Andragogic Propaedeutic Mathematics

David A. SANTOS dsantos@ccp.edu

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Preface

Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country. *-David HILBERT*

I would like to thank Eleonora Chertok, Wimayra Luy, and Alain Schremmer for their input.

I would like to thank the students who first tested these notes in the Spring of 2008: Comfort Ademuwagun, Takira Adkins, Roxan Delisi, Hai Do, Janie Drayton, Jacquelin Fulton, Sean Horton, Fatimah Johnson, Wakeelah Lloyd, Ashley Mangum, Inmaculada Martinez, Ruben Medacier, Leslie Pierre, Madeline Roman and Karina Soriano.

I have not had the time to revise, hence errors will abound, especially in the homework answers. I will be grateful to receive an email pointing out corrections.

David A. SANTOS

dsantos@ccp.edu

To the Reader

When writing these notes I set a certain number of goals and presuppositions.

- I presume that you know basic addition, subtraction, multiplication and division of whole numbers, but that you are not very efficient at it.
- My goal is to shew you how to perform these operations efficiently and to think about the algorithms used to carry out these operations.
- At times it will seem that you already know what I am talking about. This is natural, since I am not taking a linear approach to the subject. Nevertheless, pay attention and do the homework. Moreover, I would prefer that you use the methods I teach you instead of some others you may remember from before. This is because it is very likely that your memory as to why the correct methods might be faulty.
- I put a very strong emphasis on story problems. The goal is to see to teach you how to derive abstractions from concrete situations. Therefore, I would prefer a verbose response to a curt one.

These notes are provided for your benefit as an attempt to organise the salient points of the course. They are a *very terse* account of the main ideas of the course, and are to be used mostly to refer to central definitions and theorems. The number of examples is minimal. The *motivation* or informal ideas of looking at a certain topic, the ideas linking a topic with another, the worked-out examples, etc., are given in class. Hence these notes are not a substitute to lectures: **you must always attend to lectures**. The order of the notes may not necessarily be the order followed in the class.

Tutoring can sometimes help, but bear in mind that whoever tutors you may not be familiar with my conventions. Again, I am here to help! On the same vein, other books may help, but the approach presented here is at times unorthodox and finding alternative sources might be difficult.

Here are more recommendations:

- Read a section before class discussion, in particular, read the definitions.
- Class provides the informal discussion, and you will profit from the comments of your classmates, as well as gain confidence by providing your insights and interpretations of a topic. **Don't be absent!**
- I encourage you to form study groups and to discuss the assignments. Discuss among yourselves and help each other but don't be *parasites!* Plagiarising your classmates' answers will only lead you to disaster!
- Once the lecture of a particular topic has been given, take a fresh look at the notes of the lecture topic.
- Try to understand a single example well, rather than ill-digest multiple examples.
- Start working on the distributed homework ahead of time.
- **Ask questions during the lecture.** There are two main types of questions that you are likely to ask.
    1. *Questions of Correction: Is that a minus sign there?* If you think that, for example, I have missed out a minus sign or wrote P where it should have been Q,[1] then by all means, ask. No one likes to carry an error till line XLV because the audience failed to point out an error on line I. Don't wait till the end of the class to point out an error. Do it when there is still time to correct it!

[1] My doctoral adviser used to say "I said A, I wrote B, I meant C and it should have been D!

2. *Questions of Understanding: I don't get it!* Admitting that you do not understand something is an act requiring utmost courage. But if you don't, it is likely that many others in the audience also don't. On the same vein, if you feel you can explain a point to an inquiring classmate, I will allow you time in the lecture to do so. The best way to ask a question is something like: "How did you get from the second step to the third step?" or "What does it mean to complete the square?" Asseverations like "I don't understand" do not help me answer your queries. If I consider that you are asking the same questions too many times, it may be that you need extra help, in which case we will settle what to do outside the lecture.

- Don't fall behind! The sequence of topics is closely interrelated, with one topic leading to another.

- Presentation is critical. Clearly outline your ideas. When writing solutions, outline major steps and write in complete sentences. As a guide, you may try to emulate the style presented in the scant examples furnished in these notes.

Part I

Sets

1 Sets and Inclusion-Exclusion

1 Definition By a *set* we will understand any well-defined collection of objects. These objects are called the *elements* of the set. If a belongs to the set A, then we write $a \in A$, read "a is an element of A." If a does not belong to the set A, we write $a \notin A$, read "a is not an element of A."

2 Example Let $D = \{0,1,2,3,4,5,6,7,8,9\}$ be the set of the ten decimal digits. Then $4 \in D$ but $11 \notin D$.

☞ *We will normally denote sets by capital letters, say $A, B, \Omega, \mathbb{N}$, etc. Elements will be denoted by lowercase letters, say a, b, ω, r, etc. The following sets will have the special symbols below.*

$\mathbb{N} = \{0,1,2,3,\ldots\}$	*denotes the set of natural numbers.*
$\varnothing$	*denotes the empty set.*

3 Example Let $\Omega = \{1,2,\ldots,20\}$, that is, the set of integers between 1 and 20 inclusive. A subset of Ω is $E = \{2,4,6,\ldots,20\}$, the set of all even integers in Ω. Another subset of Ω is $P = \{2,3,5,7,11,13,17,19\}$, the set of primes in Ω. Observe that, for example, $4 \in E$ but $4 \notin P$.

4 Definition A *subset* is a sub-collection of a set. We denote that the set B is a subset of A by the notation $B \subseteq A$.

5 Example The set

$$S = \{\text{Comfort}, \text{Takira}, \text{Roxan}, \text{Hai}, \text{Janie}, \text{Jacquelin}, \text{Sean}, \text{Fatimah}, \text{Wakeelah}, \text{Ashley}, \text{Inmaculada}, \text{Ruben}, \text{Leslie}, \text{Madeline}, \text{Karina}\}$$

is the set of students in a particular section of Maths 016. This set can be split into two subsets: the set

$$F = \{\text{Comfort}, \text{Takira}, \text{Roxan}, \text{Hai}, \text{Janie}, \text{Jacquelin}, \text{Fatimah}, \text{Wakeelah}, \text{Ashley}, \text{Inmaculada}, \text{Madeline}, \text{Karina}\}$$

of females in the class, and the set

$$M = \{\text{Sean}, \text{Ruben}, \text{Leslie}\}$$

of males in the class. Thus we have $F \subseteq S$ and $M \subseteq S$. Notice that it is *not true* that $F \subseteq M$ or that $M \subseteq F$.

6 Example Find all the subsets of $\{a,b,c\}$.

Solution: ► *They are*

$S_1 = \varnothing$	$S_4 = \{c\}$	$S_7 = \{c,a\}$
$S_2 = \{a\}$	$S_5 = \{a,b\}$	
$S_3 = \{b\}$	$S_6 = \{b,c\}$	$S_8 = \{a,b,c\}$

◄

7 Definition The *union* of two sets A and B is the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Observe that this "*or*" is inclusive, that is, it allows the possibility of x being in A, or B, or possibly both A and B.

The *intersection* of two sets A and B, is

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The difference of sets A *set-minus* B, is

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

Figures 1.1 through 1.3 represent these concepts pictorially, through the use of *Venn Diagrams*.

8 Definition Two sets A and B are *disjoint* or *mutually exclusive* if $A \cap B = \varnothing$.

9 Definition Let $A \subseteq \Omega$. The *complement* of A with respect to Ω is $A^c = \{\omega \in \Omega : \omega \notin A\} = \Omega \setminus A$.

Observe that A^c is all that which is outside A. The complement A^c represents the event that A does not occur. We represent A^c pictorially as in figure 1.4.

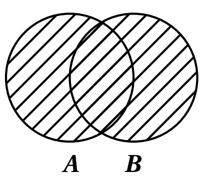


Figure 1.1: $A \cup B$

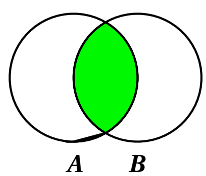


Figure 1.2: $A \cap B$

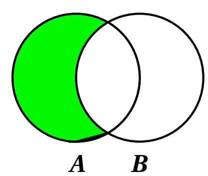


Figure 1.3: $A \setminus B$

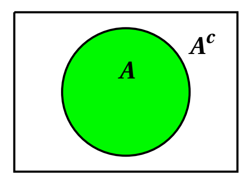


Figure 1.4: A^c

10 Example Let $\Omega = \{0,1,2,3,4,5,6,7,8,9\}$ be the universal set of the decimal digits and let $A = \{0,2,4,6,8\} \subseteq \Omega$ be the set of even digits. Then $A^c = \{1,3,5,7,9\}$ is the set of odd digits.

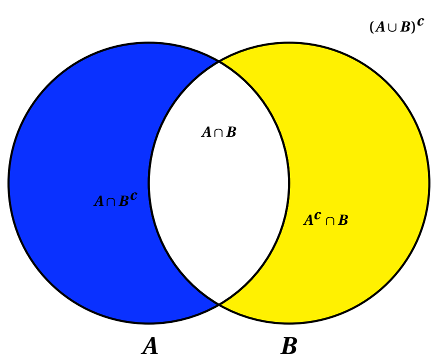


Figure 1.5: Two sets.

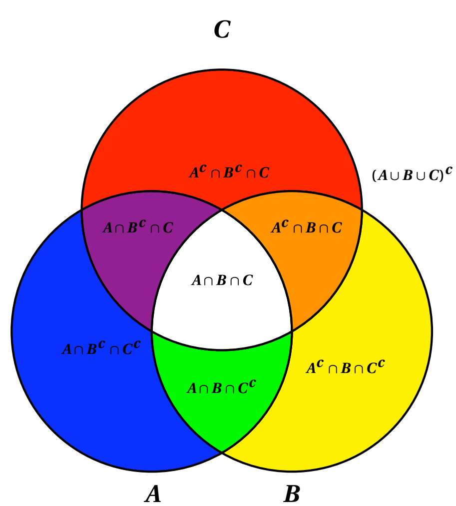


Figure 1.6: Three sets.

The various intersecting regions for two and three sets can be seen in figures 1.5 and 1.6.

11 Example Let $A = \{1,2,3,4,5,6\}$, and $B = \{1,3,5,7,9\}$. Then

$$A \cup B = \{1,2,3,4,5,6,7,9\}, \qquad A \cap B = \{1,3,5\}, \qquad A \setminus B = \{2,4,6\}, \qquad B \setminus A = \{7,9\}.$$

Homework

Problem 1.1 *Find all the subsets of* $\{a,b,c,d\}$.

2 Inclusion-Exclusion

12 Definition The *cardinality* of a set A, denoted by $\text{card}(A)$ is the number of elements that it has. If the set X has infinitely many elements, we write $\text{card}(X) = \infty$.

13 Example If $A = \{0, 1\}$ then $\text{card}(A) = 2$. Also, $\text{card}(\mathbb{N}) = \infty$.

We are now interested in finding the cardinality of a union of sets. The following examples introduce the method of *inclusion-exclusion*, which allows us to find the cardinality of a union.

14 Example Of 40 people, 28 smoke and 16 chew tobacco. It is also known that 10 both smoke and chew. How many among the 40 neither smoke nor chew?

Solution: ► *We fill up the Venn diagram in figure 2.1 as follows. Since* $\text{card}(A \cap B) = 10$*, we put a* 10 *in the intersection. Then we put a* $28 - 10 = 18$ *in the part that* A *does not overlap* B *and a* $16 - 10 = 6$ *in the part of* B *that does not overlap* A*. We have accounted for* $10 + 18 + 6 = 34$ *people that are in at least one of the set. The remaining* $40 - 34 = 6$ *are outside these sets.* ◄

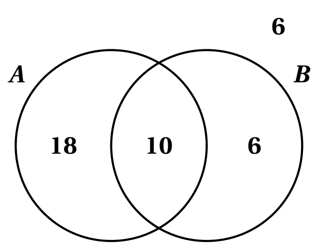


Figure 2.1: Example 14.

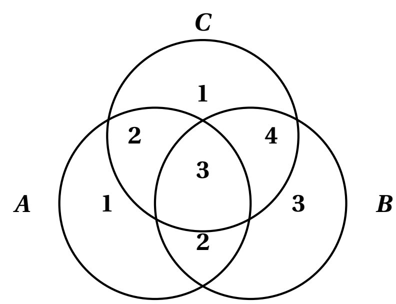


Figure 2.2: Example 15.

15 Example In a group of 30 people, 8 speak English, 12 speak Spanish and 10 speak French. It is known that 5 speak English and Spanish, 5 Spanish and French, and 7 English and French. The number of people speaking all three languages is 3. How many do not speak any of these languages?

Solution: ► *Let* A *be the set of all English speakers,* B *the set of Spanish speakers and* C *the set of French speakers in our group. We fill-up the Venn diagram in figure 2.2 successively. In the intersection of all three we put 8. In the region common to* A *and* B *which is not filled up we put* $5 - 2 = 3$*. In the region common to* A *and* C *which is not already filled up we put* $5 - 3 = 2$*. In the region common to* B *and* C *which is not already filled up, we put* $7 - 3 = 4$*. In the remaining part of* A *we put* $8 - 2 - 3 - 2 = 1$*, in the remaining part of* B *we put* $12 - 4 - 3 - 2 = 3$*, and in the remaining part of* C *we put* $10 - 2 - 3 - 4 = 1$*. Each of the mutually disjoint regions comprise a total of* $1 + 2 + 3 + 4 + 1 + 2 + 3 = 16$ *persons. Those outside these three sets are then* $30 - 16 = 14$*.* ◄

Homework

Problem 2.1 *A survey of a group's viewing habits over the last year revealed the following information:*

1. **28%** *watched gymnastics*
2. **29%** *watched baseball*
3. **19%** *watched soccer*
4. **14%** *watched gymnastics and baseball*
5. **12%** *watched baseball and soccer*
6. **10%** *watched gymnastics and soccer*
7. **8%** *watched all three sports.*

Calculate the percentage of the group that watched none of the three sports during the last year.

Problem 2.2 *In a group of* **100** *camels,* **46** *eat wheat,* **57** *eat barley, and* **10** *eat neither. How many camels eat both wheat and barley?*

Problem 2.3 *At* Medieval High *there are forty students. Amongst them, fourteen like Mathematics, sixteen like theology, and eleven like alchemy. It is also known that seven like Mathematics and theology, eight like theology and alchemy and five like Mathematics and alchemy. All three subjects are favoured by four students. How many students like neither Mathematics, nor theology, nor alchemy?*

Problem 2.4 *Rusty has* **20** *marbles of different colours: black, blue, green, and yellow. Seventeen of the marbles are not green, five are black, and* **12** *are not yellow. How many blue marbles does he have?*

Part II

Symbols

3 The Necessity of Symbols*

If a man who cannot count finds a four-leaf clover, is he entitled to happiness?
-Stanislaw Jerzey

Why bother? Symbols economise thought. For example, if we had $12 345 in our bank account, one way to keep track of this money is to have 12345 pebbles, one for each dollar. But this is silly! We know of a perfectly efficient notation to represent this, namely with numerals.

The notions of *counting* and *number* are as old as civilisation. Primitive men felt the need to count their herds.[1] Once the idea of a *quantity* or *number* was formed, symbols were introduced in order to communicate these ideas. A *numeral* is a symbol used to represent a number.

Early civilisations, like the Egyptians, used a variety of symbols to represent the abstraction of number. A number was represented by the aggregation of symbols, with no symbol appearing more than nine times. Table 3.1 gives the equivalence of the symbols to numbers.

𓁨	𓆐	𓂭	𓆼	𓍢	𓎆	𓏺
1,000,000	100,000	10,000	1,000	100	10	1

Table 3.1: Egyptian Numerals.

