After a private exchange with Warren Esty à propos Teaching about Inverse Functions*, his Spring 2005 article in these pages, Part One of* **Mathematics For Learning** *was completely reorganized with the result that* ALGEBRA *is now integrated with* ARITHMETIC *from the very beginning:*

compare
match one-to-one
relationship
leftover
count from ... to ...
is less numerous than

As serialized:	*As reorganized-rewritten:*
1. Counting With Number-Phrases	**1. *Basic* Collections of Money**
Accounting for Money	Counting up to NINE
Addition	Equalities & Inequalities
2. Accounting for Money	Equations & "Inequations"
(Decimal) Headings	Addition
Adding Under A Heading	Subtraction
Subtracting Under A Heading	Combinations
(Decimal) Number-Phrases	**2. *Extended* Collections of Money**
	Bundles and Exchanges
	The rest essentially as before.

This should result in a better course for developmental students who often feel slighted by having to spend a lot of time on ARITHMETIC *before being allowed to "graduate" to* ALGEBRA. *The following consists of the* early *introductions of* Equalities & Inequalities *and* Equations & "Inequations". *Hopefully, the whole "new improved" version will eventually be available for download.*

0.1 Comparing Collections: Equalities and Inequalities

We now want to **compare** collections—involving the *same* kind of objects.

1. We begin with the *comparison* of two collections on the *counter* and the board *procedure* for getting the *result* of the comparison.

♠ On the *counter*, what we do is to **match one-to-one** the *objects* in the two collections; the **relationship** between the two collections depends on which of the two collections the **leftover** objects are in.

❖ On the *board*, we count the two collections and then we **count from** the numerator of the first number-phrase **to** the numerator of the second number-phrase, that is, starting *after* the numerator of the *first* number-phrase, we count to the numerator of the *second* number-phrase.

Either way, we then have the following three possibilities:

a. In general,

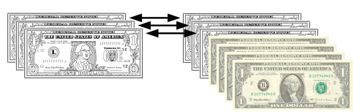
♠ When the leftover objects are in the *second* collection, the first collection **is less numerous than** the second collection[1].

[1]Educologists may question this contrived term. Of course, the issue is to have different

count forward
succeed
is more numerous than
count backward
precede
precession

❖ To count from the first numerator to the second one, *starting* with the digit *after* the first numerator, we must **count forward**, that is, we must call the digits that **succeed** it in 1, 2, 3, 4, 5, 6, 7, 8, 9 and *end* with the second numerator.
For instance, $\xrightarrow{4,\ 5,\ 6,\ 7}$ is a *forward* count that starts *after* 3 and ends with 7.

For instance,

♠ On the *counter*.	❖ On the *board*.
Jack has 	We *count* Jack's collection: $\xrightarrow{1,\ 2,\ 3}$
Jill has 	We *count* Jill's collection: $\xrightarrow{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7}$
 ***Jack***'s collection is *less numerous than* ***Jill***'s collection	$\xrightarrow{4,\ 5,\ 6,\ 7}$ We must count *forward*.

b. In general,

♠ When the leftover objects are in the *first* collection, the first collection **is more numerous than** the second collection.

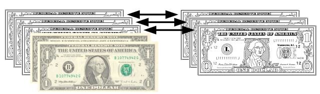
❖ To count from the first numerator to the second one, *starting* with the digit *before* the first numerator, we must **count backward**, that is, we must call the digits that **precede** it in 1, 2, 3, 4, 5, 6, 7, 8, 9 and *end* with the second numerator. For instance, $\xleftarrow{3,\ 4}$ is a *backward* count that starts *before* 5 and ends with 3.
Note. Thus, the **precession** 9, 8, 7, 6, 5, 4, 3, 2, 1 should be memorized as well as the *succession* 1, 2, 3, 4, 5, 6, 7, 8, 9[2].

For instance,

terms for use on the counter and to write on the board and only experience can tell if the difference is worth making.

