Mathematics For Learning

With Inflammatory Notes for the Mortification of Educologists and the Vindication of "Just Plain Folks"

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[Editor's note: In the Spring 2004 issue of *The AMATYC Review*, Schremmer introduced his idea for an open-source serialized text: *Mathematics For Learning.* The Preface to the text appeared in the Spring 2004 issue with a new chapter in each subsequent issue of *The AMATYC Review*. This issue contains Chapter 5.]

5. Multiplication

As children, we first encounter multiplication as **additive power**, that is as *repeated addition*, of *whole numbers* on the basis of which we are made to memorize the **multiplication tables**.

Our goal here will be to extend the notion of multiplication to *signed decimal numbers.* The difficulty[1] is that multiplication is an operation much different from addition in that, for instance and independently of what the denominator **Blob** might stand for, while 3 **Blobs** + 2 **Blobs** = 6 **Blobs** is a *meaningful*—if false—sentence, 3 **Blobs** × 2 **Blobs** = 6 **Blobs** is utterly *meaningless.*

Multiplication occurs in three very different types of situation, namely:

- As dilation, e.g. 3.2 (20.1 **Blobs**) = 64.36 **Blobs**
- As co-multiplication, e.g. 3.2 **Blobs** × 20.1 $\frac{\textsf{Dollars}}{\textsf{Blob}}$ = 64.36 **Dollars**
- As external composition law, e.g. 3.2 **Blobs** × 20.1 **Blobs** = 64.36 **SquareBlobs**

We shall develop the *procedures* for multiplication based on the *multiplication tables.* In order to do so, though, it will be convenient to have denominators for each digit in a decimal number-phrase.

Metric Headings

A heading in the **metric denominator system** involves

[1]Which, since Educolgists cannot be bothered with *denominators*, they just ignore.

- picking a **unit denominator**
- using the following **metric prefixes**:

KILO	HECTO	DEKA	–	DECI	CENTI	MILLI

We can illustrate with *money*: if we pick **Franklins** as our unit denominator, then the heading that corresponds to

Clevelands	Franklins	Hamiltons	Washingtons	Dimes

in the metric denominator system is

DEKA Franklins	Franklins	DECI Franklins	CENTI Franklins	MILLI Franklins

and, for instance, under this metric heading,

3.2 **Clevelands** becomes 3.2 **DEKAFranklins**,
3.72 **Franklins** remains 3.72 **Franklins**,
7.45 **Hamiltons** becomes 7.45 **DECIFranklins**.

But, of course, the *unit denominator* can be anything and, for instance, when dealing with 327.8 **Foot** we will be able to say that 2 is the number of **DEKAFoot**, 3 is the number of **HECTOFoot** and 8 is the number of **DECIFoot**.[2]

NOTE: The above is taken from a comprehensive chapter designed to let students familiarize themselves with the several variants of the metric denominator system such as, for instance, the **exponential denominator system** in which, say, **HECTOFoot** is replaced by × **TEN**$^{+2}$**Foot** and where × is just a **separator** or the **decimal denominator system** in which **HECTOFoot** is replaced by × 100. **Foot** where × is first seen as a *separator* and then as *multiplication symbol*. The chapter was omitted from this serialization for the sake of brevity but is available from the author.

Multiplication As Dilation

Given a rubber band that, in the initial state, is two feet long, and given a three-fold **dilation**, the rubber band in the final state will be six feet long. If we are able to assimilate this dilation to a *repeated addition* of length,[3] we then have a

[2]It is puzzling that, instead, Educologists should use TENS, HUNDREDS and TENTHS, thereby effectively obliterating, along with the distinction between *numerators* and *denominators*, any possibility of real understanding on the part of the students.

[3]This is by no means easy even though Educologists blissfully ignore the issue.

procedure to get the final length namely

$$\begin{aligned} 3\,(2\ \textsf{Foot}) &= 2\ \textsf{Foot} + 2\ \textsf{Foot} + 2\ \textsf{Foot} \\ &= 6\ \textsf{Foot} \end{aligned}$$

In any case, the issue here is to extend this to *decimal* numerators, for instance to 3.4 (20.1 **Foot**).

The idea is to decompose the decimal numerators into "single-digit" numerators by use of metric prefixes so as to be able to use the *multiplication tables*. (Recall that we use "&" for *combinations*, i.e. with *different* denominators, and "+" only for *addition*, i.e. with *identical* denominators.)