The sign for *one* represents a vertical staff, for *one thousand* a lotus flower, for *ten thousand* a pointing finger, for a *hundred thousand* a burbot, and for a *million* a man in astonishment. Egyptologists are not clear about the meaning of the other symbols.

16 Example In ancient Egyptian, **276** is 𓍢𓍢𓎆𓎆𓎆𓎆𓎆𓎆𓎆𓏺𓏺𓏺𓏺𓏺𓏺 and **4622** is 𓆼𓆼𓆼𓆼𓍢𓍢𓍢𓍢𓍢𓍢𓎆𓎆𓏺𓏺 . The examples of **276** and **4622** actually occur on a stone carving from Karnak, dating from around **1500** BC, now displayed in the Louvre in Paris.

Perhaps more familiar to you is the system of Roman numerals. These numbers were first developed by the Etruscans, passed to the Romans, and then modified in the Middle Ages to produce the system we use today. It is based on certain letters which are given values as numerals. For large numbers (five thousand and above), a bar is placed above a base numeral to indicate multiplication by **1000**.

I	V	X	L	C	D	M	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$	$\overline{D}$	$\overline{M}$
1	5	10	50	100	500	1000	5,000	10,000	50,000	100,000	500,000	1,000,000

The rules for using Roman numerals have evolved across centuries. For example, it was common to write IIII for **4**, rather than the IV we use today. At Harvard Medical School's Library one reads MDCCCCIIII for 1904, instead of MCMIV. The commonly used modern rules are the following:

[1]The word *men* is used here in its epicene sense. The author has no patience for the politically correct humbug of "inclusive language."

- Values are obtained by accretion of symbols.
- A symbol may appear at most three consecutive times.
- If a preexisting symbol exists for a number, do not aggregate symbols of lower value to represent the number.
- Preceding a higher-value symbol by a lower-value symbol indicates subtraction.

17 Example Using the rule above, the first ten digits are

$1 = I$	$2 = II$	$3 = III$	$4 = IV$	$5 = V$	$6 = VI$	$7 = VII$	$8 = VIII$	$9 = IX$	$10 = X$

We start with the symbol of least value I, in order to produce I for **1**. By accretion we produce II for **2**. Again, by accretion we produce III for **3**. Next is **4**. Since we cannot use the I more than three consecutive times, we subtract from the next highest valued symbol V, producing IV for **4**. We have a symbol for V. By accretion we produce VI for **6**. Again by accretion, we produce VII for **7**. We continue, producing VIII for **8**. Now, since we cannot use three consecutive I's, we subtract from the next highest symbol, producing IX for **9**. Finally, we have the symbol X for **10**.

Analogously we have

$10 = X$	$20 = XX$	$30 = XXX$	$40 = XL$	$50 = L$	$60 = LX$	$70 = LXX$	$80 = LXXX$	$90 = XC$	$100 = C$

and

$100 = C$	$200 = CC$	$300 = CCC$	$400 = CD$	$500 = D$	$60 = DC$	$700 = DCC$	$800 = DCCC$	$900 = CM$	$1000 = M$

18 Example Write, with explanation, **1989** in Roman numerals.

Solution: ► *Write*

$$1989 = 1000 + 900 + 80 + 9.$$

We first write **9**, *which is IX. We then write* **80**. *This is LXXX. We now write* **900**, *which is CM. Finally, we write* **1000**, *which is M. We now concatenate these strings resulting in*

$$1989 = 1000 + 900 + 80 + 9 = M + CM + LXXX + IX = MCMLXXXIX.$$

◄

19 Example In Roman numerals, we have:

$1215 = MCCXV$	$1492 = MCDXCII$
$1776 = MDCCLXXVI$	$1787 = MDCCLXXXVII$
$1863 = MDCCCLXIII$	$1966 = MCMLXVI$
$2000 = MM$	$2001 = MMI$
$2888 = MMDCCCLXXXVIII$	$2889 = MMDCCCLXXXIX$

Using letters of the alphabet as numbers can have amusing consequences. During the Catholic-Protestant wars in Europe, Protestants claimed that the Pope was the antichrist. To support this claim, the Protestants came up with the hoax that the Pope's tiara bears the inscription **VICARIVS FILII DEI**, and if one counts the Roman numerals (giving the value of **0** to the symbols with no value), one obtains **666**. The Pope's tiara, of course, does not bear that inscription, nor is this one of the Pope's titles. Again, using numeric values in vogue for the Greek letters around the time of the Apostle St. John, one obtains that the Emperor Nero's name gave **666**.

The Egyptian and the Roman numeral are but two systems developed to describe numbers. Other ancient cultures developed their own systems. For example, the ancient Hebrews attached a numerical values to the letters of the alphabet, having the following equivalences. Numbers were represented by accretion of symbols. A curiosity is that the symbols for **15** and **16**, instead of being written, as our decimal thinking dictates, $\mathbf{15 = 10 + 5}$ = יה and $\mathbf{16 = 10 + 6}$ = יו, were written $\mathbf{9 + 6}$ =טו and $\mathbf{9 + 7}$ = טז. This is to avoid writing part of the name of G-d, which is not to be uttered.

Hebrew Letter	Alef	Bet	Gimel	Dalet	Ha	Vav	Zayin	Het	Tet	Yod		
Hebrew Symbol	א	ב	ג	ד	ה	ו	ז	ח	ט	י		
Value	1	2	3	4	5	6	7	8	9	10		
Hebrew Letter	Kaf	Lamed	Mem	Nun	Samekh	Ayin	Pe	Tsadi	Qof	Resh	Shin	Tav
Hebrew Symbol	ך	ל	ם	ן	ס	ע	ף	ת	ק	ר	ש	ת
Value	20	30	40	50	60	70	80	90	100	200	300	400

Table 3.2: Hebrew Numerals.

Homework

Problem 3.1 *Given that* ♠ = ♢♢♢ *and that* Γ = ♠♠, *how many* ♢*'s are needed to write* **ΓΓΓ♠** *as a string of* ♢*'s?*

Problem 3.2 *Use Egyptian numerals to write* **1**, **12**, *and* **123**. *Are there any advantages gained by using Egyptian numerals? Any disadvantages? Discuss.*

Problem 3.3 *Use Roman numerals to write* **1914**, **1917**, **1939**, **1963**, *and* **1989**.

Problem 3.4 *Convert* $\overline{V}CCCXXI$ *to a Hindu-Arabic numeral.*

Problem 3.5 *Convert* $\overline{V}MMMDCCCLXXXVIII$ *to a Hindu-Arabic numeral.*

Problem 3.6 *A question worth pondering is how did the ancient Egyptians and the ancient Romans represent fractions. Do an internet or a library search on this topic, consulting history of Mathematics books and articles.*

Problem 3.7 *The text tacitly points to a distinction between* numerals *and* numbers. *Explain.*

References

[Eves] is an old, but reliable reference on ancient numerical systems. [Hof] contains interesting stories about the number of the beast. [Nic] is a very accessible Hebrew grammar for beginners.

[Eves] Howard EVES, "An Introduction to the History of Mathematics".

[Hof] Paul HOFFMAN, "Archimedes Revenge,"

[ManSte] Richard MANKIEWICZ and Ian STEWART , “The Story of Mathematics”.

[McL] John MCLEISH , “The Story of Numbers: How Mathematics Has Shaped Civilization”.

[Nic] Sarah NICHOLSEN, “Teach Yourself Biblical Hebrew”.

4 Positional Notation

Numbers are the beginning and end of thinking. With thoughts were numbers born. Beyond numbers thought does not reach. -Magnus Gustaf MITTAG-LEFFLER

> **Why bother?** In the preceding lecture you learned how different cultures represented the same idea of *number*. But try multiplying , say, ***MCMLXI*** times ***LXXVII*** in Roman numerals. Many ancients did! You will discover that it is painful and that our modern decimal notation is more suitable to carry out such computations.

Our system of writing numbers—the base **10** or decimal system of notation—comes from India, where it was well-established by the VIII century. Around the same time, it was independently discovered in China. It was introduced to Europe in the XIII century through Arab merchants, and thus we call them *Hindu-Arabic numbers.* Before their introduction to Europe, European merchants kept their accounting records using Roman numerals. Performing arithmetic in Roman numerals was a daunting task, and this kept civilisation from advancing.

We start with two symbols, **0** and **1**, and an operation +, adjoining the elements

$$\mathbf{1+1}, \quad \mathbf{1+1+1}, \quad \mathbf{1+1+1+1}, \quad \mathbf{1+1+1+1+1}, \quad \mathbf{\ldots .}$$

Observe that this set is *infinite* and *ordered*, that is, you can compare any two elements and tell whether one is larger than the other. We define the symbols

$$\mathbf{2=1+1}, \quad \mathbf{3=1+1+1}, \quad \mathbf{4=1+1+1+1}, \quad \mathbf{5=1+1+1+1+1}, \quad \mathbf{6=1+1+1+1+1+1},$$

$$\mathbf{7=1+1+1+1+1+1+1}, \quad \mathbf{8=1+1+1+1+1+1+1+1}, \quad \mathbf{9=1+1+1+1+1+1+1+1+1}.$$

Beyond **9** we reuse these symbols by also attaching a meaning to their place. Thus

$$\begin{aligned}
\mathbf{10} &= \mathbf{1+1+1+1+1+1+1+1+1+1}, \\
\mathbf{11} &= \boxed{\mathbf{1+1+1+1+1+1+1+1+1+1}}\mathbf{+1}, \\
\mathbf{12} &= \boxed{\mathbf{1+1+1+1+1+1+1+1+1+1}}\mathbf{+1+1}, \\
&\vdots \\
\mathbf{21} &= \boxed{\mathbf{1+1+1+1+1+1+1+1+1+1}}\mathbf{+}\boxed{\mathbf{1+1+1+1+1+1+1+1+1+1}}\mathbf{+1}, \\
&\vdots
\end{aligned}$$

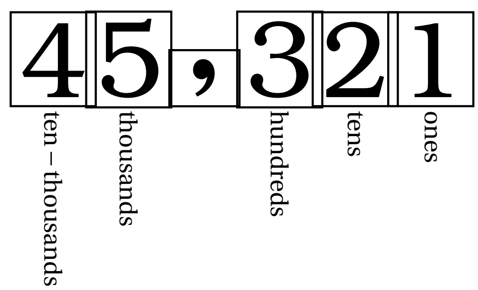
20 Definition A *positional notation* or *place-value notation system* is a numeral system in which each position is related to the next by a constant multiplier of that numeral system. Each position is represented by a limited set of symbols. The resultant value of each position is the value of its symbol or symbols multiplied by a power of the base.

To form any number, we use the ten digits:[1] **0,1,2,3,4,5,6,7,8,9**. We can represent numbers as large as we want by using their position to represent different values (place-value). The first digit right to

[1] From the Latin *digitum*, meaning *finger*.

left corresponds to the ones place, the next digit (again, right to left) corresponds to the tens place, the next digit to the hundreds place, and so on. We use addition to explain the place-value. Positional notation was such a great idea, and so greatly advanced mathematical thought, that the great XIX century German mathematician K. F. Gauss (1777-1855) wondered why the greatest mathematician of antiquity, Archimedes of Syracuse (ca 287 BC- ca 212 BC), had not come up with it. In fact, between the XII and the XVI centuries in Europe, there was a great divide between the *algorists*, the ones who advocated the new system of positional notation brought to them by the Arabs from India, doing computation by writing the operations, and the *abacists*, who used the abacus to perform the arithmetic operations.

21 Example Consider the number $45,321$. Observe that $45,321 = 40,000+5000+300+20+1.$ The place-values are explained below.



We now introduce some notation that will simplify our computations with positional notation.

22 Definition (Exponentiation) If n is a whole number greater than or equal to 1 then the *n-th power of a* is defined by

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$$

Here, a is the *base*, and n is the *exponent*.. If a is any number different from 0 then we define

$$a^0 = 1.$$

We do not attach any meaning to 0^0.[2]

Observe that it is very easy to compute powers of 10. In fact

$$10^n = 1\underbrace{0\ldots0}_{n \text{ zeroes}},$$

that is, 10^n is a 1 trailed by n zeroes. Thus

$$10^0 = 1, \quad 10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \ldots,$$

A *googol* is 10^{100}, a term that was invented by the then nine-year-old Milton Sirotta in 1920. The name of the search engine *Google* is a misspelling of googol.

Incidentally, here are the names of powers of ten and million and over:

[2] Much to the chagrin of logicians and other spawns of Satan.

10^6	million	10^9	billion	10^{12}	trillion
10^{15}	quadrillion	10^{18}	quintillion	10^{21}	sextillion
10^{24}	septillion	10^{27}	octillion	10^{30}	nonillion
10^{33}	decillion	10^{36}	undecillion	10^{39}	duodecillion
10^{42}	tredecillion	10^{45}	quattuordecillion	10^{48}	quindecillion
10^{51}	sexdecillion	10^{54}	septendecillion	10^{57}	octodecillion
10^{60}	novendecillion	10^{63}	vigintillion	10^{100}	googol

Table 4.1: Names of some powers of ten.

23 Example Powers of 2 permeate computer culture. A *bit* is a binary digit taking a value of either 0 (electricity does not pass through a circuit) or 1 (electricity passes through a circuit). We have,

$$2^1 = 2 \qquad 2^6 = 64$$
$$2^2 = 4, \qquad 2^7 = 128$$
$$2^3 = 8, \qquad 2^8 = 256$$
$$2^4 = 16, \qquad 2^9 = 512$$
$$2^5 = 32, \qquad 2^{10} = 1024$$

Since $2^{10} \approx 1000$,[3] we call 2^{10} a *kilobit*.[4]

Notice that $2^3 = 8$ and $3^2 = 9$ are consecutive powers. A 150 year old problem, called Catalan's Conjecture *asserted that these were the only strictly positive consecutive powers. This conjecture was proved by the number theorist Preda Mihailescu on 18 April 2002. This is one more example that not "everything" has been discovered in Mathematics, that research still goes on today. Again, notice that a^b* **is not** *ab. Thus $2^3 = (2)(2)(2) = 8$, and* **not** *$(2)(3) = 6$.*

Returning to the topic of this lecture, we see that we can develop any integer in decimal notation as the sum of powers of 10. For example,

$$45321 = 4 \cdot 10^4 + 5 \cdot 10^3 + 3 \cdot 10^2 + 2 \cdot 10 + 1.$$

We notice than in our system of notation, we have ten symbols and every number can be expressed uniquely as sum of a multiple of one of these symbols and a power of ten.

Suppose now that we only had two symbols: 0 and 1. Counting in this system—called the *binary* system—takes the following form:

$1 = 1_2$	$2 = 10_2$	$3 = 11_2$	$4 = 100_2$	$5 = 101_2$	$6 = 110_2$	$7 = 111_2$	$8 = 1000_2$	$9 = 1001_2$	$10 = 1010_2$

If instead we use the the five symbols: 0, 1, 2, 3 and 4, counting now becomes.

$1 = 1_5$	$2 = 2_5$	$3 = 3_5$	$4 = 4_5$	$5 = 10_5$	$6 = 11_5$	$7 = 12_5$	$8 = 13_5$	$9 = 14_5$	$10 = 20_5$

[3]The symbol $\approx$ is read "approximately."
[4]From the Greek *kilo*, meaning *thousand*.

Now that we know how to count in these systems, let us examine how to convert a number represented in them to a decimal number.

