Notes from the Mathematical Underground

by Alain Schremmer.
Mathematics Department, Community College of Philadelphia,
1700 Spring Garden Street, Philadelphia, PA 19130
SchremmerA @People-Link.com

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> A treatment for "just plain folks" should first be a mathematically correct one. Gaps will be necessary but they must be clearly delineated and not require any suspension of belief. Anything else adds insult to injury.
>
> … Popular aphorism

One arguments advanced in favor of Precalculus is that it allows the "introduction" of the elementary functions by way of simple concrete situations but nobody seems to have wondered about what the case actually might be.

In the case of the exponential for instance, Precalculus texts invariably introduce the *sequence* 2^n. But while it has been held, notably by (Staib 1966) who invoked (Hardy 1908), that "*the notion of a limit is most easily grasped in the case of sequences* [so that] *the theory of limits* [of sequences can then be] *exploited in the introduction of all other limit concepts, including integration*", this is by no means self-evident and, in any case, proceeding from a sequence to a function is a *conceptual switch* unsettling enough that it would have to be *demonstrably* worth it. As it is, extending a sequence to a function is never a trivial matter—think for instance of the extension of $n!$ to the gamma function. But, of course, we are only supposed to present the sequence as "illustrating the overall behavior of the function" much as a plot is supposed to announce a graph. Well, that says it all.

For example, Precalculus often focuses on the *sequence* Pop(nT) of the jump points of Pop(t), the *function* giving the *number* of cells in a petri dish at time t. For instance, letting the doubling period T equal 1 gives the very misleading *plot*:

n	pop(n)	Pop Change(n)
0	**1**	
1	2	1
2	4	2
3	8	4
4	16	8
5	32	16
6	64	32
7	128	64
8	256	128
9	512	256
10	1024	512

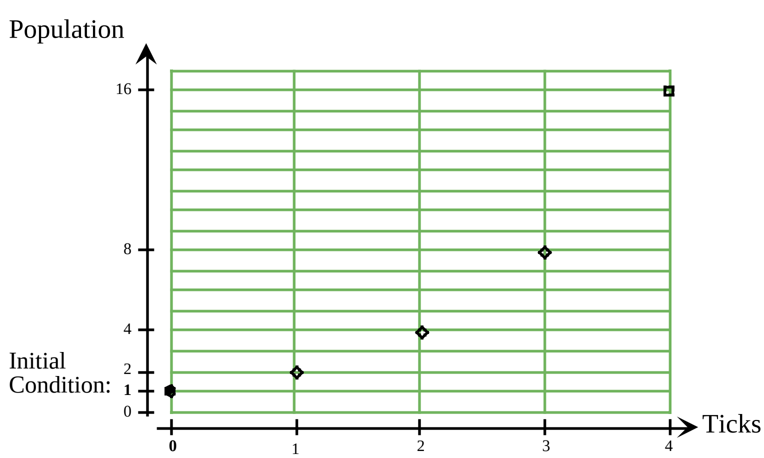

Figure 1.

Pop(n) is a solution of the sequential equation $s(n+1) = 2s(n)$ which is readily solved: From the initial condition, Pop(0) = 1 = 2^0 and from the equation, Pop(1) = 2•Pop(0) = 2•2^0 = 2^1, Pop(2) = 2•Pop(1) = 2•2^1 = 2^2, … and by induction Pop(n) = 2^n. But to conclude that Pop(t) = 2^t is as unwarranted as "joining plot points smoothly" and in fact false since Pop(t) is clearly *piecewise constant*!

What other use could a prior investigation of 2^n have? Since a Precalculus introduction should anticipate the treatment to be used in Calculus, let us first decide how we should primarily view e^x: **i.** as a function somewhat dual to a power function in that here it is the exponent that is the input? **ii.** as the inverse of the antiderivative of the reciprocal function? **iii.** as the limit of a deceptively simple power series, $\sum_0^n \frac{x^i}{i!}$? **iv.** as a function with a particular type of growth?