[2] Should Educologists ask children to do so, they might discover that children actually *love* to count backward.

is as numerous as
sentence
verb

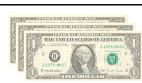
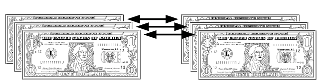
♠ On the *counter*.	❖ On the *board*.
Jack has 	We *count* Jack's collection: $\xrightarrow{1,\,2,\,3,\,4,\,5}$
Jill has 	We *count* Jill's collection: $\xrightarrow{1,\,2,\,3}$
	
Jack's collection is *more numerous than* ***Jill***'s collection.	$\xleftarrow{3,\,4}$ We must count *backward*.

c. In general,

♠ When there are *no* leftover objects, the first collection **is as numerous as** the second collection.

❖ The two numerators are the same and we must count neither forward nor backward.

For instance,

♠ On the *counter*.	❖ On the *board*.
Jack has 	We *count* Jack's collection: $\xrightarrow{1,\,2,\,3}$
Jill has 	We *count* Jill's collection: $\xrightarrow{1,\,2,\,3}$
	
Jack's collection is *equal* to ***Jill***'s collection.	We must count neither forward nor backward.

2. In order to represent on the *board* the *result* of comparing two collections, we must expand our *mathematical* language beyond *number-phrases*.

a. Given a *relationship* between two collections, we write a **sentence** involving the two *number-phrases* that represent the collections and a **verb** that represents the *relationship* between the two collections:

$<$
is smaller than
$>$
is larger than
$=$
is equal to
strict inequality
equality
bounded inequality
$\leqq$
less than or equal to

- We will use the *verb* $<$ to represent the relationship *is less numerous than* and we will read it **is smaller than** For instance, in the first of the above three examples, we will write the sentence 3 **Dollars** $<$ 7 **Dollars** which we will read "THREE ***dollars*** is smaller than FIVE ***dollars***."
- We will use the *verb* $>$ to represent the relationship *is more numerous than* and we will read it **is larger than**. For instance, in the second of the above three examples, we will write the sentence 5 **Dollars** $>$ 3 **Dollars** which we will read "FIVE ***dollars*** is larger than THREE ***dollars***."
- We will use the *verb* $=$ to represent the relationship *is as numerous as* and we will read it **is equal to**. For instance, in the third of the above three examples, we will write the sentence 3 **Dollars** $=$ 3 **Dollars** which we will read "THREE ***dollars*** is equal to THREE ***dollars***."

In other words,

When we must count *forward* $\xrightarrow{\ ,\ \cdots\ ,\ }$	we write $<$	which is read as "is *smaller* than"
When we must count *backward* $\xleftarrow{\ ,\ \cdots\ ,\ }$	we write $>$	which is read as "is *larger* than"
When we must *not* count either way	we write $=$	which is read "is *equal* to"

Note. Beware that the symbols $<$ and $>$ go in directions *opposite* to that of the arrowheads when we count from the first numerator to the second numerator. (If need be, one can think of $<$ as $\cdot :$ with $\cdot$ being "smaller" than $:$ and of $>$ as $: \cdot$ with $:$ being "larger" than $\cdot$.)

b. *Sentences* involving the *verbs* $>$ or $<$ are called **strict inequalities** while sentences involving the *verb* $=$ are called **equalities**. For example,

3 **Dollars** $<$ 7 **Dollars** and 8 **Dollars** $>$ 2 **Dollars** are *strict inequalities*
3 **Dollars** $=$ 3 **Dollars** is an *equality*

c. In English, when we say that we allow "up to" 5 **Dollars**, we mean that we allow 1 **Dollar**, 2 **Dollars**, 3 **Dollars**, 4 **Dollars** but that we do *not* allow the *endpoint* itself, 5 **Dollars**. If we do want also to allow the *endpoint*, 5 **Dollars**, we say "up to and including" 5 **Dollars**.

In mathematics we shall also need to make this distinction, that is, to allow or not to allow the *endpoint*, and, when we do allow it, we will say that the inequality is a **bounded inequality**:

- We will use the verb $\leqq$ to represent the relationship *is less numerous than or as numerous as* and we will read it **less than or equal to**.