In order to overcome their initial "fear of proving", the students first need to reduce a great many specifying-phrases down to where they can use multiplication tables as in, say,

$$\begin{aligned} 2\,(\,0.3\ \textsf{Foot}\,) &= 2\,(\,3\ \textsf{DeciFoot}\,) \\ &= (2\times 3)\ \textsf{DeciFoot} \\ &= 6\ \textsf{DeciFoot} \\ &= 0.6\ \textsf{Foot} \end{aligned}$$

where:

i. 0.3 **Foot** is read under a heading as 3 **DeciFoot**

ii. the "repeated addition" 2 (3 **DeciFoot**) is turned into the multiplication (2 × 3) **DeciFoot**,

iii. the multiplication (2 × 3) **DeciFoot** gives, by way of the multiplication table, 6 **DeciFoot**

iv. the result, 6 **DeciFoot**, is changed under the heading back to the original denominator: 0.6 **Foot**.

Then they should go through a sequence of progressively more complicated instances, all the way up to, say,

$$3.4\,(20.1\ \textsf{Foot}) = 3.\,(20.1\ \textsf{Foot}) + 0.4\,(20.1\ \textsf{Foot})$$

and, decomposing 20.1 **Foot** into the *combination* 2 **DekaFoot** & 1 **DeciFoot**,

$$\begin{aligned} &= 3\,(2\ \textsf{DekaFoot}\ \&\ 1\ \textsf{DeciFoot})\ \&\ 4\ \textsf{Deci}\ (2\ \textsf{DekaFoot}\ \&\ 1\ \textsf{DeciFoot}) \\ &= (3\times 2)\ \textsf{DekaFoot}\ \&\ (3\times 1)\ \textsf{DeciFoot})\ \&\ (4\times 2)\ \textsf{Foot}\ \&\ (4\times 1)\ \textsf{CentiFoot} \end{aligned}$$

$$= 6\ \textsf{DekaFoot}\ \&\ 3\ \textsf{DeciFoot}\ \&\ 8\ \textsf{Foot}\ \&\ 4\ \textsf{CentiFoot}$$

and, changing back to the initial **Foot** as *common* denominator,

$$\begin{aligned} &= 60.\ \textsf{Foot} + 0.3\ \textsf{Foot} + 8.\ \textsf{Foot} + 0.04\ \textsf{Foot} \\ &= 68.34\ \textsf{Foot} \end{aligned}$$

Finally, the students must learn to streamline the above to, say:

$$\begin{aligned} 3.4\,(20.1\ \textsf{Foot}) &= 34.\,(201.\ \textsf{Centi Foot}) \\ &= (34. \times 201.)\ \textsf{Centi Foot} \\ &= 6834.\ \textsf{Centi Foot} \\ &= 68.34\ \textsf{Foot} \end{aligned}$$

In other words, on the first line, we moved the two decimal points one place each to the right and, on the last line, we *undid* what we had done on the first line by moving the decimal point two places to the left.[4]

Multiplication as Co-multiplication

We seldom deal with collections of *items* without wanting to know their **worth** given the **unit-worth** of the items. For instance, given a collection of THREE ***apples***—represented on the board[5] as 3 **Apples**—with a *unit-worth* of SEVEN ***dimes-per-apple***—represented as $7\ \frac{\textsf{Dimes}}{\textsf{Apple}}$, its *worth* is TWENTY-ONE ***dimes***.

In order to obtain the *value* 21 **Dimes** on the board, we **co-multiply** as follows[6]

$$\begin{aligned} \textsf{Collection's Value} &= [3\ \cancel{\textsf{Apples}}] \times \left[7\ \frac{\textsf{Dimes}}{\cancel{\textsf{Apple}}}\right] \\ &= [3 \times 7]\ \textsf{Dimes} \\ &= 21\ \textsf{Dimes} \end{aligned}$$

[4]Which Educologists seem, rather less than helpfully, to consider "self-evident".

[5]In keeping with our distinguishing between what *is* on the desk versus what we *write* on the board.

[6]Educologists will surely recognize the importance of the concept since, after all, "cancelling" *denominators* is central both to DIMENSIONAL ANALYSIS and to LINEAR ALGEBRA. More modestly, co-multiplication also arises with **percentages**:

$$[3\ \cancel{\textsf{Dollars}}] \times \left[7\ \frac{\textsf{Cents}}{\cancel{\textsf{Dollar}}}\right] = [3 \times 7]\ \textsf{Cents} = 21\ \textsf{Cents}$$

The extension to decimal number-phrases goes essentially as in the case of dilation.