For students fresh out of Basic Algebra, it is tempting, after having extended exponents from N to Z to Q, to go to R by exponentiating whatever limit process was used to define irrationals. The problem is that, while it is not too difficult to define irrational powers, the extension of the usual *operations* to irrational powers is surprisingly tricky and we cannot hope to give such students even a remote idea of the difficulties and the preliminary investigation of 2^n is of no help whatsoever.

The second viewpoint became the preferred one some thirty years ago because, even though the motivation is not exactly … immediate, the proofs—it was still OK to worry about such things—are much simpler. But, from that viewpoint too, starting with 2^n serves no purpose.

The third point of view, *not to be confused with that of Lagrange*, is excellently presented in four of the 158 pages (Levi 1968). The way Levi sets the problem is "*to seek a function f, defined for every real value of x, for which $f(x)f(y) = f(x+y)$. We require that this function should generalize our notion of exponent (a) by agreeing with it for rational values of x and (b) by not giving strange results for irrational values of x. Towards meeting this last requirement we require that f be continuous.* (…) *If there were such a function, its series expansion at* 0 *would be* … . Not even a hint of 2^n. Indeed, Q is dense in R but N isn't! By the way, isn't it telling that the expansion of e^x should be so much more … immediate than that of 2^x?

With the current popularity of Data Analysis, the fourth viewpoint has gained favor. For instance, following (Hughes-Hallett, Gleason et al. 1994), it has become fashionable to define *exponential* behavior "in contrast" with *linear* (what they really mean is *affine*) behavior:

> To recognize that a function $y = f(x)$ given by a table of data is linear, look for differences in y values that are constant for equal differences in x.
>
> To recognize that a function $P = f(t)$ given by a table of data is exponential, look for ratios of P values that are constant for equally spaced t values.

But the phrase "a function given by a table" simply obliterates the essential problem, namely that of the distinction between sequences and functions.

More appropriately, the fourth viewpoint should present EXP as solution of the initial value problem $f'(x) = f(x)$, $f(0) = 1$ (as is now often done to supplement the second viewpoint) and the only way the investigation of 2^n would then make sense is in bringing about this initial value problem in a natural manner.

Note that if Lucky Larry rewrites $\text{Pop}(n_0+1) = 2\text{Pop}(n_0)$ as $\text{Pop}(n_0+1) = \text{Pop}(n_0) + \text{Pop}(n_0)$ and remarks that, as soon as n_0 is large, $\text{Pop}(n_0+1)$ is of the form $\text{Pop}(t_0+h)$ so that $\text{Pop}(t_0+h) = \text{Pop}\,(t_0) + \text{Pop}\,(t_0) \bullet h$, the Lagrangian definition of the derivative will give him $\text{Pop}'(t) = \text{Pop}(t)$. Poor Larry!

In fact, things are bad. We may try to characterize $\text{Pop}'(t)$ by $\text{PopChange}(n) = \dfrac{\text{Pop}(n) - \text{Pop}(n-1)}{n - (n-1)} = \text{Pop}(n) - \text{Pop}(n-1)$. Since, from the sequential equation, $\text{Pop}(n-1) = \frac{1}{2}\,\text{Pop}(n)$, we get $\text{PopChange}(n) = \frac{1}{2}\,\text{Pop}(n)$ but, from this *difference equation*, we very fortunately cannot conclude that $\text{Pop}'(t) = \frac{1}{2}\,\text{Pop}(t)$.

If we look at the difference equation for a faster clock, say $T = 4$, the plot points dutifully line up on a piecewise *constant* function:

n	Pop(*n*)	Pop Change(*n*)
0	**1**	
1	**1**	0
2	**1**	0
3	**1**	0
4	2	1
5	2	0
6	2	0
7	2	0
8	4	2
9	4	0
10	4	0
11	4	0
12	8	4