24 Example Which (decimal) number is represented by the binary number 11110100001001000000_2?

Solution: ► *The above counting method would not be very efficient here. Notice that* 11110100001001000000_2 *has twenty digits. We have*

$$\begin{aligned} 11110100001001000000_2 &= 1\cdot 2^{19}+1\cdot 2^{18}+1\cdot 2^{17}+1\cdot 2^{16}+1\cdot 2^{14}+1\cdot 2^{9}+1\cdot 2^{6} \\ &= 524288+262144+131072+65536+16384+512+64 \\ &= 1000000. \end{aligned}$$

◄

25 Example Convert the octal (base 8) number 777_8 to a decimal.

Solution: ► *We have*

$$777_8 = 7\cdot 8^2+7\cdot 8+7 = 511.$$

◄

26 Example In the hexadecimal (base 16) system, one uses the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 and the letters A ($= 10$ decimal), B ($= 11$ decimal), C ($= 12$ decimal), D ($= 13$ decimal), E ($= 14$ decimal), and F ($= 15$ decimal). The hexadecimal system is commonly used in the computer industry to store the "names" of different shades of colour. For example, it is used in HTML (the language in which most internet web pages are written), where colour names are six-hexadecimal digits long. Translate the hexadecimal numbers $AB21$ and $21AB$ into decimal numbers.

Solution: ►

$$AB21 = 10\cdot 16^3+11\cdot 16^2+2\cdot 16+1 = 43809,$$

and

$$21AB = 2\cdot 16^3+1\cdot 16^2+10\cdot 16+11 = 8619.$$

◄

We now tackle the problem of converting a decimal into a number of a different base. Consider the problem of converting 401, say, to base 3. What is the largest power of 3 that is smaller than 401? We quickly see that

$$3^5 = 243 < 401 < 729 = 3^6.$$

By division,

$$401 = 1\cdot 243+158 = 1\cdot 3^5+158.$$

Now we do the same for the remainder 158, finding

$$3^4 = 81 < 158 < 243 = 3^5.$$

As $150 = 1\cdot 81+77 = 1\cdot 3^4+77$, we deduce that

$$401 = 1\cdot 243+158 = 1\cdot 3^5+158 = 1\cdot 3^5+1\cdot 3^4+77.$$

Now we do the same for the remainder 77, finding

$$3^3 = 27 < 77 < 81 = 3^4.$$

As $\mathbf{77 = 2 \cdot 27 + 23 = 2 \cdot 3^3 + 23}$, we deduce that

$$\mathbf{401 = 1 \cdot 243 + 158 = 1 \cdot 3^5 + 158 = 1 \cdot 3^5 + 1 \cdot 3^4 + 77 = 1 \cdot 3^5 + 1 \cdot 3^4 + 2 \cdot 3^3 + 23.}$$

Now we do the same for the remainder $\mathbf{23}$, finding

$$\mathbf{3^2 = 9 < 23 < 27 = 3^3.}$$

As $\mathbf{23 = 2 \cdot 9 + 5 = 2 \cdot 3^2 + 5}$, we deduce that

$$\mathbf{401 = 1 \cdot 243 + 158 = 1 \cdot 3^5 + 158 = 1 \cdot 3^5 + 1 \cdot 3^4 + 77 = 1 \cdot 3^5 + 1 \cdot 3^4 + 2 \cdot 3^3 + 23 = 1 \cdot 3^5 + 1 \cdot 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 5.}$$

Finally, we do the same for the remainder $\mathbf{5}$, finding

$$\mathbf{3^1 = 3 < 5 < 9 = 3^2.}$$

As $\mathbf{5 = 1 \cdot 3 + 2}$, we deduce that

$$\begin{aligned} \mathbf{401} &= \mathbf{1 \cdot 243 + 158 = 1 \cdot 3^5 + 158} \\ &= \mathbf{1 \cdot 3^5 + 1 \cdot 3^4 + 77} \\ &= \mathbf{1 \cdot 3^5 + 1 \cdot 3^4 + 2 \cdot 3^3 + 23} \\ &= \mathbf{1 \cdot 3^5 + 1 \cdot 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 5} \\ &= \mathbf{1 \cdot 3^5 + 1 \cdot 3^4 + 2 \cdot 3^3 + 2 \cdot 3^2 + 1 \cdot 3 + 2} \\ &= \mathbf{112212_3.} \end{aligned}$$

The above procedure may seem laborious, but we can actually display it in a very convenient form that renders it trivial. The key is to observe that finding the powers of **3** that fit into **401** is equivalent to successively dividing **401** by **3**. Thus we display

3	**401**	**2**
3	**133**	**1**
3	**44**	**2**
3	**14**	**2**
3	**4**	**1**
3	**1**	**1**

Explanation: Divide **401** by **3**: we get a quotient of **133** and remainder **1**. Divide **133** by **3**: we get a quotient of **44** and a remainder of **2**. Keep dividing until the quotient is smaller than **3**, then stop. Read the successive remainders from the last to the first one obtained.

27 Example Convert **1492** to base **7**.

Solution: ► *We divide successively by* **7**, *noting each successive quotient in the central column, and each successive remainder on the rightmost column, obtaining*

7	1492	1
7	213	3
7	30	2
7	4	4

We conclude that $1492 = 4231_7$. *To check our result, we perform*

$$4231_7 = 4 \cdot 7^3 + 2 \cdot 7^2 + 3 \cdot 7 + 1 = 1372 + 98 + 21 + 1 = 1492.$$

◄

Even though the decimal system dominates our arithmetic, this was not always the case. The ancient Babylonians used a sexagesimal (base **60**) system, the vestiges of which we can see in the fact that a minute has **60** seconds and an hour **60** minutes. To represent numbers is such a system one needs **60** different symbols.

Homework

Problem 4.1 *Give the value of the underlined digit.*

1. $1\,23\underline{4}$
2. $1\,2\underline{3}4$
3. $1\,\underline{2}34$
4. $1\,2\underline{3}4\,056\,789$

Problem 4.2 *Light travels at approximately* **six hundred sixty nine million six hundred thousand** *miles per hour. Give a decimal numerical representation for this quantity expressed in English.*

Problem 4.3 *Are* **10** *and* **010** *the same decimal number? Explain.*

Problem 4.4 *Write in decimal numerals: "twelve million twelve thousand twelve hundred and twelve."*

Problem 4.5 *The names of the whole numbers from one to twelve are written down in* lexicographical *order, that is, in the order in which they appear in the dictionary. What is the fourth number on the list?*

Problem 4.6 *Convert* 11111_2 *to decimal.*

Problem 4.7 *Convert* 11111_3 *to decimal.*

Problem 4.8 *Convert* **11111** *to binary.*

Problem 4.9 *Convert* 123_4 *to base* **5**.

Problem 4.10 *What integer follows* 1266_7 *in base* **7**?

Problem 4.11 *Convert* 11111_3 *to decimal.*

Problem 4.12 *In the Microsoft software* Excel, *columns are labelled by letters. The first twenty six columns are labelled A through Z, the twenty seventh is labelled AA, the twenty eighth AB, etc. What label appears on the 2007th column?*

Problem 4.13 *How many (decimal) whole numbers are there between* **1** *and* **1000** *whose digits add up to* **12**?

Problem 4.14 *A* palindrome *is a natural number whose decimal expansion is symmetric, that is, it reads the same backwards as forwards. For example,* **1**, **11**, **345543**, *are all palindromes. If the sequence of palindromes is written in ascending order*

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 22, \ldots, 99, 101, 111, 121, \ldots,$$

which palindrome follows **1991**?

Problem 4.15 *A natural number is called a* palindrome *if it is read forwards as backwards, e.g.,* **1221**, **100010001**, *etc., are palindromes. The palindrome* **10001** *is strictly between two other palindromes. Which two?*

Problem 4.16 *Are there advantages and disadvantages of using the decimal system over the binary system? Discuss.*

Problem 4.17 *Without converting either number to decimal, decide which number is larger:* $\mathbf{1111_3}$ *or* $\mathbf{1111_4}$.

Problem 4.18 *I am thinking of a whole number in the range from* **0** *to* **15**, *inclusive. You are allowed to ask me* **4** *yes or no questions, to which I will respond truthfully. Can you always guess my number in at most four such questions?*

References

[HaKn] is the book I learned Algebra from. Written in the XIX century, I still think is unsurpassed in clarity of exposition, careful selection of examples, and breadth of material. It covers the material of this section more deeply than what was presented here. Binary (base **2**), octal (base **8**) and hexadecimal (base **16**) representations are essential in computer science, and [Lip] is a good reference for this.

[HaKn] H.S. HALL, and S.R. KNIGHT, "Elementary Algebra for Schools ".

[Lip] Seymour LIPSCHUTZ, "Schaum's Outline of Essential Computer Mathematics".

5 Symbolical Expression

Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different. *-GOETHE*

Why bother? Part of the goal of this course is to prepare you for *Algebra*. In *Arithmetic* we deal with specific quantities, say, we compute $2+3$. The idea of Algebra is to generalise from Arithmetic, and in order to accomplish that level of abstraction, is necessary to have an efficient way of notation, so that we don't have to refer to specific instances.

In our arithmetical excursions it will be necessary to introduce the abstraction of symbols. We will use *letters* to denote arbitrary numbers. This will free us from a long enumeration of cases. For example, suppose we notice that

$$1+0=1, \quad 2+0=2, \quad \frac{1}{2}+0=\frac{1}{2}, \ldots,$$

etc. Since numbers are infinite, we could not possibly list all cases. Here abstraction provides some economy of thought: we could say that if x is a number, then

$$x+0=x,$$

with no necessity of knowing what the arbitrary number x is. Moreover, the introduction of this symbolism will serve as prelude to Algebra.

We will normally associate the words *increase, increment, augment*, etc., with addition. Thus if x is an unknown number, the expression "a number increased by seven" is translated into symbols as $x+7$. We will later see that we could have written the equivalent expression $7+x$.

We will normally associate the words *decrease, decrement, reduce, excess, diminish, difference* etc., with subtraction. Thus if x is an unknown number, the expression "a certain number decreased by seven" is translated into symbols as $x-7$. We will later see that this differs from $7-x$, which is "seven decreased by a certain number."

We will normally associate the word *product* with multiplication. Thus if x is an unknown number, the expression "the product of a certain number and seven" is translated into symbols as $7x$. Notice here that we use *juxtaposition* to denote the multiplication of a letter and a number, that is, we do not use the × (times) symbol, or the · (central dot) symbol. This will generally be the case, and hence the following are all equivalent,

$$7x, \quad 7\times x, \quad 7(x), \quad (7)(x), \quad 7\cdot x.$$

Notice again that a reason for *not* using × when we use letters is so that we do not confuse this symbol with the letter x. We could have also have written $x7$, but this usage is just plain weird.

☞ *We do need symbols in order to represent the product of two numbers. Thus we write the product* $5\cdot 6 = 5\cdot 6 = (5)(6) = 30$ *so that we do not confuse this with the number* 56.

A few other words are used for multiplication by a specific factor. If the unknown quantity is a, then *twice* the unknown quantity is represented by $2a$. *Thrice* the unknown quantity is represented by $3a$. To *treble* a quantity is to triple it, hence "treble a" is $3a$. The *square* of a quantity is that quantity multiplied by itself, so for example, the square of a is aa, which is represented in short by a^2. Here a is the *base* and 2 is the *exponent*. The *cube* of a quantity is that quantity multiplied by its square, so for example, the cube of a is aaa, which is represented in short by a^3. Here a is the *base* and 3 is the *exponent*.

The word *quotient* will generally be used to denote division. For example, the quotient of a number and 7 is denoted by $x \div 7$, or equivalently by $\frac{x}{7}$ or $x/7$.

Here are some more examples.

28 Example If a number x is trebled and if to this new number we add five, we obtain $3x+5$.

29 Example If x is the larger between x and y, the difference between x and y is $x-y$. However, if y is the larger between x and y, the difference between x and y is $y-x$.

30 Example If a and b are two numbers, then their product is ab, which we will later see that it is the same as ba.

31 Example The sum of the squares of x and y is x^2+y^2. However, the square of the sum of x and y is $(x+y)^2$.

32 Example If n is an integer, its predecessor is $n-1$ and its successor is $n+1$.

33 Example You begin the day with E eggs. During the course of the day, you fry O omelettes, each requiring A eggs. How many eggs are left?

Solution: ► *$E-OA$, since OA eggs are used in frying O omelettes.* ◄

34 Example An even natural number has the form $2a$, where a is a natural number. An odd natural number has the form $2a+1$, where a is a natural number.

35 Example A natural number divisible by 3 has the form $3a$, where a is a natural number. A natural number leaving remainder 1 upon division by 3 has the form $3a+1$, where a is a natural number. A natural number leaving remainder 2 upon division by 3 has the form $3a+2$, where a is a natural number.

Homework

Problem 5.1 *If a person is currently N years old, what was his age 20 years ago?*

Problem 5.2 *If a person is currently N years old, what will his age be in 20 years?*

Problem 5.3 *You start with x dollars. Then you treble this amount and finally you increase what you now have by 10 dollars. How many dollars do you now have?*

Problem 5.4 *You start with x dollars. Then you add \$10 to this amount and finally you treble what you now have. How many dollars do you now have?*

Problem 5.5 *A knitted scarf uses three balls of wool. I start the day with b balls of wool and knit s scarves. How many balls of wool do I have at the end of the day?*

Problem 5.6 *What is the general form for a natural number divisible by 4? Leaving remainder 1 upon division by 4? Leaving remainder 2 upon division by 4? Leaving remainder 3 upon division by 4?*

Problem 5.7 *If you have a quarters and b dimes only, how many coins do you have?*

Problem 5.8 *If you have a quarters and b dimes only, how much money, in cents, do you have?*

Problem 5.9 *If d is a whole number, what is the next larger whole number?*

References

Again, [HaKn] has much of this material, in very old-fashioned language. So does [Boy], a book that is accessible online for free, via Project Gutenberg.

[HaKn] H.S. HALL, and S.R. KNIGHT, “Elementary Algebra for Schools ”.

[Boy] Wallace C. BOYDEN, “A First Course in Algebra”.

Part III

The Natural Numbers

6 Addition and Subtraction

Why bother? Addition and subtraction of natural numbers are the most fundamental operations in Arithmetic. We will introduce some fancy vocabulary terms in this section, in order to facilitate our road to abstraction.

36 Definition The set of *natural numbers* $\mathbb{N}$, is the set

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$$

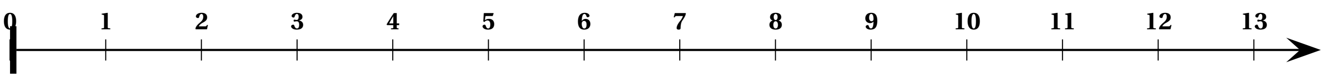


Figure 6.1: The Natural Numbers $\mathbb{N}$.

We will use the symbol $\in$, read *is in*, or *is an element of*, to indicate that a certain element belongs to a certain set. The negation of $\in$ is $\not\in$. For example, $1 \in \mathbb{N}$ because 1 is a natural number, but $\frac{1}{2} \not\in \mathbb{N}$.

Natural numbers are used for two main reasons:

1. counting, as for example, "there are 10 sheep in the herd",
2. or ordering, as for example, "Los Angeles is the second largest city in the USA."

We can interpret the natural numbers as a linearly ordered set of points, as in figure 6.1. This interpretation of the natural numbers induces an order relation as defined below.