We now extend the concept of *co-multiplication* to obtain the *gain* or *loss* caused by a *transaction*. Suppose for instance that we look at *transactions* occurring in an ***apple*** store where, for whatever reason to be left to the reader's imagination,

- ***apples*** can *appear in* or *disappear from* the store,
  For instance,

♠ We can *have*:	❖ We then *write*:
THREE ***apples*** *appearing in* the store,	[+3 **Apples**]
or	
FIVE ***apples*** *disappearing from* the store.	[−5 **Apples**]

- ***apples*** can be either *good*, with therefore a sale profit, or *bad*, with therefore a disposal cost.
  For instance,

♠ We can *have*:	❖ We then *write*:
apples that are ***good*** and could be *sold* at a unit-*profit* of, say, SEVEN ***cents-per-apple***	$+7\ \frac{\text{Cents}}{\text{Apple}}$
or	
apples that are ***bad*** and must be *disposed of* at a unit-*loss* of, say, SEVEN ***cents-per-apple***	$-7\ \frac{\text{Cents}}{\text{Apple}}$

Then, no student has ever contested that:

THREE ***apples*** that *appear* in the store	[+3 **Apples**]
and are ***good***, with a unit-*profit* of SEVEN ***cents-per-apple***,	$\left[+7\ \frac{\text{Cents}}{\text{Apple}}\right]$
result in	$[+3\ \cancel{\text{Apples}}] \times \left[+7\ \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [+3] \times [+7]$ **Cents**
a *profit* of TWENTY-ONE ***cents***.	$= +21$ **Cents**

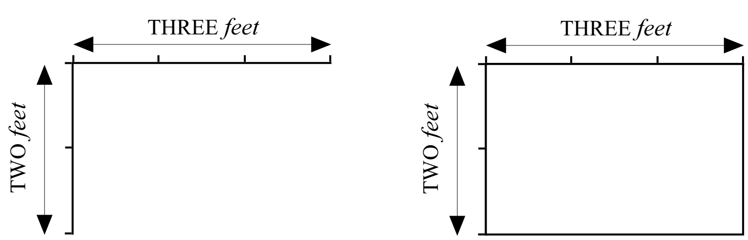
THREE ***apples*** that *appear* in the store	[+3 **Apples**]
and are ***bad***, with a unit-*loss* of SEVEN ***cents-per-apple***,	$\left[-7\ \frac{\text{Cents}}{\text{Apple}}\right]$
result in	$[+3\ \cancel{\text{Apples}}] \times \left[-7\ \frac{\text{Cents}}{\cancel{\text{Apple}}}\right]$
	$= [+3] \times [-7]$ **Cents**
a *loss* of TWENTY-ONE ***cents***	$= -21$ **Cents**

THREE ***apples*** that *disappear* from the store	$[-3\ \textsf{Apples}]$
and are ***good***, with a unit-*profit* of SEVEN ***cents-per-apple***,	$\left[+7\ \frac{\textsf{Cents}}{\textsf{Apple}}\right]$
result in	$[-3\ \cancel{\textsf{Apples}}] \times \left[+7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right]$
	$= [-3] \times [+7]\ \textsf{Cents}$
a *loss* of TWENTY-ONE ***cents***	$= -21\ \textsf{Cents}$
THREE ***apples*** that *disappear* from the store	$[-3\ \textsf{Apples}]$
and are ***bad***, with a unit-*loss* of SEVEN ***cents-per-apple***,	$\left[-7\ \frac{\textsf{Cents}}{\textsf{Apple}}\right]$,
result in	$[-3\ \cancel{\textsf{Apples}}] \times \left[-7\ \frac{\textsf{Cents}}{\cancel{\textsf{Apple}}}\right]$
	$= [-3] \times [-7]\ \textsf{Cents}$
a *profit* of TWENTY-ONE ***cents***.	$= +21\ \textsf{Cents}$

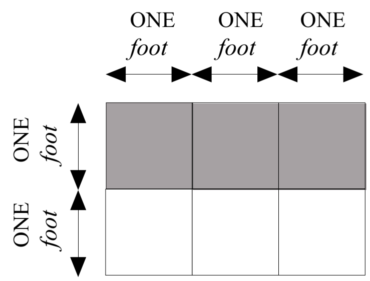
Multiplication as Area of a Rectangle

Finally, we look at multiplication as an **external composition law**, for instance 2 **Foot** × 3 **Foot**.