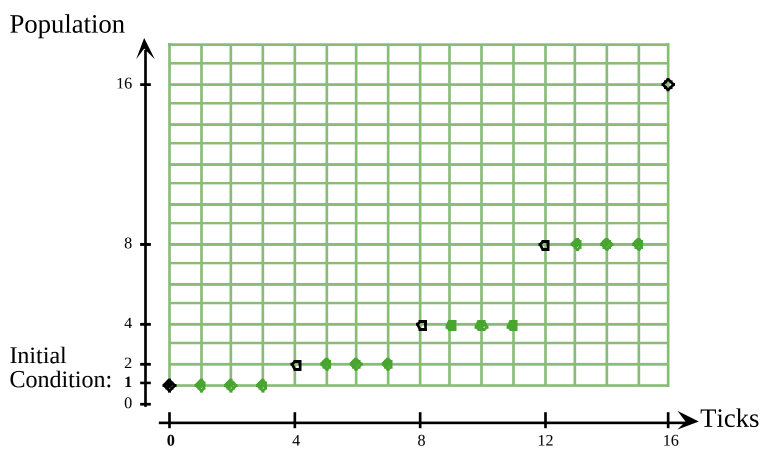

Figure 2.

The functional equation is now $\text{Pop}(n) = 2\text{Pop}(n-4)$ with the initial conditions $\text{Pop}(0) = \text{Pop}(1) = \text{Pop}(2) = \text{Pop}(3) = 1$ but this is not likely to get us the differential equation any more easily.

Of course, starting with just one cell was not exactly realistic and we should try to start with more. But if they are all at the same mitosis stage, the plot will remain the same with just the output ruler rescaled. We thus take 4 cells at different stages so that there will be a mitosis at each tick:

n	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*15*	*16*
Cell #1	1	1	1	1	**2**	2	2	2	**4**	4	4	4	**8**	8	8	8	**16**
Cell #2	1	1	1	**2**	2	2	2	**4**	4	4	4	**8**	8	8	8	**16**	16
Cell #3	1	1	**2**	2	2	2	**4**	4	4	4	**8**	8	8	8	**16**	16	16
Cell #4	1	**2**	2	2	2	**4**	4	4	4	**8**	8	8	8	**16**	16	16	16
Totals:	**4**	5	6	7	**8**	10	12	14	**16**	20	24	28	**32**	40	48	56	**64**

This gives us a more realistic sequence in that the plot points now line up on a piecewise *affine* function in which the *rate of change* doubles from piece to piece:

n	Pop(n)	Pop Change(n)
0	4	
1	5	1
2	6	1
3	7	1
4	8	1
5	10	2
6	12	2
7	14	2
8	16	2
9	20	4
10	24	4
11	28	4
12	32	4

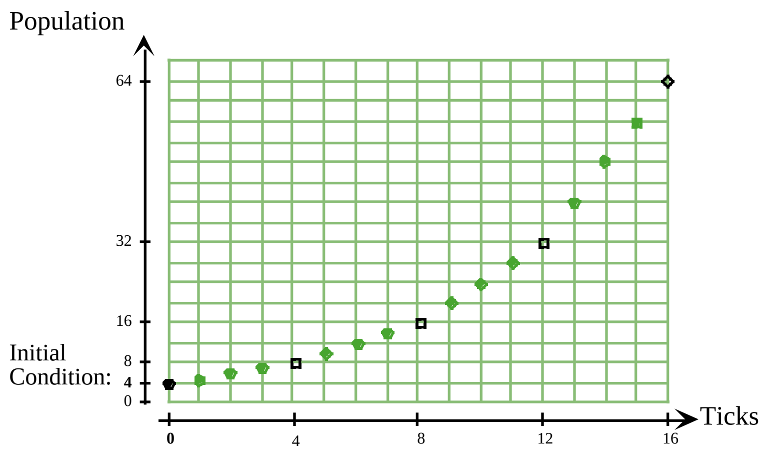

Figure 3.

However, when we start with more cells (and increase the clock speed accordingly), we find that the new plot points fall on the *same* piecewise affine function so that we cannot get better approximations that way.