37 Definition Let a and b be two natural numbers. We say that a *is (strictly) less than* b, if a is to the left of b on the natural number line. We denote this by $a < b$. Some other order symbols follow:

$>$ *is strictly greater than*,

$\leq$ *is less than or equal to* and,

$\geq$ *is greater than or equal to.*

We also mention the symbols $=$, *equal to*, and $\neq$, *not equal to.*

38 Example The farther to the right a number is on the number line, the larger it is. Thus we have

$$1914 < 1939, \quad 666 > 100, \quad 1 \leq 1.$$

We now introduce the operation of $+$ or *addition* in $\mathbb{N}$. We can think of $+$ as *joining*, *concatenating*, *appending*, that is, *and*. In fact, the $+$ used today evolved from the fact that in Latin manuscripts people used to write *et*—meaning *and*—for "plus." This was shortened to just "*t*", and then this was slightly altered to give the modern sign of plus. Let us introduce some fancy terminology.

39 Definition In the expression $a + b = s$, a is called the *augend*, b is called the *addend* and s is called the *sum*. Both the augend and the addend are called *summands.*

Thus in $1+4=5$, 1 and 4 are the summands and 5 is their sum. Now, joining one marble to four marbles gives five marbles, that is, 1 marble $+4$ marbles $=5$ marbles. Concatenating a line segment of length b inches to one of length a inches gives a line segment of length $a+b$ inches. See figures 6.2 and 6.3

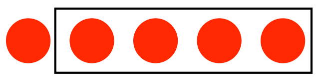


Figure 6.2: Joining.

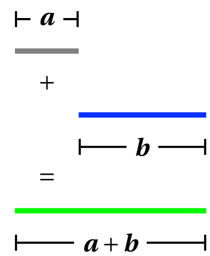


Figure 6.3: Concatenation.

Observe that in the examples above we added like quantities. Thus marbles were added to marbles and inches to inches. We did not dare to add marbles to inches! We call these similar entities the *denominator* of the term. Hence in the sum 1 marble $+4$ marbles $=5$ marbles, marbles are the denominator. In the sum 1 inch $+4$ inches $=5$ inches, inches are the denominator. It should not be a stretch of one's imagination to progressively reach the following levels of abstraction:

- $1m+4m=5m$
- $1A+4A=5A$
- $\frac{1}{401}+\frac{4}{401}=\frac{5}{401}$
- $1+4=5$

In the first sum, m is the common denominator, we are adding two like objects. Similarly, A is the common denominator for the second sum, and $\frac{\cdot}{401}$ is the one for the third sum. The fourth sum is more abstract, in the sense that even though we do not have a common denominator, we are asserting that if to a unit of whatever we add four units of the same whatever, the result is five of the whatever. It is important to realise that this last abstraction came from our agreement that can only add terms with a common denominator. Here are more challenging examples.

40 Example Ariel walked x miles and rode 10 miles. How far did he go?

> **Solution:** ► *The total of miles he went was* $x+10$. ◄

41 Example Collect like terms:

$$a+2b+3a+4b.$$

> **Solution:** ► *We combine the* a*'s with the* a*'s and the* b*'s with the* b*'s:*
>
> $$a+2b+3a+4b=4a+6b.$$
>
> ◄

☞ *In the preceding example there is the temptation to further combine the* $4a$ *with the* $6b$*. But this temptation has no basis: we do not know what the* a*'s are and what the* b*'s are. They are different denominators and we cannot combine them without further information. So, if the* a*'s were* apples *and the* b*'s* bananas, *we cannot combine them because they are different fruit! But you may object and say: "Well,* 4 *apples and* 6 *bananas do give* 10 *fruit! There! I combined them! But you have cheated! You have created a common denominator,* fruit, *for apples and bananas, and that was not asked in the problem. You cannot deviate from what is given to you!*

Again, consider the following way of adding 731 and 695. Since

$$731 = 7 \cdot 10^2 + 3 \cdot 10 + 1, \qquad 6 \cdot 10^2 + 9 \cdot 10 + 5,$$

we could add in the following fashion, without worrying about carrying:

$$(7 \cdot 10^2 + 3 \cdot 10 + 1) + (6 \cdot 10^2 + 9 \cdot 10 + 5) = 13 \cdot 10^2 + 12 \cdot 10 + 6 = 1300 + 120 + 6 = 1426.$$

The above reasoning can be extended to other than powers of ten.

42 Example Collect like terms:

$$(7x^2 + 3x + 1) + (6x^2 + 8x + 5).$$

Solution: ► *We match the squares with the squares, the x to the first power with the x to the first power, and the constant term with the constant term to get*

$$(7x^2 + 3x + 1) + (6x^2 + 8x + 5) = 13x^2 + 11x + 6.$$

◄

As we begin to see more examples of addition of natural numbers, the following properties emerge.

43 Axiom (Closure) The sum of any two natural numbers is a natural number, that is, $\mathbb{N}$ is *closed under addition*: if $a \in \mathbb{N}$ and $b \in \mathbb{N}$ then also $a + b \in \mathbb{N}$.

☞ *Thus if we add two natural numbers, we don't get eggs, we don't get pastrami, we get a natural number.*

44 Axiom (Additive Identity) If 0 is added to any natural number, the result is unchanged, that is $0 \in \mathbb{N}$ is the *additive identity* of $\mathbb{N}$. It has the property that for all $x \in \mathbb{N}$ it follows that

$$x = 0 + x = x + 0.$$

Again, it is easy to see that when two natural numbers are added, the result does not depend on the order. This is encoded in the following axiom.

45 Axiom (Commutativity) Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ be arbitrary. Then $a + b = b + a$.

The last axiom has to do with grouping.

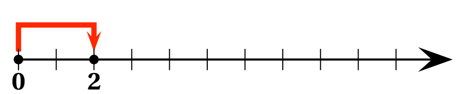
46 Axiom (Associativity) Let a, b, c be arbitrary natural numbers. Then the order of parentheses when performing addition is irrelevant, that is,

$$a + (b + c) = (a + b) + c = a + b + c.$$

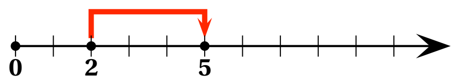
How do we actually *add* in decimal notation? Let us first present a geometric method to add integers along the number line.

47 Example Use the number line to perform the addition $2 + 3$.

Solution: ► *First locate 2 by starting from 0 and moving two units right.*



Now, starting at **2** *move three units right.*



Since we landed at **5**, *we conclude that* $2+3=5$. ◄

The above method of addition can, in principle, accomplish all the sums of natural numbers that we think of. It suffers from the limitation of us having to locate numbers on the number line and counting at least twice. If the numbers are really large, then we might miscount. Countless cultures that did not know our decimal positional notation were able to carry out these computations. But the cumbersomeness of these calculations meant that the operations could be carried out by expert accountants. Today, one hopes, even a child can perform these operations.

In order to explain an algorithm for addition, let us first think of how we can give the decimal representation of an arbitrary number. Let us agree that the notation

$$\underline{d_n d_{n-1} \ldots d_2 d_1 d_0},$$

where the d's are digits, means

$$\underline{d_n d_{n-1} \ldots d_1 d_0} = d_n \cdot 10^n + d_{n-1} \cdot 10^{n-1} + \cdots + d_2 \cdot 10^2 + d_2 \cdot 10 + d_0.$$

For example, if the numeral is **60637**, then this has **5** digits, and we have,[1]

$$\underline{d_4 d_3 d_2 d_1 d_0} = 60637 \implies d_4 = 6, \quad d_3 = 0, \quad d_2 = 6, \quad d_1 = 3, \quad d_0 = 7.$$

If the numeral is **1234567**, then this has **7** digits, and we have

$$\underline{d_6 d_5 d_4 d_3 d_2 d_1 d_0} = 1234567 \implies d_6 = 1, \quad d_5 = 2, \quad d_4 = 3, \quad d_3 = 4, \quad d_2 = 5, \quad d_1 = 6, \quad d_0 = 7.$$

Suppose now that we have two arbitrary decimal integers, say

$$\underline{d_n d_{n-1} \ldots d_2 d_1 d_0}, \qquad \underline{e_p e_{p-1} \ldots e_2 e_1 e_0}.$$

Here it may be the case that $n = p$, that is, the integers have the same number of digits, but we do not know that in advance. Let us suppose for the sake of argument that $n < p$. To add the integers, line them up at the units:

$$\begin{array}{ccccccccc} & & & & d_n & d_{n-1} & \ldots & d_2 & d_1 & d_0 \\ + & e_p & e_{p-1} & \ldots & e_n & e_{n-1} & \ldots & e_2 & e_1 & e_0 \\ \hline \end{array}$$

The algorithm is then as follows:

1. Add $d_0 + e_0$. If the sum is smaller than **10**, put the result below in this first column. If the result is **10** or more, put the units of the result below in the first column and carryover the the tens of the result to the next column.

2. Add $d_1 + e_1$ and any carryover from the units (if there is any). If the sum is smaller than **10**, put the result below in this second column. If the result is **10** or more, put the units of the result below in the second column and carryover the the tens of the result to the next column. Continue a similar process until the digits of the longest number have been exhausted.

48 Example Use the above algorithm to perform the addition $17066 + 1492$.

[1]The arrow $\implies$ is read *implies*.

Solution: ► *We first line up the numbers at their last digit:*

$$\begin{array}{r} 17066 \\ +\ \ 1492 \\ \hline \bullet\bullet\bullet\bullet\bullet \end{array}$$

Now we add the units: $6+2=8$ *and there is no carrying since the result is smaller than* 10*:*

$$\begin{array}{r} 17066 \\ +\ \ 1492 \\ \hline \bullet\bullet\bullet\bullet 8 \end{array}$$

We add the tens: $6+9=15$*. We put* 5 *below on the column of tens and we carry the* 1 *to the column of hundreds:*

$$\begin{array}{r} {}^{1}\ \ \ \\ 17066 \\ +\ \ 1492 \\ \hline \bullet\bullet\bullet 58 \end{array}$$

We add the hundreds, without forgetting the carry: $1+0+4=5$*. We put* 5 *below on the column of hundreds, and there is no carrying, since the result is smaller than* 10*:*

$$\begin{array}{r} {}^{1}\ \ \ \\ 17066 \\ +\ \ 1492 \\ \hline \bullet\bullet 558 \end{array}$$

We add the thousands: $7+1=8$*. We put* 8 *below on the column of thousands, and there is no carrying, since the result is smaller than* 10*:*

$$\begin{array}{r} {}^{1}\ \ \ \\ 17066 \\ +\ \ 1492 \\ \hline \bullet 8558 \end{array}$$

Finally, we add the ten thousands: $1+0=1$ *(remember that* 1492 *is the same as* 01492*). We put* 1 *below on the column of ten thousands, and there is no carrying, since the result is smaller than* 10*:*

$$\begin{array}{r} {}^{1}\ \ \ \\ 17066 \\ +\ \ 1492 \\ \hline 18558 \end{array}$$

◄

49 Example Add: $(x^4+7x^3+6x+6)+(x^3+4x^2+9x+2)$.

Solution: ► *We line up the terms that have the same exponents and add vertically.*

$$\begin{array}{rrrrrrrrrr} & x^4 & + & 7x^3 & & & + & 6x & + & 6 \\ + & & & x^3 & + & 4x^2 & + & 9x & + & 2 \\ \hline & x^4 & + & 8x^3 & + & 4x^2 & + & 15x & + & 8 \end{array}$$

◄

Having now an idea of what it means to add natural numbers, we define subtraction of natural numbers by means of addition. This is often the case in Mathematics: we define a new procedure in terms of old procedures.

50 Definition (Definition of Subtraction) Let m, s, d be natural numbers, with $m \geq s$. Then the statement $m - s = d$ means that $m = s + d$. In the expression $m - s = d$, m is called the *minuend*, s the *subtrahend*, and d the *difference*.

Of course, our link to reality and subtraction is to think of subtraction as *taking away*.

51 Example To compute $15 - 3$ we think of which number when added 3 gives 15. Clearly then $15 - 3 = 12$ since $15 = 12 + 3$.

Notice that subtraction of natural numbers satisfies neither closure, nor commutativity, nor associativity. For example,

$$3 - 5 \notin \mathbb{N},$$

and so subtraction is not close in $\mathbb{N}$. Since

$$5 - 3 \neq 3 - 5,$$

subtraction is not commutative. Lastly, since

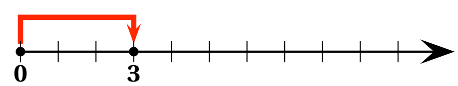
$$10 - (7 - 4) \neq (10 - 7) - 4,$$

subtraction is not associative.

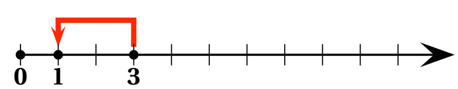
Let us use a geometric method similar to the additive one in order to perform subtraction.

52 Example Use the number line to perform the subtraction $3 - 2$.

Solution: ▸ *First locate* 3 *by starting from* 0 *and moving three units right.*



Now, starting at 3 *move two units left.*



Since we landed at 1*, we conclude that* $3 - 2 = 1$. ◂

We raise the same objections that we raised against the geometric method for addition: it becomes cumbersome for large values and it is prone to error if we miscount. Let us now discuss an algorithm for subtraction.

Suppose again that we have two arbitrary decimal integers, say

$$\underline{d_n d_{n-1} \ldots d_2 d_1 d_0}, \qquad \underline{e_p e_{p-1} \ldots e_2 e_1 e_0},$$

with $n \leq p$ and $\underline{d_n d_{n-1} \ldots d_2 d_1 d_0} \leq \underline{e_p e_{p-1} \ldots e_2 e_1 e_0}$. To subtract the integers, line them up at the units:

$$\begin{array}{rcccccccc} & e_p & e_{p-1} & \ldots & e_n & e_{n-1} & \ldots & e_2 & e_1 & e_0 \\ - & & & & d_n & d_{n-1} & \ldots & d_2 & d_1 & d_0 \\ \hline \end{array}$$

The algorithm is then as follows:

1. If it follows that $d_k \le e_k$ for all $k \le n$, then there is no regrouping and we simply put the result of $e_k - d_k$ in the kth column.
2. If the previous item is not true, consider a string $i \le j \le n$ for which $d_i > e_i$, $d_{i+1} \ge e_{i+1}$, ..., $d_j \ge e_j$ and $d_{j+1} < e_{j+1}$. Starting at the index $i+1$, successively regroup so that now in the i-th column you have $10 + e_i - d_i$, on the $i+1$th column you have $10 + e_{i+1} - 1 - d_{i+1}$, and so on, until the jth column where you will have $10 + e_j - 1 - d_j$ and on the $j+1$th column you will have $e_{j+1} - d_{j+1}$. Now all these subtractions can be accomplished. Put the result below in the respective column. Repeat again after the $j+1$th column if the situation arises again.