- We will use the verb $\geqq$ to represent the relationship *is more numerous than or as numerous as* and we will read it **more than or equal to**.

more than or equal to
true
false
realize
negation
$\neg$ [ ]
slashing
(linguistic) duality
(linguistic) symmetry

d. Inasmuch as the sentences that we wrote above represented *actual* relationships between collections on the counter, they were **true** but there is of course nothing to prevent us from writing sentences that are **false** in the sense that there is no way that we could come up with *situations* that these sentences would represent. For example, the sentence

$$5 \textsf{ Dollars} = 3 \textsf{ Dollars}$$

is *false* because there is no way that we could **realize** this on the counter with actual collections.

However, while occasionally useful, it is usually not very convenient to write sentences that are *false* because we must not forget to say so when writing and we may miss where it says so when reading and, so, inasmuch as possible, we shall write only sentences that are true and use the *default rule*:

WHEN NO INDICATION OF TRUTH OR FALSEHOOD IS GIVEN, MATHEMATICAL SENTENCES WILL BE UNDERSTOOD TO BE TRUE AND THIS WILL GO WITHOUT SAYING.

Moreover, when a sentence is *false*, rather than writing *it*, what we shall usually do is to write *its* **negation**—which is *true*—which we can do either by placing the false sentence within the symbol $\neg$ [ ] or by **slashing** the *verb*.

For instance, instead of writing that

the sentence $5 \textsf{ Dollars} = 3 \textsf{ Dollars}$ is *false*

we will write either the (true) sentence

$$\neg\,[5 \textsf{ Dollars} = 3 \textsf{ Dollars}]$$

or the (true) sentence

$$5 \textsf{ Dollars} \neq 3 \textsf{ Dollars}$$

3. The **(linguistic) duality** that exists between $<$ and $>$ must not to be confused with **(linguistic) symmetry**, a concept which we tend to be more familiar with[3].

a. Examples of linguistic *symmetry* include:

- **Jack** is a child of **Jill** versus **Jill** is a child of **Jack**
- **Jill** beats **Jack** at poker versus **Jack** beats **Jill** at poker
- **Jack** loves **Jill** versus **Jill** loves **Jack**
- $9 \textsf{ Dimes} > 2 \textsf{ Dimes}$ versus $9 \textsf{ Dimes} < 2 \textsf{ Dimes}$

[3]This confusion is a most important *linguistic* stumbling block for students and one that Educologists utterly fail to take into consideration.

dual
specify
requirement
satisfy

In each example, the two sentences represent *opposite* relationships between the two people/collections because, even though the verbs *are the same*, the two people/collections are *mentioned in opposite order*.
Observe that just because one of the two sentences is *true* (or *false*) does not, by itself, *automatically* force the other to be either *true* or *false* and that whether or not it does depends on the *nature* of the relationship.

b. Examples of linguistic *duality* include:

- **Jack** is a *child* of **Jill** versus **Jill** is a *parent* of **Jack**
- **Jill** *beats* **Jack** at poker versus **Jack** *is beaten by* **Jill** at poker
- **Jack** *loves* **Jill** versus **Jill** *is loved by* **Jack**
- 9 **Dimes** $>$ 2 **Dimes** versus 2 **Dimes** $<$ 9 **Dimes**

In each example, the two sentences represent the *same* relationship between the two people/collections because, even though the people/collections are *mentioned in opposite order*, the two *verbs* are **dual** of each other which "undoes" the effect of the order so that only the *emphasis* is different.
Observe that, as a result, if one of the two sentences is *true*(or *false*) this *automatically* forces the other to be *true* (or *false*) and this regardless of the *nature* of the relationship.

c. The following are examples of simultaneous *linguistic symmetry* and *linguistic duality* because the *verbs are the same* and the order does *not* matter.

- **Jack** is a sibling of **Jill** versus **Jill** is a sibling of **Jack**
- 2 **Nickels** $=$ 1 **Dime** versus 1 **Dime** $=$ 2 **Nickels**

Observe that, in that case, it looks as if as soon as one sentence is *true* (or *false*), by itself, this *automatically* forces the other to be *true* (or *false*) and that it does not seem to depend on the *nature* of the relationship.

0.2 Specifying Collections: Equations and "Inequations"

In real life, we often have to **specify** things by stating some **requirement(s)** that the things we want must **satisfy**.