We assume that we know how to "construct" a **rectangle** with **width**, say, TWO ***feet***, and **length**, say, THREE ***feet***.



The *process* for tiling this rectangle with ***one-foot-by-one-foot tiles***, also known as ***squarefeet***, is to lay two rows of three tiles:

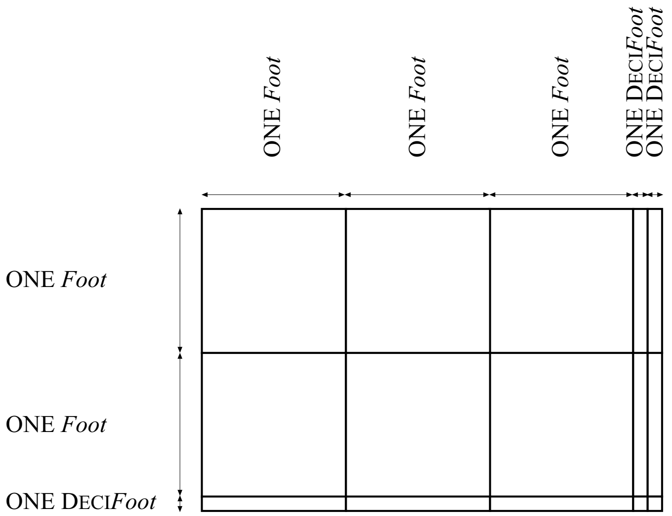


which uses up SIX ***squarefeet***.

The *procedure* to identify the specifying-phrase, 2 **Feet** $\times$ 3 **Feet**, corresponds to the above *process* in that it is multiplication as "repeated addition" and we get

$$\begin{aligned} 2 \textbf{ Foot} \times 3 \textbf{ Foot} &= 2\ [3 \textbf{ Foot} \times \textbf{Foot}] \\ &= [2 \times 3] \textbf{ SquareFoot} \\ &= 6 \textbf{ SquareFoot} \end{aligned}$$

Just as in the case of *dilation*, we can extend this to decimal number-phrases, for instance 2.1 **Foot** $\times$ 3.2 **Foot**, and since, again, the procedure follows the real life process, we give just a figure using the denominators and the corresponding computation.



$$
\begin{aligned}
2.1\ \textbf{Foot} \times 3.2\ \textbf{Foot} &= [2\ \textbf{Foot}\ \&\ 1\ \textbf{DeciFoot}] \times \left[3\ \textbf{Foot}\ \&\ 2\ \textbf{DeciFoot}\right] \\
&= [2\ \textbf{Foot} \times 3\ \textbf{Foot}]\ \&\ \left[2\ \textbf{Foot} \times 2\ \textbf{DeciFoot}\right]\ \&\ [1\ \textbf{DeciFoot} \times 3\ \textbf{Foot}] \\
&\quad \&\ \left[1\ \textbf{DeciFoot} \times 2\ \textbf{DeciFoot}\right] \\
&= 6\ \textbf{Foot}\ \textit{by}\ \textbf{Foot}\ \&\ 4\ \textbf{Foot}\ \textit{by}\ \textbf{DeciFoot}\ \&\ 3\ \textbf{DeciFoot}\ \textit{by}\ \textbf{Foot} \\
&\quad \&\ 2\ \textbf{DeciFoot}\ \textit{by}\ \textbf{DeciFoot}
\end{aligned}
$$

and, if the tiles are without pattern so that **DeciFoot** *by* **Foot** = **Foot** *by* **DeciFoot**

$$
\begin{aligned}
&= 6\ \textbf{Foot}\ \textit{by}\ \textbf{Foot}\ \&\ 7\ \textbf{Foot}\ \textit{by}\ \textbf{DeciFoot}\ \&\ 2\ \textbf{DeciFoot}\ \textit{by}\ \textbf{DeciFoot} \\
&= 6\ \textbf{SquareFoot}\ \&\ 7\ \textbf{DeciSquareFoot}\ \&\ 6\ \textbf{CentiSquareFoot} \\
&= 6\ \textbf{SquareFoot} + 0.7\ \textbf{SquareFoot} + 0.06\ \textbf{SquareFoot} \\
&= 6.72\ \textbf{SquareFoot}
\end{aligned}
$$

where we use