Notes From The Mathematical Underground

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The opinions expressed are those of the author, and should not be construed as representing the position of AMATYC, its officers, or anyone else.

I recently received the following communication:

Dear Mr. Schremmer:

Even though we, at the American Authorial Association for Appropriately Amathematical Textbooks (AAAAAT), have been reading your Notes with extreme attention from the very beginning, we find that, to this day, we remain totally unmoved by ideas which, in fact, we find absolutely undemocratic if not downright elitist or even worse.

For instance, since the more sets there are in a collection the smaller their intersection is likely to be, what can be common to all of our customers must be asymptotically close to 0. Hence the first of our Ten Commandments for American Associated Authors of Appropriately Amathematical Textbooks (TCAAAAAT):

You shall strive for zero contents.

as in "0 = 0" even if, of course, this is not entirely without content. Indeed, this first of our TCAAAAAT is the most difficult to observe as it would be difficult, at this point in time, to sell a textbook of 700 blank pages for, say, eighty or ninety some dollars. There is hope, though.

Next, to insist that certain statements are true and, even worse, to prove them to be such, represents a clearly cohercive invasion of the customers' privacy. Hence the second of our TCAAAAAT:

You shall not prove anything.

As in "Clearly, 0 = 0" even though the term "clearly" is clearly coercive.

Everything is culture dependant and, therefore, represents only the instructor's particular prejudice which, as Educators, we must not impose on our customers. Hence the third of our TCAAAAAT:

You shall always speak in alternatives: Either … or …

as in "Either $1 = 0$ or $1 \neq 0$". However, since most of us do not want to exclude anyone or anything, this Commandment is certain to be amended at our forthcoming meeting to:

You shall always speak in disjunctions: Either … or … or both.

as in: "Either $1 = 0$ or $1 \neq 0$ or both".

Since it is equally self-evident that mathematics by itself is not relevant to the needs of

Most deplorably, I somehow lost both the rest of this communication and the address of the AAAAAT so that I must leave the remaining CAAAAAT to the reader's imagination.

…

It would seem that we might finally be recognizing that "just plain folks" might not necessarily want to major in mathematics and it is true that, as Jeffrey O. Bennett and William L. Briggs wrote in these pages last Fall, “many institutions have given little serious thought to the development of appropriate mathematics curricula for [these] students”. While I am not sure about the first two of the needs they list (page 6), I completely agree with them on the third one, part of which reads: “Students need mathematics to understand (…) issues”.

What they advocate in consequence is “Logic, Critical Thinking, and Problem Solving” in “A Context-Driven Approach” which “begins by establishing the important mathematical ideas behind logic such as sets, truth tables, and Venn diagrams and then show students how these ideas are useful.” I would like to propose here that the most useful ideas in that regard are to be found in First Order Predicate Logic rather than in Sentential Logic and that *inference* can be understood only as the syntactic counterpart of *entailment*, that is in the context of Model Theory. Since this is clearly even more bizarre an idea than is my wont, I must amplify a bit.

When we want to discuss a part of “reality”, we need to be precise about it inasmuch as reality involves extremely complex, endless ramifications. For things to become manageable, it is thus necessary, even at the risk of oversimplifying, to “abstract” reality to a **situation**, that is to ignore those features and interconnections that we deem to be irrelevant to our purpose.

Suppose, for instance, that we are interested in a relationship such as love. Say that Albert loves Bill who does not love Alan at all but does love very much Cathy and Dorothy who both love Alan a little bit and Eva very much but who…. To get moving, we must begin by deciding who we are prepared to take into consideration and, also, how much love is love enough to count as love in our situation.

Thus, to specify a situation involving, say a relationship, we specify a number of **objects** and a number of (ordered) **related pairs of objects**, namely those for which the relationship is deemed to hold. Similarly, to specify a situation involving an attribute, we specify the **select objects**, that is the objects deemed to have the attribute.

We now define a **structure** as a sort of sub-situation. The advantage is that we can take a sizable situation and then cut it down to various small structures. Let us take the following as our situation:

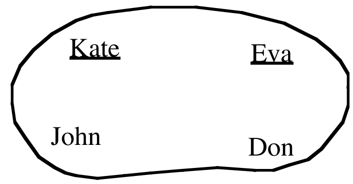
	Tom	Mary	Kate	John	Pete	Eva	Carol	Don
Weight	150	121	110	115	125	105	140	180
Age	21	19	23	24	17	28	16	35
Gender	Male	Female	Female	Male	Male	Female	Female	Male
Eyes	Blue	Brown	Brown	Blue	Blue	Blue	Brown	Brown

We may extract the following structure involving a **unary relation** (better known as **property**) by listing the objects and the select objects. For example:

Objects:	Tom, Kate, John, Eva, Don
Select objects:	Kate, Eva.

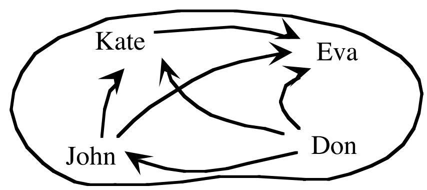
We can represent it by a table or, more picturesquely, by a line-diagram.