Here, we will *specify* collection(s) by the *requirement* that they stand in a given *relationship*, namely one or the other of the following,

- *is more numerous than* the *given* collection,
- *is less numerous than* the *given* collection.
- *is as numerous as* the *given* collection,

with a given *collection*.
For instance, say that

efficient
data

Jack has THREE ***dollars***,
Jill has SEVEN ***dollars***,
Dick has THREE ***dollars***,
Jane has FOUR ***dollars***.

and that we specify the collection(s) that satisfy the *requirement* that they be *more numerous than* ***Jack***'s collection.

1. We could of course proceed as we did in Section **??**:

♠ On the counter, matching ***Jack***'s collection one-to-one with each one of the collections of ***Jill***, ***Dick*** and ***Jane*** shows that this specifies the collections of ***Jill*** and ***Jane***.

❖ On the board, counting *from* ***Jack***'s collection each one of the collections of ***Jill***, ***Dick*** and ***Jane*** would give the same result.

This approach, though, is somewhat short of ideal if only because it would become very time-consuming with large numbers of collections to compare. So, what we want is to develop a board procedure that is more **efficient** in that the time it requires will not go up appreciably as the number of collections and of objects in the collections goes up.

2. Before we do that, though, we need a way to phrase *requirements* that lends itself to *procedural manipulations*.

a. Essentially, what we will do is to introduce the *mathematical* version of something common in everyday life, namely *forms* such as

[   ] was President of the United States.

which, when we *fill* it it with some **data**, say,

Kissinger

produces a *sentence*, namely

[Kissinger] was President of the United States.

which happens to be *false* while, when we fill it with the *data*

Bill Clinton

it produces the *sentence*

[Bill Clinton] was President of the United States.

which happens to be *true*.

b. In the case of the above example,

♠ On the counter, we want the collections of ***dollars*** that *satisfy* the *requirement* that they be more numerous than THREE ***dollars***.

solution
non-solution
unspecified numerator
equations
strict inequation
bounded inequation
replace
instruction
:=
specifying-phrase
identity
identify

❖ On the board, we want the **solutions** of the corresponding form

$$\square\ \mathsf{Dollars} > 3\ \mathsf{Dollars}$$

For instance, we found above that
- the *data* 7 produces the *sentence* $\boxed{7}\ \mathsf{Dollars} > 3\ \mathsf{Dollars}$ which is *true*,
- the *data* 4 produces the *sentence* $\boxed{4}\ \mathsf{Dollars} > 3\ \mathsf{Dollars}$ which is *true*,
- the *data* 3 produces the *sentence* $\boxed{3}\ \mathsf{Dollars} > 3\ \mathsf{Dollars}$ which is *false*.

Thus 7 and 4 are *solutions* of the *form* $\square\ \mathsf{Dollars} > 3\ \mathsf{Dollars}$ while 3 is a **non-solution**.

c. Boxes, though, would soon turn out to be impossibly difficult to use and, instead, we will use **unspecified numerators**, such as for instance the letter x, as in

$$x\ \mathsf{Dollars}$$

and, instead of the form $\square\ \mathsf{Dollars} > 3\ \mathsf{Dollars}$ we shall write

$$x\ \mathsf{Dollars} > 3\ \mathsf{Dollars}$$

We shall call:
- **equations** those forms whose *verb* is $=$,
- **strict inequations** those forms whose *verb* is either $<$ or $>$,
- **bounded inequations** those forms whose *verb* is either $\leqq$ or $\geqq$.

d. Instead of *filling the box* with the data, say, 3, we **replace** x by 3 and the **instruction** to do so will be

$$\big|_{\text{where } x:=3}$$

in which the symbol **:=**, borrowed from a computer language called Pascal, is read as "is to be replaced by." Thus

$$x\ \mathsf{Dollars}\big|_{\text{where } x:=3}$$

is a **specifying-phrase** in that it *specifies*

$$3\ \mathsf{Dollars}$$

The following sentence

$$x\ \mathsf{Dollars}\big|_{\text{where } x:=3} = 3\ \mathsf{Dollars}$$

is therefore "trivially" *true*; it is an example of a type of sentence called **identity** because it **identifies** the numerator specified by the *specifying-phrase*.