53 Example Use the above algorithm to perform the subtraction $9876666789 - 1234554321$.

Solution: ► *Since* $9876666789 > 1234554321$, *the subtraction is permissible. Notice that each digit of* 1234554321 *is smaller than each corresponding digit of* 9876666789, *and so there is no carrying. The result is displayed thus:*

$$\begin{array}{rcccccccccc} & 9 & 8 & 7 & 6 & 6 & 6 & 6 & 7 & 8 & 9 \\ - & 1 & 2 & 3 & 4 & 5 & 5 & 4 & 3 & 2 & 1 \\ \hline & 8 & 6 & 4 & 2 & 1 & 1 & 2 & 4 & 6 & 8 \end{array}$$

◄

54 Example Use the above algorithm to perform the subtraction $98723112389 - 72430014467$.

Solution: ► *Since* $98723112389 > 72430014467$, *the subtraction is permissible. The subtraction on the units and the tens is effected without complications, since both digits in the subtrahend are smaller than the corresponding digits in the minuend:*

$$\begin{array}{rccccccccccc} & 9 & 8 & 7 & 2 & 3 & 1 & 1 & 2 & 3 & 8 & 9 \\ - & 7 & 2 & 4 & 3 & 0 & 0 & 1 & 4 & 4 & 6 & 7 \\ \hline & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & 2 & 2 \end{array}$$

When we turn to the hundreds, we find that $3 < 4$, *and so, we must regroup from the digits to the left. In the thousands we find* $2 < 4$. *In the ten thousands we find* $1 \le 1$. *We stop at the hundred thousands, because* $1 > 0$. *Thus we take* 1 *from the hundred thousands (where we now have* 0*) to the ten thousands. In the ten thousands now we have* $10 + 1 = 11$, *but we take* 1 *to the thousands, leaving* 10 *in the ten thousands. We now have* $10 + 2 = 12$ *in the thousands, but we take* 1 *to the hundreds, leaving* 11 *in the thousands. Finally, we have* $10 + 3 = 13$ *in the hundreds. We subtract until the hundred thousands getting,*

$$\begin{array}{rccccccccccc} & & & & & & {\scriptstyle 0} & {\scriptstyle 10} & {\scriptstyle 11} & {\scriptstyle 13} & & \\ & 9 & 8 & 7 & 2 & 3 & 1 & 1 & 2 & 3 & 8 & 9 \\ - & 7 & 2 & 4 & 3 & 0 & 0 & 1 & 4 & 4 & 6 & 7 \\ \hline & \bullet & \bullet & \bullet & \bullet & \bullet & 0 & 9 & 7 & 9 & 2 & 2 \end{array}$$

We now turn to the millions. Since $3 < 0$, *subtraction goes without any complications:*

$$\begin{array}{rccccccccccc} & & & & & & {\scriptstyle 0} & {\scriptstyle 10} & {\scriptstyle 11} & {\scriptstyle 13} & & \\ & 9 & 8 & 7 & 2 & 3 & 1 & 1 & 2 & 3 & 8 & 9 \\ - & 7 & 2 & 4 & 3 & 0 & 0 & 1 & 4 & 4 & 6 & 7 \\ \hline & \bullet & \bullet & \bullet & \bullet & 3 & 0 & 9 & 7 & 9 & 2 & 2 \end{array}$$

In the ten millions we find that $2 < 3$, *so we must regroup from the digits to the left. In the hundred millions we see that* $7 > 4$ *so we stop there. We take a* 1 *from the hundred millions, and so we have a* 6 *in the hundred millions. In the ten millions we now have* $10 + 2 = 12$. *We perform the subtractions:*

$$\begin{array}{rccccccccccc} & & & {\scriptstyle 6} & {\scriptstyle 12} & & {\scriptstyle 0} & {\scriptstyle 10} & {\scriptstyle 11} & {\scriptstyle 13} & & \\ & 9 & 8 & 7 & 2 & 3 & 1 & 1 & 2 & 3 & 8 & 9 \\ - & 7 & 2 & 4 & 3 & 0 & 0 & 1 & 4 & 4 & 6 & 7 \\ \hline & \bullet & \bullet & 2 & 9 & 3 & 0 & 9 & 7 & 9 & 2 & 2 \end{array}$$

Finally, the subtraction in the billions and the ten billions occur without major complications. We display the result:

$$
\begin{array}{cccccccccccc}
 & & & {\scriptstyle 6} & {\scriptstyle 12} & & {\scriptstyle 0} & {\scriptstyle 10} & {\scriptstyle 11} & {\scriptstyle 13} & & \\
 & 9 & 8 & 7 & 2 & 3 & 1 & 1 & 2 & 3 & 8 & 9 \\
- & 7 & 2 & 4 & 3 & 0 & 0 & 1 & 4 & 4 & 6 & 7 \\
\hline
 & 2 & 6 & 2 & 9 & 3 & 0 & 9 & 7 & 9 & 2 & 2
\end{array}
$$

◄

55 Example A box of raisins was bought for a dollars, and a firkin of butter for b dollars. If both were sold for c dollars, how much was gained?

Solution: ► *The amount of money spent on raisins and butter gives an original cost of* $a+b$ *dollars. Since we sold it for* c*, we must take away our original cost, and hence we made a profit of* $c-(a+b)$ *dollars.* ◄

Homework

Problem 6.1 Without availing from a calculator, *perform the addition* $123456789+987654321$.

Problem 6.2 Without availing from a calculator, *perform the subtraction* $987654321-123456789$.

Problem 6.3 *Suppose we arrange the five numbers* $1,2,3,4,5$ *in each of the five squares so that the horizontal and vertical lines both add to* 8*. Which number must go in the middle square?*

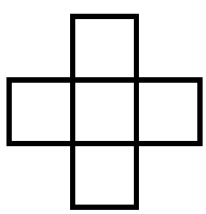
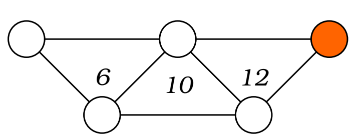


Figure 6.4: Problem 6.3.

Problem 6.4 *An oak tree is* 42 *feet high. The oak tree is* 18 *feet taller than the fir tree. How tall is the fir tree?*

Problem 6.5 *Fill in the numbers from* 1 *up to* 5 *in the five little circles, in such a way that in each triangle the given number is the sum of the numbers on the vertices. Which number appears in the shaded circle?*



Problem 6.6 *Can we find five even integers whose sum is* 25*?*

Problem 6.7 *A bottle of wine and its cork cost* $\$1$*. The bottle of wine costs* 80¢ *more than than the cork. What is the price of the cork, in cents?*

Problem 6.8 *Ingrid made* 686 *biscuits. She sold some of them. If* 298 *were left over, how many biscuits did she sell?*

Problem 6.9 *What is the value of*

$$1+2+3+\cdots+7+8+9+8+7+\cdots+3+2+1,$$

where all of the integers from 1 *through* 9 *and then back down to* 1 *are added together?*

Problem 6.10 *How many numbers are there in the set*

$$\{2,4,6,\ldots,2006\},$$

that is, in the set of all strictly positive natural numbers between 2 *and* 2006*? What is*

$$(2+4+\cdots+2006)-(1+3+\cdots+2005),$$

that is, what is the difference between the sum of all the odd positive integers up to 2005 *and the sum of all the even positive integers up to (and including)* 2006*?*

Problem 6.11 *Luigi saved* $\$184$*. He saved* $\$63$ *more than Brian. How much did Brian save?*

Problem 6.12 *Alice is* 4 *inches taller than Betty. Caroline is* 8 *inches shorter than Alice. Betty is* 69 *inches tall. How tall is Caroline, in inches?*

Problem 6.13 *A tree was planted* 54 *years before* 1961*. How old was that tree in* 2008*?*

Problem 6.14 *Lilliam had* 20 *pieces of candy. She gave two pieces to her sister.*

1. *How many did she have left?*
2. *If she gave away 2 pieces each to 4 more people, how many pieces would she have left?*

Problem 6.15 *Buses need to be rented for* **27** *children going on a field trip. Each bus can take* **12** *children in addition to the driver. How many buses must be rented?*

Problem 6.16 *Collect like terms:*

1. $(2a+8b)+(a+3b)$
2. $2a+8b-a-3b$
3. $(x^2+x+1)+(3x^2+2x+1)$

Problem 6.17 *Bobby was playing on the elevator of a very tall building. Starting from the floor where he was, he went five floors up, four down, three up, four up, and two down. If he is now in the* **30***th floor, what was the original floor from where he started?*

Problem 6.18 *How many addition signs should be put between digits of the number* **987654321** *and where should we put them to get a total of* **99***?*

Problem 6.19 *You have a seven inch gold bar, that is already segmented into seven equal pieces. You are allowed to make two cuts to it. How can you pay an employee that demands to be paid one gold piece daily for the seven days that he works for you?*

Problem 6.20 *Fill in the blank using either of the symbols '=' , '≠', '∈', '∉' as appropriate:*

1. $3\_\_\_\_\_3+0$
2. $5+1\_\_\_\_\_\mathbb{N}$
3. $1+3\_\_\_\_\_2+2$
4. $64\_\_\_\_\_604$
5. ✠$\_\_\_\_\_\mathbb{N}$
6. $5+6\_\_\_\_\_7+8$

Problem 6.21 *How many different sums can be made when two non-necessarily distinct numbers from the set* $\{1,3,4,5,7\}$ *are taken?*

Problem 6.22 *A frog is in a* **10** *ft well. At the beginning of each day, it leaps* **5** *ft up, but at the end of the day it slides* **4** *ft down. After how many days, if at all, will the frog escape the well?*

Problem 6.23 *Judith was imprisoned by a band of mathematicians and sent to Guantánamo for crimes against Mathematics. Through the mercy of the Brahmin mathematician, she was given the choice of being released after* **10** *years or be given freedom if she climbed the* **100** *steps of a* **100***-step staircase subject to the following rules:*

1. *She climbs up or down only one step per day.*
2. *She climbs up on every day of January, March, May, July, September, and November.*
3. *She goes down on every day of February, April, June, August, October, and December.*

Being adept at climbing, she chose this later option. If Judith started on January 1 2001, when will she gain her freedom?

Problem 6.24 *Using all the digits* $\{0,1,2,3,4,5,6,7,8,9\}$*, form two* **5***-digit numbers so that their difference is as large as possible.*

Problem 6.25 *Using all the digits* $\{0,1,2,3,4,5,6,7,8,9\}$*, form two* **5***-digit numbers so that their difference is as small as possible.*

Problem 6.26 *You and I play the following game. I tell you to write down three* **2***-digit integers between* **10** *and* **89***. Then I write down three* **2***-digit integers of my choice. The answer comes to* **297***, no matter which three integers you choose (my choice always depends on yours). For example, suppose you choose* **12,23,48***. Then I choose* **87,76,51***. You add*

$$12+23+48+87+76+51=297.$$

Again, suppose you chose **33,56,89***. I then choose 66, 43, 10. Observe that*

$$33+56+89+66+43+10=297.$$

Explain how I choose my numbers so that the answer always comes up to be **297** *(!!!).*

Problem 6.27 *A merchant bought* a *barrels of sugar and* p *barrels of molasses. How many barrels in all did he buy?*

Problem 6.28 *Jill bought a silk dress for* m *dollars, a muff for* l *dollars, a shawl for* v *dollars, and a pair of gloves for* c *dollars. What was the entire cost?*

Problem 6.29 *You start the day with* q *quarters and* d *dimes. How much money do you have? Answer in cents. If by the end of the day you have lost* a *quarters and* b *dimes, how much money do you now have? Answer in cents. Write a one or two line explanation for your answers.*

Problem 6.30 *Brian tore out several successive pages from a book. The first page that he tore up was page number* **143***. The last page that he tore up is also a three-digit number written with the same digits* $\{1,4,3\}$ *but in a different order. How many pages did he tear up?*

Problem 6.31 *A man bought a hat for* h *dollars. He then bought a jacket and a pair of trousers. If the jacket is thrice as expensive as the hat and the trousers are* **8** *dollars cheaper than jacket, how much money did he spend in total for the three items?*

Problem 6.32 *Peter is x years old, Paul is y, and Mary is z years. What is the sum of their ages?*

Problem 6.33 *A boy bought a pound of butter for y cents, a pound of meat for z cents, and a bunch of lettuce for s cents. How much did they all cost?*

Problem 6.34 *A man sells a carriage for m dollars and loses x dollars. What was the cost of the carriage?*

Problem 6.35 *I paid c¢ for a pound of butter, and f¢ for a lemon. How much more did the butter cost than the lemon?*

Problem 6.36 *Sold a lot of wood for b dollars, and received in payment a barrel of flour worth e dollars. How many dollars remain due?*

Problem 6.37 *Complete the following addition:*

$$\begin{array}{r} 3\bullet 7\bullet \\ +\ \bullet 283 \\ \hline 89\bullet 2 \end{array}$$

Problem 6.38 *Find the missing digits:*

$$\begin{array}{r} 7\bullet 4 \\ +\ \bullet 8\bullet \\ \hline \bullet 03 \end{array}$$

Problem 6.39 *Complete the following subtraction:*

$$\begin{array}{r} 5\bullet 71 \\ -\ 192\bullet \\ \hline \bullet 7\bullet 1 \end{array}$$

Problem 6.40 *By inspection (without any calculation) determine whether the following are true or false. Mention which property makes the true statements true.*

1. $1298+7459=7459+1298$
2. $1298+7459=7460+1298$
3. $(547+1250)+3=547+(1250+3)$
4. $(547+1250)+3=547+(3+1250)$

Problem 6.41 *Each square represents a digit. Find the value of each missing digit.*

$$\begin{array}{r} \blacksquare\ 7\ 5\ \blacksquare\ 6 \\ -\quad \blacksquare\ 5\ 6\ \blacksquare \\ \hline 2\ 4\ \blacksquare\ 7\ 5 \end{array}$$

Problem 6.42 *Is it possible to replace the letter A in the square below so that every row has the same sum of every column?*

1	2	5
3	3	2
A	3	1

Problem 6.43 *Fill each square with exactly one number from*

$$\{1,2,3,4,5,6,7,8,9\}$$

so that the square becomes a magic square, that is, a square where every row has the same sum as every column, and as every diagonal.

Is there more than one solution?

Problem 6.44 *Vintik and Shpuntik agreed to go to the fifth car of a train. However, Vintik went to the fifth car from the beginning, but Shpuntik went to the fifth car from the end. How many cars has the train if the two friends got to one and the same car?*

Problem 6.45 *The desks in a classroom are arranged in straight rows. José is in the third row from the front and the fourth row from the back. He is also third from the left end of a row and fifth from the right. How many desks are in the classroom?*

Problem 6.46 *In a contest to guess the number of balloons in a bunch, Adam guessed **25**, Bob guessed **31**, Carlos guessed **29**, Daniel guessed **23** and Edward guessed **27**. Two guesses were wrong by **2**, and two guesses were wrong by **4**. The other guess was correct. How many balloons are there?*

Problem 6.47 *In the following array, some entries are* *missing. Find the value of* x.

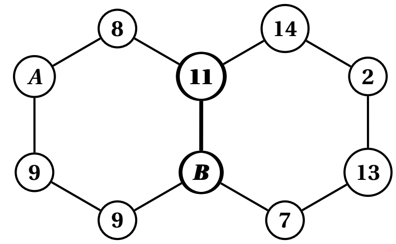
					Row Totals
	?	?	5	7	24
	5	?	?	1	21
	7	8	?	?	26
	9	3	2	x	?
Column Totals	29	21	22	19	

Problem 6.48 *An urn contains* **28** *blue marbles,* **20** *red marbles,* **12** *white marbles,* **10** *yellow marbles, and* **8** *magenta marbles. How many marbles must be drawn from the urn in order to assure that there will be* **15** *marbles of the same color?*

Problem 6.49 *In a* **4** *by* **4** *magic square the sum of the four entries in each row, in each column, and in each of the two main diagonals are all the same. If the magic square shewn below is completed, find* $A+B$.

		7	12
A	4	9	
10	5		3
8	11		B

Problem 6.50 *In each of the following overlapping rings the sum of the entries is* **55**. *What is* $A-B$?