What all of this amounts to is that *counting* cells gets us nowhere. Instead, what we should try to describe, if only approximately, is their *continuous* growth. But since this would require knowledge from biology, we turn to the future value of a principal P during a time interval T with ρ the interest earned on 1 unit of money during 1 unit of time. There are two issues: the *function* FutVal(t) and the *value* FutVal(T) but the game plan is clear:

a. Start with simple interests, that is accumulate interests on P so that $\mathrm{FutVal}_0(t) = P + P\rho t = P(1+\rho t)$ is an affine function and $\mathrm{FutVal}_0(T) = P(1+\rho T)$.

b. Now compound at time τ. We get a *piecewise* affine function as, for $t > \tau$, we accumulate interests on $P(1+\rho\tau)$ rather than on just P:

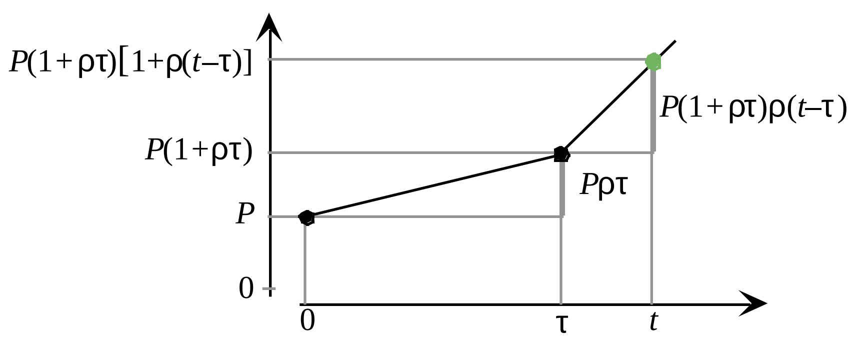

Figure 4.

Compounding at $\tau = \frac{T}{2}$, we get $\mathrm{FutVal}_1(T) = P\left(1+\rho\frac{T}{2}\right)^2$.

c. Compounding twice, at $\frac{T}{3}$ and $\frac{2T}{3}$, gives a 3-piece affine function:

for $0 < t < \frac{T}{3}$, $\mathrm{FutVal}_2(t) = P(1+\rho t)$ and $\mathrm{FutVal}_2\left(\frac{T}{3}\right) = P\left(1+\rho\frac{T}{3}\right)$

for $\frac{T}{3} < t < \frac{2T}{3}$, $\text{FutVal}_2(t) = P\left(1+\rho\frac{T}{3}\right)\left(1+\rho\left(t-\frac{T}{3}\right)\right)$ and $\text{FutVal}_2\left(\frac{2T}{3}\right) = P\left(1+\rho\frac{T}{3}\right)^2$

for $\frac{2T}{3} < t < T$, $\text{FutVal}_2(t) = P\left(1+\rho\frac{T}{3}\right)^2\left(1+\rho\left(t-\frac{2T}{3}\right)\right)$ and $\text{FutVal}_2(T) = P\left(1+\rho\frac{T}{3}\right)^3$

d. Compounding n times gives a $(n+1)$-piece affine function FutVal_n whose limit as n becomes large makes graphic sense as does the limit of $P\left(1+\rho\frac{T}{n}\right)^n$.

e. What of the differential equation? The function that gives the rate of change—let us call it the derivative—is piecewise constant. For example, compounding 9 times 3% interests on $P = 100$ during $T = 60$ gives:

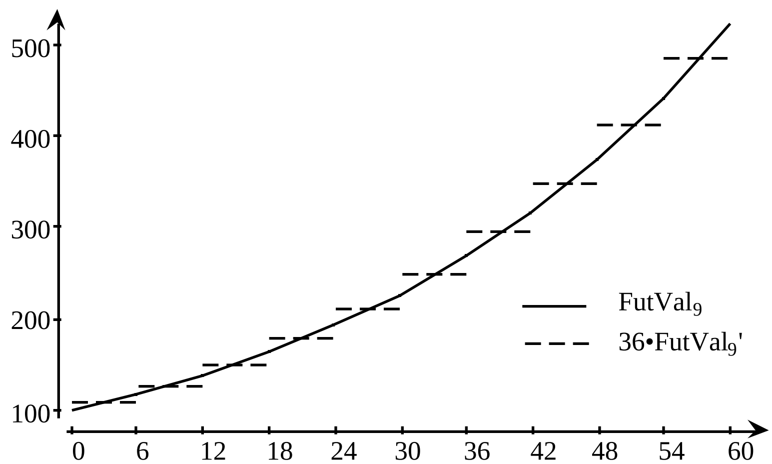

Figure 5.