Objects	Selected
Kate	*yes*
John	*no*
Eva	*yes*
Don	*no*



Similarly, we could extract from the above situation a structure involving a **binary relation** by specifying the objects and selecting the related pairs of objects. We can represent it by a double entry table or by an arrow diagram. For instance, taking the same objects as above and selecting those pairs in which the first person *is heavier than* the second person, we have:

	Kate	John	Eva	Don
Kate	*no*	*no*	*yes*	*no*
John	*yes*	*no*	*yes*	*no*
Eva	*no*	*no*	*no*	*no*
Don	*yes*	*yes*	*yes*	*no*



We can of course have structures involving more than one relation. For instance, we will use below the structure that combines the above two[1].

In order to discuss a structure, we need a **language**, that is a **vocabulary** including **names** and **verbs** to **denote** the objects and the relations together with a **syntax** to decide which arrangements of these will be acceptable as **sentences**. To discuss the above structure, we could use:

Vocabulary		Denotation	
Names:	Verbs:	*a* denotes Kate	_P denotes "__ is female"
a, *b*, *c*, *d*, *e*	_P, _R_.	*b* denotes John	_R _ denotes "__ is heavier than__"
		c denotes Eva	
		d denotes Don	
		e denotes Eva	

where *b*P and *d*R*b* are sentences but *ca*P, *d*P*a*, *a*r , *db*r or r *db* are not

When the language is given a priori, a structure, together with a denotation, is looked upon as an **interpretation** of that language. Thus, under the above interpretation, *b*p **means** "John is female" with the **truth value F** because the meaning does not agree with the structure while *d*r *b* means "Don is heavier than John" with the truth value **T** because the meaning does agree with the structure. We shall say that the above interpretation is a **model** for *d*r *b* but not for *b*p .

The more interesting statements about structures involve **formulae**, i.e. verb-phrases in which some places have not been filled in with names but are tagged with **place-holders** such as *x*, *y*, *z*, subject to the condition that places held by *identical* placeholder *must* be filled with the same name whereas places held by *different*

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[1] There are of course relations with more than two places such as, for instance, "_ is between _ and _" as well as *functions* and *operations*. However, to keep things simple, we shall not deal with these here.

place-holders *may* be filled with the same name. Compare, for instance, the **solution set** of $x\mathrm{r}\ \ y$ with that of $x\mathrm{r}\ \ x$ which, here, is empty.

Then there are symbols called **logical** because their meaning does *not* depend on the interpretation. Two are the **quantifiers**, $\exists$ to mean "at least one" and $\forall$ to mean "no matter who". For instance $\forall x[d\mathrm{r}\ \ x]$ means "No matter what the person, Don is heavier than her/him" which is **F** since the solution set of the formula $d\mathrm{r}\ \ x$ is not full and $\exists x[x\mathrm{p}]$ means "There is at least one person who is female" which is **T** since the solution set of the formula $x\mathrm{p}$ is not empty.

Other logical symbols are the **connectors**, usually $\neg$, $\wedge$, $\vee$, with respectively the meanings "not", "and", "and/or", each given by a **truth table**. Thus, for instance, $a\mathrm{r}\ \ d \wedge b\mathrm{p}$ is a sentence meaning "Kate is heavier than Don and John is female" with value **F**. But there is also the (in?)famous abbreviation $\to$. For instance, $a\mathrm{r}\ \ d \to b\mathrm{p}$ stands, *by definition*, for $\neg[a\mathrm{r}\ \ d \wedge \neg b\mathrm{p}]$, as in "you can't have both your cake and *not* eat it" but we really use $\to$ to restrict the **scope** of $\forall$ as in $\forall x[x\mathrm{p} \to x\mathrm{r}\ \ d]$ just as we use $\wedge$ to restrict the scope of $\exists$ as in $\exists x[x\mathrm{p} \wedge x\mathrm{r}\ \ d]$.

Finally, **=** is a verb with an intermediate status; it indicates that two names denote the *same* object. For instance, $c = e$ is **T** but $a = b$ is **F**.

It takes from three to six hours to reach a point where students are comfortable enough to cope with a *set of interpretations*[2] relative to which we can then define: A sentence is **valid** if all the interpretations are models of it and a sentence **entails** another sentence if any model of the first is also a model of the other, i.e. if there is no **counter example**. Naturally, **logical validity** and **logical entailment**[3] correspond to the set of *all* possible interpretations. Note that, sententially, $\to$ is the "difference operation" for entailment: S_1 entails S_2 if and only if the sentence $\mathrm{S}_1 \to \mathrm{S}_2$ is valid. (As in: $x > y$ if and only if x-y is positive.)

It does take some time for the concept of entailment to really sink in but, once assimilated, it is quite natural to want to introduce *syntactic* **rules of inference** such that S_1 **yields**[4] S_2 (through a finite use of these rules) if and only if S_1 entails (logically) S_2. Gödel's Completeness Theorem is that the usual rules do the job. One may, but need not, require from the students that they *develop* proofs. One certainly can require them to either check that a given proof is correct or to construct a counterexample. In any case, just the idea that Gödels' Theorem allows us to replace by a *finite* proof checking entailment under an *infinite* number of interpretations is already quite useful.

I once developed such a course which was taught over a few years to some two thousands students by a number of my colleagues as well as by myself. The course died when the then Provost decided that the students were wasting their time and

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[2] Here I find the term **ontology** convenient.

[3] aka **logical consequence** but I prefer the flexibility given by the *verb* "entail".

[4] aka S_2 **is inferred from** S_1.

that, instead, Johnny should learn to add. Whether Johnny actually did under the new regime and, if so, was better off than under the previous one remains unclear to this day.

Comments, criticisms and rebuttals are very welcome and should be sent to:

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