Problem 6.51 *Let* $\mathbb{E} = \{0, 2, 4, 6, \ldots\}$ *be the set of even natural numbers. Is this set closed under addition? Is this set closed under multiplication.*

Problem 6.52 *Let* $\mathbb{O} = \{1, 3, 5, 7, \ldots\}$ *be the set of odd natural numbers. Is this set closed under addition? Is this set closed under multiplication.*

Problem 6.53 *Define the operation* $\oplus$ *of "sooper-dooper addition" by* $a \oplus b = a + b + 10$, *where* a *and* b *are natural numbers. For example* $2 \oplus 3 = 2 + 3 + 10 = 15$ *and* $4 \oplus 10 = 4 + 10 + 10 = 24$. *Is sooper-dooper addition a commutative operation? Is sooper-dooper addition an associative operation?*

Problem 6.54 *Consider* **2005** *hexagonal "domino" pieces with the numbers* **1, 2, 3, 4, 5, 6** *written on the edges in clockwise fashion, as in figure 6.5. A "chain" is formed so that domino rules are observed. This means that two edges from different pieces are joined only when the numbers on the edges agree. The numbers on the edges joining two different domino pieces are now deleted and the remaining numbers are now added. What is the maximum value of this sum?*

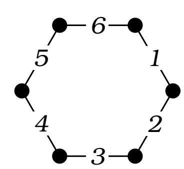


Figure 6.5: Problem 6.54.

7 Arithmetic Progressions

56 Definition An arithmetic progression is a succession of numbers where you start with a given number and form the next element by always adding the same fix quantity, that is, there are numbers a and d such that the succession of numbers takes the form:

$$a, \quad a+d, \quad a+d+d=a+2d, \qquad a+d+d+d=a+3d, \ldots,$$

Here a is called the *first term*, $a+d$ is called the *second term*, etc., and d is the *common* difference.

57 Example The arithmetic progression

$$3, 10, 17, 24, \ldots,$$

has first term 3 and common difference 7.

Observe that in example 57, the pattern of additions is

$$3 = 3+7\cdot 0, \quad 10 = 3+7\cdot 1, \quad 17 = 3+7\cdot 2, \quad 24 = 3+7\cdot 3, \ldots,$$

Thus if we wanted to find the fifth term we would compute $3+7\cdot 4 = 3+28 = 31$. If we wanted to compute the hundredth term we would find $3+7\cdot 99 = 696$.

58 Example Consider the arithmetic progression

$$5, 11, 17, 23, \ldots.$$

1. Find its next term.
2. Find its hundredth term.
3. Find a formula for the term in position n.
4. Is 100 a term in this progression? If so, which term is it?
5. Is 101 a term in this progression? If so, which term is it?

Solution: ► *The first term is* 5 *and the common difference is* 6. *The term that follows* 23 *is* $23+6=29$. *Notice that*

$$5 = 5+6\cdot 0, \quad 11 = 5+6\cdot 1, \quad 17 = 5+6\cdot 2, \quad 23 = 5+6\cdot 3, \ldots,$$

and so the hundredth term is $5+6\cdot 99 = 599$. *From this pattern we infer that the term in position* n *is* $5+6(n-1)$.

Each time we are adding a multiple of 6 *to* 5. *Hence the numbers in this progression leave remainder* 5 *upon division by* 6. *Since* $100 = 4+16\cdot 6$ *leaves remainder* 4 *upon division by* 6, *the number* 100 *is not in this progression. However,* $101 = 5+16\cdot 6$, *which means that* 101 *is the* 17*th term of this progression. We have in fact,*

$$5, 11, 17, 23, 29, 35, 41, 47, 53, 59, 65, 71, 77, 83, 89, 95, 101, \ldots.$$

so we explicitly see how to get to 101. ◄

59 Example How many terms are in the arithmetic progression

$$7, \quad 15, \quad 23, \quad \ldots, \quad 1207?$$

Solution: ► *Observe that the first term is* 7 *and that the common difference is* 8. *The terms have the law of formation*

$$7 = 7 + 8 \cdot 0, \quad 15 = 7 + 8 \cdot 1, \quad 23 = 7 + 8 \cdot 2, \quad \ldots,$$

so each time we are adding 7 *to a multiple of* 8. *Now, simply divide* 1207 *by* 8 *to find*

$$1207 = 7 + 8 \cdot 150.$$

This means that 1207 *is the* 151*st position, that is, that there are* 151 *terms in this progression.* ◄

The terms of an arithmetic progression display a symmetry that makes summing consecutive terms easy. The following classic example is reputed to have been solved in seconds by the then seven-year-old K. F. Gauss (a XIX century mathematician).

60 Example Find the sum of all the natural numbers from 1 to 100, inclusive.

Solution: ► *The trick is to consider the fifty pairs*

$$100 + 1, \quad 99 + 2, \quad 98 + 3, \quad \ldots, \quad 51 + 50.$$

Each pair adds up to 101 *and there are* 50 *of them, hence*

$$1 + 2 + 3 + \cdots + 100 = 101 \cdot 50 = 5050.$$

◄

The series of multiples of a natural number n forms an arithmetic progression with common difference n. For example, the multiples of 3 are

$$0, 3, 6, 9, 12, 15, \ldots,$$

an arithmetic progression with common difference 3, and the multiples of 4 are

$$0, 4, 8, 12, 16, \ldots,$$

an arithmetic progression with common difference 4. The *least common multiple* $\mathrm{LCM}(3,4)$ is the smallest strictly positive number that is common in both lists, in this case $\mathrm{LCM}(3,4) = 12$.

61 Example Find $\mathrm{LCM}(12,45)$.

Solution: ► *We form the progressions of the multiples of both numbers and circle the first nonzero repeat:*

$$0, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, \boxed{180}, 192, \ldots,$$

$$0, 45, 90, 135, \boxed{180}, 205, \ldots$$

whence $\mathrm{LCM}(12,45) = 180$. ◄

The following sort of arithmetic progressions will be useful when we study the division algorithm. Every natural number is either even or odd, so we write

$$\mathbb{N} = \{0, 2, 4, 6, \ldots\} \cup \{1, 3, 5, 7, \ldots\},$$

where the symbol $\cup$ (read *union*) means that both sets are joined. Notice that both sets are arithmetic progressions with common difference 2. Again, if the divisor is 3, then the possible remainders when we divide by 3 are 0, 1, and 2, and we find

$$\mathbb{N} = \{0, 3, 6, 9, \ldots\} \cup \{1, 4, 7, 10, \ldots\} \cup \{2, 5, 8, 11, \ldots\}.$$

We say that these sets *partition* the natural numbers.

62 Example Partition the natural numbers into five infinite arithmetic progressions.

Solution: ► *Upon division by* 5, *every natural number leaves remainder* 0, 1, 2, 3, *or* 4. *Hence we may take*

$$\mathbb{N} = \{0,5,10,15,\ldots\} \cup \{1,6,11,16,\ldots\} \cup \{2,7,12,17,\ldots\} \cup \{3,8,16,21,\ldots\} \cup \{4,9,13,18,\ldots\}.$$

◄

Homework

Problem 7.1 *Find the* 20*th number on the list*

$$8,11,14,\ldots$$

assuming that the pattern is preserved.

Problem 7.2 *Find the* 1000*th number on the list*

$$1,10,19,\ldots$$

assuming that the pattern is preserved.

Problem 7.3 *How many terms are there in the arithmetic progression*

$$8,11,14,\ldots 3005?$$

What is their sum?

Problem 7.4 *Find a formula for the* n*-th term of the arithmetic progression*

$$2,7,12,17,\ldots.$$

Problem 7.5 *Find a general formula for the* n*-th term of the arithmetic progression*

$$1,7,13,19,25,\ldots.$$

Problem 7.6 *You start counting by* 11*s, starting with* 3*:*

$$3,14,25,36,\ldots.$$

Which number appears in the 100*th position? Give a formula for the number appearing in the* N*th position.*

Problem 7.7 *Find* $\mathrm{LCM}(30,36)$.

Problem 7.8 *Find* $\mathrm{LCM}(24,36)$.

Problem 7.9 *Find* $\mathrm{LCM}(30,24)$.

Problem 7.10 *Partition the natural numbers into the union of four infinite arithmetic progressions.*

8 Multiplication

63 Definition Given natural numbers m and n, we define their *product* mn as

$$mn = \underbrace{n + n + \cdots + n}_{m \text{ times}}.$$

The operation between m and n is called *multiplication*. Here m is called the *multiplier*, n the *multiplicand*. Both m and n are called *factors*.

☞ *Ordinarily in Algebra, we use* juxtaposition, *that is, putting one letter next to another, to indicate multiplication. If we were multiplying two numbers, we would use Leibniz's raised dot, as in* $2 \cdot 3$. *We do not use the cross symbol* $\times$ *so as to avoid confusion with the letter* x.

A geometrical interpretation of multiplication is the following. Consider an array with identical squares of area 1 of width m squares and height n, as in figure 8.1. The product mn is the number of squares, which is also the area of the rectangle obtained once the squares are packed together and all the white space is eliminated.

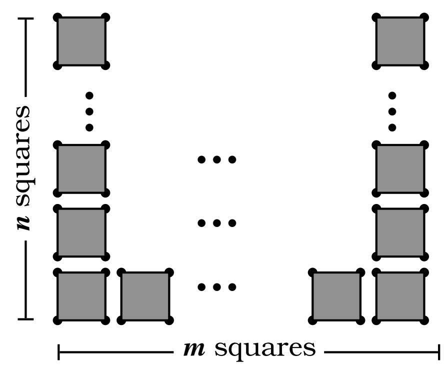


Figure 8.1: Multiplication in $\mathbb{N}$.

Just like addition of natural numbers, multiplication satisfies the following axioms.

64 Axiom (Closure) The product of any two natural numbers is a natural number, that is, $\mathbb{N}$ is *closed under multiplication*: if $a \in \mathbb{N}$ and $b \in \mathbb{N}$ then also $ab \in \mathbb{N}$.

65 Axiom (Multiplicative Identity) If 1 is multiplied to any natural number, the result is unchanged, that is $1 \in \mathbb{N}$ is the *multiplicative identity* of $\mathbb{N}$. It has the property that for all $x \in \mathbb{N}$ it follows that

$$x = 1 \cdot x = x \cdot 1.$$

66 Axiom (Commutativity) Let $a \in \mathbb{N}$ and $b \in \mathbb{N}$ be arbitrary. Then $ab = ba$.

67 Axiom (Associativity) Let a, b, c be arbitrary natural numbers. Then the order of parentheses when performing multiplication is irrelevant, that is,

$$a(bc) = (ab)c = abc.$$

Since multiplication is stenography for addition, it is to be understood that in a computation that involves sums and multiplications, multiplications must be carried out first, unless some other computation is coerced by parentheses. For example, in

$$2+3\cdot 4 = 2+12 = 14,$$

we perform the multiplication first, but in

$$(2+3)\cdot 4 = 5\cdot 4 = 20,$$

the addition is coerced by the parentheses, and hence, we perform it first.

The next property links multiplication and addition.

68 Axiom (Distributive Law) Let a, b, c be natural numbers. Then

$$a(b+c) = ab+ac,$$

and

$$(a+b)c = ac+bc.$$

For example,

$$3(4+3) = 3\cdot 7 = 21, \qquad 3\cdot 4+3\cdot 3 = 12+9 = 21.$$

In the case when the factors have each two summands, the distribute law takes the form

$$(a+b)(c+d) = a(c+d)+b(c+d) = ac+ad+bc+bd. \tag{8.1}$$

For example,

$$(3+4)(5+6) = 3\cdot 5+3\cdot 6+4\cdot 5+4\cdot 6 = 15+18+20+24 = 77.$$

☞ *Needless to say, we could perform in a faster manner* $(3+4)(5+6) = 7\cdot 11 = 77$, *but our desire was to illustrate the distributive law.*

Recall now the operation of exponentiation, which we have already defined as, for $a \in \mathbb{N}$ and $n \in \mathbb{N}$, $n > 0$,

$$a^n = \underbrace{a\cdot a\cdots a\cdot a}_{n\ a'\text{s}}.$$

We will have the convention that $a^0 = 1$ when $a \neq 0$, that is, any non-zero number raised to the zeroth power is 1 and that $a^1 = a$, that is, if the exponent is 1 we do not write the exponent.

We now analyse how to deal with products of exponential expressions with the same base. For example, observe that

$$2^2 = 4, \quad 2^3 = 8, \quad 2^5 = 32, \implies 2^2\cdot 2^3 = 4\cdot 8 = 32 = 2^5,$$

and we deduce that $2^2\cdot 2^3 = 2^{2+3} = 2^5$, that is, when multiplying two exponential expressions with the same base, we added the exponents. In general we have the following result.

69 Theorem (First Law of Exponents) Let a be a real number and m, n natural numbers. Then

$$a^m a^n = a^{m+n}.$$

Proof: *We have*

$$\begin{aligned} a^m a^n &= \underbrace{a\cdot a\cdots a}_{m\ a'\text{s}}\cdot\underbrace{a\cdot a\cdots a}_{n\ a'\text{s}} \\ &= \underbrace{a\cdot a\cdots a}_{m+n\ a'\text{s}} \\ &= a^{m+n}. \end{aligned}$$

❑

70 Example We have: $(5x)(2x) = 10x^{1+1} = 10x^2$.

71 Example We have: $(3x^2)(6x^4) = 18x^{2+4} = 18x^6$.

Using the first law of exponents and the distributive law, we will now give an algorithm for multiplication. Before explaining the multiplication algorithm in general, let us consider an example. Suppose we wanted to multiply 34 to 56. We go through the digits of 56, starting left to right. We multiplied every digit of 34 by 6, giving us

$$6 \cdot 34 = 6(30+4) = 6 \cdot 30 + 6 \cdot 4 = 180 + 24 = 204.$$

Now we move to the digit 5 of 56. But the value of this digit is 50, and so we multiply

$$50 \cdot 34 = 50(30+4) = 50 \cdot 30 + 50 \cdot 4 = 1500 + 200 = 1700.$$

We conclude that

$$34 \cdot 56 = 34(50+6) = 34 \cdot 50 + 34 \cdot 6 = 1700 + 204 = 1904.$$

It is customary to display the result in the following manner:

$$\begin{array}{r} 3\,4 \\ \times\quad 5\,6 \\ \hline 2\,0\,4 \\ 1\,7\,0 \\ \hline 1\,9\,0\,4 \end{array}$$

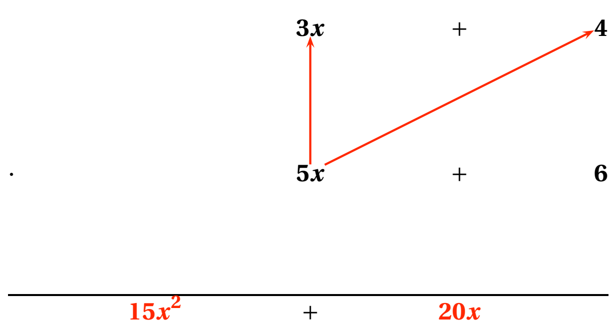
Notice that here we shifted the multiplication by 5 one unit left, and pretended that we were in fact multiplying by 5 and not 50. The two approaches have the same effect. If we wanted to see more gory detail, and perhaps have a glimpse of how this algorithm generalises to Algebra, we would probably proceed as follows. Since

$$34 = 3 \cdot 10 + 4, \quad 56 = 5 \cdot 10 + 6,$$

using the distributive law (8.1) we obtain,

$$(3 \cdot 10 + 4)(5 \cdot 10 + 6) = 3 \cdot 10 \cdot 5 \cdot 10 + 3 \cdot 10 \cdot 6 + 4 \cdot 5 \cdot 10 + 4 \cdot 6 = 1500 + 180 + 200 + 24 = 1904.$$

The above "vertical multiplication" algorithm generalises to algebraic expressions. Suppose now that we wanted to multiply $(3x+4)(5x+6)$. We first multiply every term of $3x+4$ by $5x$:



We now multiply every term of $3x+4$ by 6, lining up the terms of the same exponent of x (in our case, exponent 1 and exponent 0) with those of the first partial product:

$$\begin{array}{cccccc} & & 3x & + & 4 & \\ \cdot & & 5x & + & 6 & \\ \hline & 15x^2 & + & 20x & & \\ & & + & \color{red}{18x} & + & \color{red}{24} \end{array}$$

Finally, we add (collect) like terms:

$$\begin{array}{cccccc} & & 3x & + & 4 & \\ \cdot & & 5x & + & 6 & \\ \hline & 15x^2 & + & 20x & & \\ & & + & 18x & + & 24 \\ \hline & \color{red}{15x^2} & + & \color{red}{38x} & + & \color{red}{24} \end{array}$$

This multiplication can also be displayed horizontally, in a more obvious display of (8.1):

$$(3x+4)(5x+6) = (3x)(5x)+(3x)(6)+4(5x)+4(6) = 15x^2+18x+20x+24 = 15x^2+38x+24.$$

Observe that if we substitute $x=10$ in the above we obtain,

$$15\cdot 10^2+38\cdot 10+24 = 1500+380+24 = 1904.$$

It is now easy to describe a multiplication algorithm akin to the one used for addition. Let

$$\underline{d_n d_{n-1}\dots d_2 d_1 d_0}, \qquad \underline{e_p e_{p-1}\dots e_2 e_1 e_0},$$

be two arbitrary decimal integers. To multiply them:

1. Multiply every digit of $\underline{d_n d_{n-1}\dots d_2 d_1 d_0}$ by e_0, taking into account all the possible carries.
2. Multiply every digit of $\underline{d_n d_{n-1}\dots d_2 d_1 d_0}$ by e_1, taking into account all the possible carries, and write the result below the partial result for e_0, but shift this partial result one unit to the left.
3. Keep multiplying by the e_k, taking care of all carries, and each time shifting to the left each partial products.
4. Add all these partial products.