where FutVal_9' was dilated by a factor of about 36. Since $36 \bullet \text{FutVal}_9'$ approximates FutVal_9, we get that FutVal_9 is an approximate solution of $f' = \frac{1}{36} f$. Indeed, the graph of $100 \bullet \exp\left(\frac{t}{36}\right)$ is almost undistinguishable from that of FutVal_9.

In the case of COS and SIN, we begin by investigating the projections COS_n and SIN_n of a point winding on a n-gon. Winding around rectangles, we see how the projections inherit their symmetries from those of the circuit, around a square why they should translated from each other and around a diamond why the input ruler should be tick-marked with non-integers—here multiples of $\sqrt{2}$. Because COS_n and SIN_n are piecewise affine, COS_n' and SIN_n' are piecewise constant and, as n gets large, COS_n' begins to look like an approximation of $-\text{SIN}_n$ and SIN_n' like an approximation of COS_n. When $n = 12$, we have:

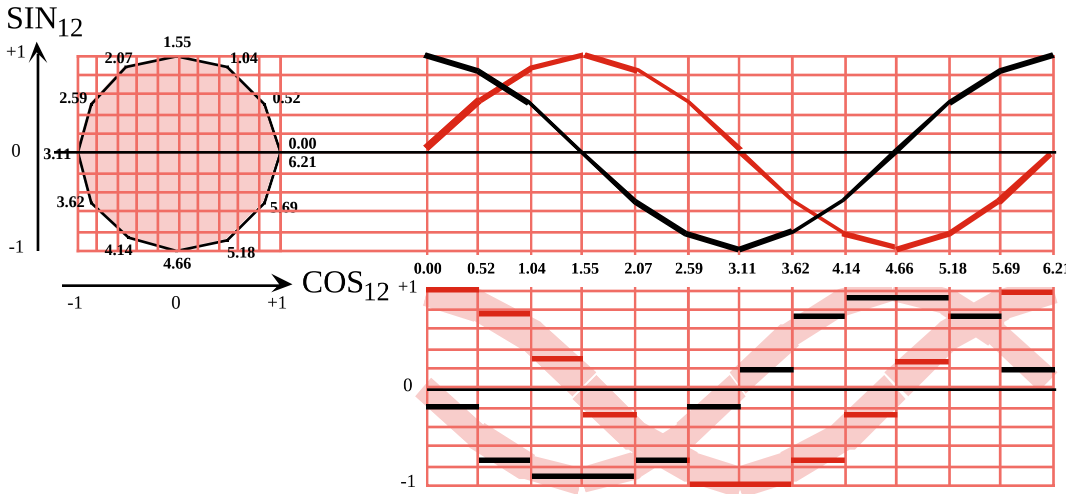

Figure 6.

Whence both the global behavior of COS and SIN and their differential equations.

Altogether then, if Precalculus is really to help "just plain folks" by providing a *natural* transition from Basic Algebra to the differential calculus, Precalculus should be the calculus of piecewise affine and constant functions. Indeed,

i. They are computable by students with just a background in Basic Algebra.
ii. They provide lucid examples for functional terminology.
iii. They give approximate solutions for significant problems.
iv. Sequences of increasingly better approximate solutions can be constructed.
v. They provide a simple initiation to first order differential equations.

Then, Calculus One can concentrate on the convergence of the approximate solutions to the elementary functions, that is on what happens when the number of pieces goes to infinity just as when we look at a circle as a regular polygon with infinitely many vertices.

References

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