We also have

- x **Dollars**$|_{\text{where } x:=7} > 3$ **Dollars**,
- x **Dollars**$|_{\text{where } x:=4} > 3$ **Dollars**,
- x **Dollars**$|_{\text{where } x:=3} \not> 3$ **Dollars**.

general
solution set
break-even point
associated equation

3. We now turn to the simplest possible instance of a more **general** problem which is that we shall now want *all* the collection(s), if any, that stand in a given relationship with a given collection.
For example,

♠ Say ***Jack*** has FIVE ***dollars*** on the *counter*. We then want to find *all* collections of ***dollars*** that satisfy whichever one of the following three *requirements*:
 i. *is less numerous* than ***Jack***'s collection,
 ii. *is more numerous* than ***Jack***'s collection,
 iii. *is as numerous* as ***Jack***'s collection.
 (In other words, we are looking here at *three* distinct problems at once.)

❖ On the *board*, we are looking for the **solution set** of the corresponding *inequation/equation*:
 i. x **Dollars** <5 **Dollars**
 ii. x **Dollars** >5 **Dollars**
 iii. x **Dollars** $=5$ **Dollars**

We now proceed to do just so.

a. Regardless of which one of the three requirements we are trying to satisfy, we begin by considering the *equation*

$$x \textsf{ Dollars} = 5 \textsf{ Dollars}$$

whose *solution set* contains of course one, and only one, numerator: 5.

b. If it was the *equation* we were trying to solve, we are of course done.

If it was either one of the *inequations*

$$x \textsf{ Dollars} < 5 \textsf{ Dollars} \quad \text{or} \quad x \textsf{ Dollars} > 5 \textsf{ Dollars},$$

that we were trying to solve, we need to determine which side of the **break-even point** is the *solution set* of the *inequation*. (The *break-even point* is the solution of their **associated equation**, x **Dollars** $=5$ **Dollars**, that is, of the *equation* obtained from the *inequation* by replacing the verb, $<$ or $>$, by the verb $=$.)

That the solution set must be a *complete side* of the break-even point is because, if there were both a *solution* and a *non-solution* on the *same* side of the break-even point, there would then have to be *another* break-even point in-between the solution and the non-solution. But that cannot be since a

pick
test
test-point
curly brackets
endpoint

break-even point is a solution of the *associated equation* x **Dollars** $=5$ **Dollars** which can have only *one* solution, namely 5.

So, on each side of the *break-even* point, all we need to do is to **pick** *one* numerator and **test** it against the wanted requirement, that is to ask whether this **test-point** *is* or is a solution or a non-solution: every numerator on the same side of the break-even point as the test-point will then be the same.

For instance, say we are looking at the *inequation*

$$x \textsf{ Dollars} > 5 \textsf{ Dollars}$$

The *associated equation* is

$$x \textsf{ Dollars} = 5 \textsf{ Dollars}$$

so that the *break-even point* of the *inequation* is 5. Then, on each side of 5, we pick a *test-point*. Say we *pick* 3 and 7. Since to count from 3 to 5 we have to count *forward*, 3 is *not* a solution and *all* numerators on the same side of 5 as 3 will *not* be solutions either. Since to count from 7 to 5 we have to count *backward*, 7 *is* a solution and *all* numerators on the same side of 5 as 7 *will* also be solutions so that the solution set of the *inequation*

$$x \textsf{ Dollars} > 5 \textsf{ Dollars}$$

is 6, 7, 8, ...

Note. It is customary, though, to write solutions sets in-between **curly brackets** as in $\{6, 7, 8, \ldots\}$ and we shall follow the custom.

Observe that the time we spent with the above procedure does *not* depend anymore on the number of collections we are dealing with.

Observe that, here, the *break-even point* is also an **endpoint** in that *all* the numerators on the *one* side of the break-even point *are* solutions and *all* the numerators on the *other* side of the break-even point are *not* solutions. This, though, will *not* be always the case and we will encounter *break-even points* that will turn out *not* to be *endpoint*s.