72 Example Multiply **123** by **456**, and display your result using the "vertical" multiplication algorithm. Then multiply x^2+2x+3 by $4x^2+5x+6$ "horizontally."

Solution: ▶ *We have,*

$$\begin{array}{r} 1\,2\,3 \\ \times \quad 4\,5\,6 \\ \hline 7\,3\,8 \\ 6\,1\,5 \\ 4\,9\,2 \\ \hline 5\,6\,0\,8\,8 \end{array}$$

and

$$\begin{aligned}(x^2+2x+3)(4x^2+5x+6) &= x^2(4x^2+5x+6)+2x(4x^2+5x+6)+3(4x^2+5x+6) \\ &= 4x^4+5x^3+6x^2+8x^3+10x^2+12x+12x^2+15x+18 \\ &= 4x^4+13x^3+28x^2+27x+18.\end{aligned}$$

Observe that if we substitute x *for* **10** *we obtain*

$$4\cdot 10^4+13\cdot 10^3+28\cdot 10^2+27\cdot 10+18 = 40000+13000+2800+270+18 = 56088.$$

◀

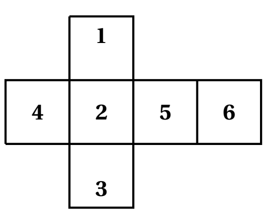
To conclude this section we will introduce the symbol **!**.

73 Definition If $n \in \mathbb{N}$ we define $n!$ (*n factorial*) as follows:

$$0! = 1, \quad 1! = 1, \quad 2! = 1\cdot 2 = 2, \quad 3! = 1\cdot 2\cdot 3 = 6, \quad 4! = 1\cdot 2\cdot 3\cdot 4 = 24, \quad n! = 1\cdot 2\cdots n.$$

Homework

Problem 8.1 *If the figure shewn is folded to form a cube, then three faces meet at every vertex. If for each vertex we take the product of the numbers on the three faces that meet there, what is the largest product we get?*



Problem 8.2 *Pencils are* **8** *¢each. How much would* **7** *pencils cost?*

Problem 8.3 *Mr. Schremmer's class was doing a science experiment. There were* **7** *groups in the class. Each group got* **4** *test tubes. How many test tubes did the class use?*

Problem 8.4 *In some years an extra second is added on* **31** *December to keep the clocks in time with the rotation of the earth. How many seconds does* **31** *December have when this happens?*

Problem 8.5 *Sound travels at approximately* **330** *meters per second. The sound of an explosion took* **28** *seconds to reach a person. Estimate how far away was the person from the explosion.*

Problem 8.6 *According to experts the first* **4** *moves in a chess game can be played in* **197299** *totally different ways. If it takes* **30** *seconds to make one move, what time, in seconds, would it take one player to try every possible set of* **4** *moves?*

Problem 8.7 *When writing all the natural numbers from* **1** *to* **999** *in a row, as follows,*

$$123456789101112\ldots998999$$

how many digits have I used?

Problem 8.8 *The integers from* **1** *to* **1000** *are written in succession. Find the sum of all the digits.*

Problem 8.9 *All the natural numbers—starting with* **1***—are listed consecutively*

$$123456789101112131415161718192021\ldots$$

Which digit occupies the **1002***nd place?*

Problem 8.10 *Compute the following:*

$$1^2,\quad 11^2,\quad 111^2,\quad 1111^2,\quad 11111^2,\quad 111111^2.$$

Do you notice a pattern?

Problem 8.11 *Multiply the numbers* 12 *and* 45. *Then multiply* $x+2$ *and* $4x+5$.

Problem 8.12 *Compute* $5!-4!$.

Problem 8.13 *Peter is* x *years old, Paul is* y, *and Mary is* z *years. What is the product of their ages?*

Problem 8.14 *A farmer has* 7 *ducks. He has* 5 *times as many chickens as ducks. How many more chickens than ducks does he have?*

Problem 8.15 *The sum of three consecutive integers is* 2007. *Find their product.*

Problem 8.16 *Multiply:* $(2x^2)(4x^3)$.

Problem 8.17 *Find the missing digits in the following product.*

$$\begin{array}{rrrr} & & 2 & \blacksquare \\ \times & & \blacksquare & 9 \\ \hline & \blacksquare & \blacksquare & \blacksquare \\ & \blacksquare & 5 & \\ \hline & 4 & \blacksquare & 5 \end{array}$$

Problem 8.18 *Find the numerical value of* $3\cdot 4+4^2$.

Problem 8.19 *Find the numerical value of* $1^1 2^2 3^3$.

Problem 8.20 *Prove that*

$$(3+2)^2 = 25.$$

Problem 8.21 *Find the last two digits of the integer*

$$1!+2!+3!+\cdots+100!.$$

Problem 8.22 *Prove that*

$$(2)(4)+3^3 = 35.$$

Problem 8.23 *Prove that*

$$3^2+2^3 = 17.$$

Problem 8.24 *Prove that*

$$(3^2)(4)(5) = 180.$$

Problem 8.25 *Prove that*

$$(5+(3+2(4))^2)^3 = 2000376.$$

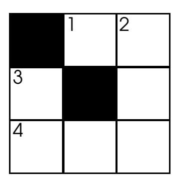
Problem 8.26 *Calculate:*

$$(123^2\cdot 456^2-789)^0+3\cdot 2^3.$$

Problem 8.27 *Dale should have divided a number by* 4, *but instead he subtracted* 4. *He got the answer* 48. *What should his answer have been?*

Problem 8.28 *When a number is multiplied by* 3 *and then increased by* 16, *the result obtained is* 37. *What is the original number?*

Problem 8.29 *A crossnumber is like a crossword except that the answers are numbers with one digit in each square. Complete the following crossnumber.*



ACROSS
1 *See 3D.*
3 *Cube.*
4 *5 times 3D.*

DOWN
2 *Square*
3 *4 times 1A.*

Problem 8.30 *Which of the digits* $\{3,4,6,8,9\}$ *cannot appear as one of the missing digits in the product*

$$\begin{array}{rrrrrrr} & & & 5 & 6 & \bullet & \bullet \\ & & \times & 1 & 9 & \bullet & \bullet \\ \hline & & 1 & 1 & 3 & 4 & 2\cdot \\ & 5 & 1 & 0 & 3 & 9 & \\ & 5 & 6 & 7 & 1 & & \\ \hline 1\,0 & 8 & 8 & \bullet & 3 & \bullet & 0 \end{array}$$

Problem 8.31 *A quiz has* 25 *questions with four points awarded for each correct answer and one point deducted for each incorrect answer, with zero for each question omitted. Anna scores* 77 *points. How many questions did she omit?*

Problem 8.32 *Doing only* one *multiplication, prove that*

$$\begin{aligned} &(666)(222)+(1)(333)+(333)(222) \\ &+(666)(333)+(1)(445)+(333)(333) \\ &+(666)(445)+(333)(445)+(1)(222) = 1000000. \end{aligned}$$

Problem 8.33 *A certain calculator gives as the result of the product*

$$987654 \cdot 745321$$

the number $7.36119E11$, *which means* $736,119,000,000$. *Explain how to find the last six missing digits.*

Problem 8.34 *How many digits does* $4^{16}5^{25}$ *have?*

Problem 8.35 *Each element of the set*

$$\{10, 11, 12, \ldots, 19, 20\}$$

is multiplied by each element of the set

$$\{21, 22, 23, \ldots, 29, 30\}.$$

If all these products are added, what is the resulting sum?

9 Primes and Factorisation

We have already explored the multiplication of natural numbers. We will now consider the reverse procedure, that is, given a number, we would like to obtain two (or more) numbers whose product is the given number.

74 Definition To *factor* a natural number greater than 1 means to find two or more natural numbers whose product is equal to the original number. The multiplication itself is called a *factorisation* of the number. The numbers in the product are called *factors* of the original number.

75 Example One factorisation of 20 is $20 = 4\cdot 5$. Here 4 and 5 are the factors. Other factorizations of 20 are $20 = 1\cdot 20$, or $20 = 2\cdot 10$, or $20 = 2\cdot 2\cdot 5$.

76 Definition A *prime number* is a natural number greater than 1 with exactly two factors: 1 and the number itself. If the natural number $n > 1$ is not prime, we say that it is *composite*. Lastly, we say that 1 is a *unit*.

Notice that the list of the primes starts as $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$.

Every natural number is either a unit, prime, or composite.

77 Definition We write $a|b$, if the natural number $a \neq 0$ divides the natural number b. We say that a is a *divisor* of b. Notice that 1 and b (if $b \neq 0$) are always divisors of b. A divisor of b is *proper*, if it is smaller than b.

A procedure, called *Eratosthenes' Sieve* allows us to find out the small primes. For example, let us find all the prime numbers smaller than 25. We first display these numbers:

2	3	4	5	6	7
8	9	10	11	12	13
14	15	16	17	18	19
20	21	22	23	24	25.

We leave 2 uncrossed, and cross out every other multiple of 2:

2	3	~~4~~	5	~~6~~	7
~~8~~	9	~~10~~	11	~~12~~	13
~~14~~	15	~~16~~	17	~~18~~	19
~~20~~	21	~~22~~	23	~~24~~	25.

The next uncrossed number after **2** is **3**. Hence, we leave **3** uncrossed and cross out every other multiple of **3** (if a multiple of **3** is already crossed, it does not matter):

2	**3**	~~**4**~~	**5**	~~**6**~~	**7**
~~**8**~~	~~**9**~~	~~**10**~~	**11**	~~**12**~~	**13**
~~**14**~~	~~**15**~~	~~**16**~~	**17**	~~**18**~~	**19**
~~**20**~~	~~**21**~~	~~**22**~~	**23**	~~**24**~~	**25.**

The next uncrossed number after **3** is **5**. Hence, we leave **3** uncrossed and cross out every other multiple of **5** (if a multiple of **5** is already crossed, it does not matter):

2	**3**	~~**4**~~	**5**	~~**6**~~	**7**
~~**8**~~	~~**9**~~	~~**10**~~	**11**	~~**12**~~	**13**
~~**14**~~	~~**15**~~	~~**16**~~	**17**	~~**18**~~	**19**
~~**20**~~	~~**21**~~	~~**22**~~	**23**	~~**24**~~	~~**25**~~**.**

We continue this procedure with the next uncrossed numbers. We see that after **13** we are already out of range. The numbers that remain uncrossed are the primes. Hence the primes smaller than **25** are It

$$\mathbf{2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots.}$$

Table 9.1 lists all primes under **1014**. Euclid proved (around 300 BC) that there are infinitely many

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
283	293	307	311	313	317	331	337	347	349
353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463
467	479	487	491	499	503	509	521	523	541
547	557	563	569	571	577	587	593	599	601
607	613	617	619	631	641	643	647	653	659
661	673	677	683	691	701	709	719	727	733
739	743	751	757	761	769	773	787	797	809
811	821	823	827	829	839	853	857	859	863
877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013

Table 9.1: Some Prime Numbers

prime numbers, hence we can find arbitrarily large prime numbers, which means that the table 9.1 can be made as long as we wish.

☞ *All prime numbers greater than* **2** *are odd. But not every odd number is prime. For example,* **9** *is not prime, since it has as three factors,* **1**, **3**, *and* **9**.

78 Definition The factorisation of a natural number greater than **1** in which every factor greater than **1** is prime, is called the *prime factorisation* of the number.

It can be proved that the factorisation of a natural number into primes is unique, up to the order of the factors. This means that if you and I start with the same number and we factor it into primes, then we will end up with the same factorisations, except perhaps in the order in which display the factors.

79 Example Find the prime factorisations of **360** and of **216**.

Solution: ► *A schematic procedure is shown in figures 9.1. The idea is to start small and then substitute equals by equals. Then choose all the prime factors in the circles. We conclude that*

$$\mathbf{360 = 2\cdot 2\cdot 2\cdot 3\cdot 3\cdot 5 = 2^3\cdot 3^2\cdot 5,}$$

and that

$$\mathbf{216 = 2^3 3^3.}$$

◄

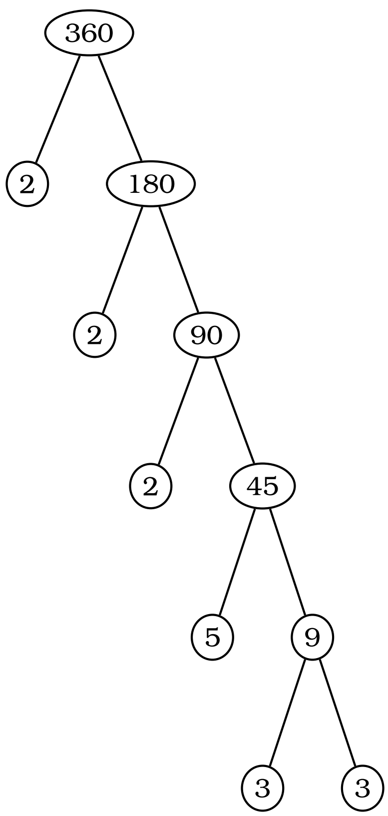


Figure 9.1: Prime factorisation of **360**.

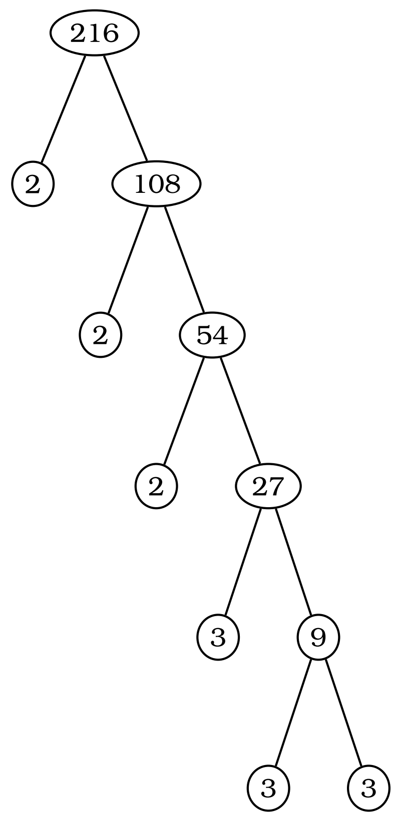


Figure 9.2: Prime factorisation of **216**.

