.
I feel sorry for your students, who will evidently and sadly be misled.
Yours truly.

Already on several occasions I have raised the issue of contents architecture. Here is another instance. In his preface, the author of (Valenza, 1993) asks:

> "What is the nature of linear algebra? One might give two antipodal and complementary replies; like wave and particle physics, both illuminate the truth:
>
> THE STRUCTURAL REPLY. Linear algebra is the study of vector spaces and linear transformations. A vector space is a structure which abstracts and generalizesw certain familiar notions of both geometry and algebra. A linear transformation is a function between vector spaces that preserves elements of this structure. In some sense, this discipline allows us to import some long-familiar and well understood ideas of geometry into settings which are not geometric in any obvious way.
>
> THE COMPUTATIONAL REPLY. Linear Algebra is the study of linear systems and, in particular, of certain tehniques of matrix algebra that arise in connection with such systems. The aims of the discipline are largely computational, and the computations are indeed complex."

Indeed, the computational approach, a historically well founded architecture, deriving as it does from what used to be called "Higher Algebra", starts with systems of equations to end with linear transformations. This has of course the merit that, for students thoroughly familiar with systems of at least two or three equations, as was generally the case once upon a time, it can make sense to engage in a careful, detailed generalization via Gauss elimination in matrix form. This is the route taken, inter alia, by (Anton & Rorres, 1994) (and I must admit that I rather liked this opus of the illustrious author of "*In defense of the Fat Calculus Text*") as well as by (Strang, 1980), an author I respect enormously—the introduction alone justifies acquiring the book. However, I contend that the only reason this route appears natural is that it is the historical one. (Just as when we teach fractions ahead of decimals and integers.) Then, there is the unfortunate fact that none of *my* students, even the very sharp ones, is well-grounded in the solution of systems of equations.

The much more recent structural approach usually begins with the study of R^n as vector space and of matrix transformations $\mathrm{R}^m \to \mathrm{R}^n$. But a really natural route would start with the coordinate-free notion of a geometric transformation, $t : \boldsymbol{V}^{(m)} \to \boldsymbol{W}^{(n)}$, as distinguished, *once bases have been chosen* in $\boldsymbol{V}^{(m)}$ and $\boldsymbol{W}^{(n)}$, from its matrix representation $T : \mathrm{R}^m \to \mathrm{R}^n$:



Then, given b $W^{(n)}$, and to quote Strang, "*(*the) *goal is a genuine understanding, deeper than elimination can give, of the equation* t (x) = b"—as distinguished from the system of equations $\sum T_{ij} x_i = b_j$.

Admittedly, linear algebra is a very tough subject to organize *and* to make transparent. What I would like to see is something like *A proposed Sophomore-Level … based on …*", say, the first part of (Lax, 1997). No doubt, this route has structural problems of its own and this should be cause enough for an "exchange" in this space.

Re mathematics' *relevance*: How come the issues that we absolutely avoid are absolutely the most relevant ones, namely those that regard the *polis*? Why couldn't we, for instance and at least those of us who have tenure, give to an arithmetic class the following "word problem" adapted from (Bartlett & Steele, 1996), a rich source of "applications":

> A Federal Reserve Board study shows that the top 1 percent of households in the USA controls 30.4 of the nation's net worth and that the next 9 percents holds 36.8 percent of the nation's wealth. How much does the remaining 90 percent account for?

Re computers: I just read (Gordon, Fusaro & Meyer, 1989). It is worth (re)reading. Not because it purports to explain why I should computerize my classes but because it is such an excellent illustration of those "*methods of proof which do not appear in treatises* [on formal mathematics], *but, none the less, are used in many mathematics classrooms and texbooks: 1. Manus Flutterus, 2. Proof by Intimidation, 3. Proof by Circumvention, 4. Proof by Coercion.*" (Browne, 1989). There is not even a single reference to suppporting research. Could it be that there isn't any? That the emperor has no clothes?

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