80 Example How many divisors does **100** have?

Solution: ► *Clearly* $100 = 2^2 5^2$. *Thus every divisor will be of the form* $2^a 5^b$, *with* $a = 0, 1,$ *or* 2, *and* $b = 0, 1$ *or* 2. *Taking* $a = 0, b = 0$ *we get* $2^0 5^0 = 1$. *Taking* $a = 1, b = 0$ *we get* $2^1 5^0 = 2$. *Taking* $a = 2, b = 0$ *we get* $2^2 5^0 = 4$. *Taking* $a = 0, b = 1$ *we get* $2^0 5^1 = 5$. *Taking* $a = 1, b = 1$ *we get* $2^1 5^1 = 10$. *Taking* $a = 2, b = 1$ *we get* $2^2 5^1 = 20$. *Taking* $a = 0, b = 2$ *we get* $2^0 5^2 = 25$. *Taking* $a = 1, b = 2$ *we get* $2^1 5^2 = 50$. *Taking* $a = 2, b = 2$ *we get* $2^2 5^2 = 100$. *The divisors of* 100 *are thus*

$$1, 2, 4, 8, 10, 20, 25, 50, 100,$$

9 *in number.* ◄

☞ *In the preceding example there was no necessity of listing all the divisors in order to know how many there are. Since there are three choices for the exponent* a, *and three choices for the exponent* b, *there is a total of* $3 \cdot 3 = 9$ *divisors of a* 100.

81 Definition The *greatest common divisor* $\gcd(a, b)$ of two natural numbers a and b is the largest natural number that divides both a and b.

For example, the divisors of 12 are

$$\{1, 2, 3, 4, 6, 12\},$$

and those of 30 are

$$\{1, 2, 3, 5, 6, 15, 30\},$$

and hence $\gcd(12, 30) = 6$, because 6 is the largest number that appears in both lists.

Using prime factorisations, we may find the greatest common divisor of two numbers without having to list all their divisors. We look at which primes are common to both numbers and then select the least power appearing of the primes.

82 Example Find $\gcd(12, 30)$ by finding the prime factorisation of 12 and 30 first.

Solution: ► *First observe that* $12 = 2^2 \cdot 3$ *and that* $30 = 2^1 \cdot 3^1 \cdot 5^1$. *The primes* 2 *and* 3 *appear in both numbers. The* 2 *appears with least exponent* 1 *and so does the* 3, *and so, therefore,* $\gcd(12, 30) = 2 \cdot 3 = 6$ ◄

83 Example Find $\gcd(300, 360)$.

Solution: ► *First observe that* $300 = 2^2 \cdot 3 \cdot 5^2$ *and that* $360 = 2^3 \cdot 3^2 \cdot 5$. *The primes* 2, 3, *and* 5 *appear in both numbers. The* 2 *appears with least exponent* 2, *the* 3 *appears with least exponent* 1, *and the* 5 *appears with least exponent* 1. *Therefore,* $\gcd(300, 360) = 2^2 \cdot 3 \cdot 5 = 60$. ◄

84 Example Find $\gcd(132, 504)$.

Solution: ► *First observe that* $132 = 2^2 \cdot 3 \cdot 11$ *and that* $504 = 2^3 \cdot 3^2 \cdot 7$. *Only the primes* 2 *and* 3 *appear in both numbers. The* 2 *appears with least exponent* 2, *the* 3 *appears with least exponent* 1. *Therefore,* $\gcd(132, 504) = 2^2 \cdot 3 = 12$. ◄

85 Example Find $\gcd(100, 343)$.

Solution: ► *Observe that* $100 = 2^2 5^2$ *and* $343 = 7^3$. *They have no prime factor in common, their only common divisor is* 1. *Therefore,* $\gcd(100, 343) = 1$. ◄

86 Definition Two natural numbers a and b are said to be *relatively prime* if $\gcd(a, b) = 1$.

A similar procedure for finding the least common multiple of two numbers can now be outlined. We first find their prime factorisations, take all the primes appearing in them, and choose the powers with the maximum exponents for this primes.

87 Example Find LCM$(12,45)$.

Solution: ► *Notice that this is example 61. The prime factorisations are* $12 = 2^2 \cdot 3$ *and* $45 = 3^2 \cdot 5$. *All primes occurring at least once are* 2, 3, *and* 5. *The prime* 2 *appears with maximum exponent* 2, *the primes* 3 *appears with maximum exponent* 2 *and the prime* 5 *appears with maximum exponent* 1. *Hence* LCM$(12,45) = 2^2 3^2 5 = 180$. ◄

88 Example Find LCM$(300,360)$.

Solution: ► *First observe that* $300 = 2^2 \cdot 3 \cdot 5^2$ *and that* $360 = 2^3 \cdot 3^2 \cdot 5$. *The primes* 2, 3, *and* 5 *appear at least once. The* 2 *appears with maximum exponent* 3, *the* 3 *appears with maximum exponent* 2, *and the* 5 *appears with maximum exponent* 2. *Therefore,* LCM$(300,360) = 2^3 \cdot 3^2 \cdot 5^2 = 1800$. ◄

89 Example Find LCM$(132,504)$.

Solution: ► *First observe that* $132 = 2^2 \cdot 3 \cdot 11$ *and that* $504 = 2^3 \cdot 3^2 \cdot 7$. *The primes* 2, 3, 7, *and* 11 *appear at least once. The* 2 *appears with maximum exponent* 3, *the* 3 *appears with maximum exponent* 2, *the* 7 *appears with maximum exponent* 1, *and the* 11 *appears with maximum exponent* 1. *Therefore,*

$$\text{LCM}(132,504) = 2^3 \cdot 3^2 \cdot 7^1 \cdot 11^1 = 5544.$$

◄

Homework

Problem 9.1 *Give the prime factorisation of* 24.

Problem 9.2 *Give the prime factorisation of* 36.

Problem 9.3 *How many prime numbers less than* 10000 *have digits adding up to* 2?

Problem 9.4 *Demonstrate that*

$$\gcd(24,36) \cdot \text{LCM}(24,36) = 24 \cdot 36.$$

Problem 9.5 *How many divisors does* 36 *have?*

Problem 9.6 *How many divisors does* 38 *have?*

Problem 9.7 *How many divisors does* 40 *have?*

Problem 9.8 *An unsolved problem in Number Theory is the* Goldbach Conjecture, *which asserts that every even natural number greater than* 6 *can be written as the sum of distinct primes. Verify Goldbach's conjecture from every even integer between* 8 *and* 36, *inclusive.*

Problem 9.9 *Given that there is a unique digit* $d \in \{0,1,2,3,4,5,6,7,8,9\}$ *so that the nine-digit number* $19700019d$ *is prime, find it.*

Problem 9.10 *A natural number is called* perfect *if it is the sum of all its proper divisors. For example, the proper divisors of* 6 *are* $1,2,3$ *and* $6 = 1+2+3$. *Also, the proper divisors of* 28 *are* $1,2,4,7,14$ *and* $28 = 1+2+4+7+14$, *so* 28 *is perfect. All perfect numbers known to date are even, and it is a conjecture, still unsolved, that there are no odd perfect numbers. Prove that* 496 *is a perfect number by finding all its proper divisors and adding them.*

Problem 9.11 *How many divisors does a prime number have?*

Problem 9.12 *Factor* 111111 *(six* 1*'s) into primes.*

Problem 9.13 *Let* p *be a prime number. List all the divisors of* p^2.

Problem 9.14 *Prove that among any* 101 *integers taken from the set* $\{1,2,\ldots,200\}$, *there will always be* 2 *which are relatively prime.*

Problem 9.15 *Which numbers have an odd number of divisors?*

Problem 9.16 *Four comrades are racing side by side down a dusty staircase. Frodo goes down two steps at a time, Gimli three, Legolas four, and Aragorn five. If the only steps with all four's footprints are at the top and the bottom, how many steps have just one footprint?*

Problem 9.17 (The Locker-room Problem) *A locker room contains* 100 *lockers, numbered* 1 *through* 100. *Ini-*

tially all doors are open. Person number **1** *enters and closes all the doors. Person number* **2** *enters and opens all the doors whose numbers are multiples of* **2**. *Person number* **3** *enters and if a door whose number is a multiple of* **3** *is open then he closes it; otherwise he opens it. Person number* **4** *enters and changes the status (from open to closed and viceversa) of all doors whose numbers are multiples of* **4**, *and so forth till person number* **100** *enters and changes the status of door number* **100**. *Which doors are locked?*

Problem 9.18 *Find an infinite set of positive integers such that the sum of any finite number of distinct elements of the set is not a square.*

References

Much of the material of this section belongs to a branch of Mathematics formerly called "the Higher Arithmetic", nowadays called Number Theory. Some references follow.

[NiZuMo] I. Niven, H. Zuckermann, H. Montgomery "Introduction to Number Theory".

10 Division

90 Definition (Definition of Division) Let m, n, x be natural numbers, with $n \neq 0$. Then the statement $m \div n = x$ means that $m = xn$.

91 Example To compute $15 \div 3$ we think of which number when multiplied 3 gives 15. Clearly then $15 \div 3 = 5$ since $15 = 5 \cdot 3$.

☞ *Division is neither closed in* $\mathbb{N}$, *nor commutative, nor associative. For example,*

$$4 \div 8 \notin \mathbb{N}, \quad 4 \div 8 \neq 8 \div 4, \quad 8 \div (4 \div 2) \neq (8 \div 4) \div 2.$$

Our links to division and reality are the ideas of *sharing* and *containment*. The example above can be thought as the solution to the following problem: 15 customers are going to be handled equally by 3 waitresses, how many people will each waitress handle? The result 5 means that $5 + 5 + 5 = 15$, that is, that each of the 3 waitresses will handle or receive an *equal share* of 5 customers. This corresponds to the idea of division as repeated subtraction:

$$\underbrace{15 - 3 = 12}_{\text{first instance}} \to \underbrace{12 - 3 = 9}_{\text{second instance}} \to \underbrace{9 - 3 = 6}_{\text{third instance}} \to \underbrace{6 - 3 = 3}_{\text{fourth instance}} \to \underbrace{3 - 3 = 0}_{\text{fifth instance}}$$

Since the subtraction was able to be evenly carried out five times, $15 \div 3 = 5$.

One may also think of the problem in the following way. If there are 15 customers to be served, and each waitress receives 5, how many waitresses are there? In this case we are asserting that $3 + 3 + 3 + 3 + 3 = 15$.

The above definition of division depends on the existence of a natural number x so that $m = xn$, which effectively means, that we have division without remainder. What happens in general? Let us first introduce some notation.

92 Definition By $\lfloor x \rfloor$ (pronounced *floor of* x) we mean either x if x is an integer, or the integer just to the left of x.

For example, $\lfloor 3 \rfloor = 3$, $\lfloor 3.1 \rfloor = 3$, $\lfloor 3.5 \rfloor = 3$, $\lfloor 3.9 \rfloor = 3$, $\lfloor 4.1 \rfloor = 4$. Notice that this is not rounding.

Observe that the following inequality is satisfied:

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1. \tag{10.1}$$

Let us now consider the task of defining $a \div b$, by considering first an example. Consider the problem of performing the division $23 \div 5$. We look for multiples of 5 that straddle 23. We readily see that

$$20 < 23 < 25 \implies 5 \cdot 4 < 23 < 5 \cdot 5.$$

We divide through by 5 and assume that inequalities are preserved:

$$5 \cdot 4 < 23 < 5 \cdot 5 \implies (5 \cdot 4) \div 5 < 23 \div 5 < (5 \cdot 5) \div 5 \implies 4 < 23 \div 5 < 5.$$

This means that

$$\lfloor 23 \div 5 \rfloor = 4.$$

In fact, we have

$$23 = 5 \cdot 4 + 3 \implies 23 = 5 \cdot \lfloor 23 \div 5 \rfloor + 3.$$

The excess 3 over the multiple of 5 just to the left of 23 is the *remainder*.

We are ready to state the following result, which we will admit without proof.

93 Theorem (Division Algorithm) Given two natural numbers a and b with $b \neq 0$, we may find two unique integers $\llbracket a \div b \rrbracket$ and r with $0 \leq r < b$ such that

$$a = b\llbracket a \div b \rrbracket + r.$$

Here a is called the *dividend*, b the *divisor*, $\llbracket a \div b \rrbracket$ the *quotient* and r the *remainder*.

Since $\llbracket a \div b \rrbracket = \dfrac{a-r}{b}$,

$$\llbracket a \div b \rrbracket \leq a \div b < \llbracket a \div b \rrbracket + 1 \implies \frac{a-r}{b} \leq a \div b < \frac{a-r}{b} + 1.$$

The division algorithm gives us a way of classifying natural numbers according to the remainder they leave. If $b > 0$ is the divisor, then the possible remainders are the b numbers 0, 1, 2, . . . , $b-1$.

94 Example Take $b = 2$ as the divisor. The possible remainders are 0 and $2-1 = 1$. Thus every natural number comes in one of the two flavours

$$\{0,2,4,6,\ldots\} \quad \text{or} \quad \{1,3,5,7,\ldots\},$$

that is, as a number that leaves remainder 0 upon division by 2 or as a number that leaves remainder 1 upon division by 2. In fact, we see that

$$\mathbb{N} = \{0,2,4,6,\ldots\} \cup \{1,3,5,7,\ldots\},$$

and so the division algorithm serves to partition the natural numbers into two parts. We can also put this in algebraic terms, by saying that every natural number is of the form $2a$ or $2a+1$ for some natural number a.

95 Example Take $b = 3$ as the divisor. The possible remainders are 0, 1, and $3-1 = 2$. Thus every natural number comes in one of the three flavours

$$\{0,3,6,9,\ldots\} \quad \text{or} \quad \{1,4,7,10,\ldots\} \quad \text{or} \quad \{2,5,8,11,\ldots\}.$$

that is, as a number that leaves remainder 0 upon division by 3, as a number that leaves remainder 1 upon division by 3 or as a number that leaves remainder 2 upon division by 2. In fact, we see that

$$\mathbb{N} = \{0,3,6,9,\ldots\} \cup \{1,4,7,10,\ldots\} \cup \{2,5,8,11,\ldots\}.$$

and so the division algorithm serves to partition the natural numbers into three parts. We can also put this in algebraic terms, by saying that every natural number is of the form $3a$, $3a+1$ or $3a+2$ for some natural number a.

96 Example Prove that the sum of two even natural numbers is even. Can you find five natural numbers whose sum is 25?

Solution: ► *Let $2x$ and $2y$ be two even natural numbers. Then using the distributive law in reverse,*

$$2x + 2y = 2(x+y).$$

Since $x+y$ is a natural numbers, $2(x+y)$ is twice a natural number and so even.

The answer to our second question is clearly NO, since 25 is an odd number and the sum of five even numbers would be even. ◄

We now discuss criteria for a given natural number to be (evenly) divisible by another number. Granted, we will only consider a few cases—just the criteria for divisibility by 2, 3, 5, and 9—but these are important in their own merit.

The task we have at hand is the following. Suppose that we had a very large number, let us say $\mathbf{1234509876.}$ How do we know, *without actually performing the division*, whether it is divisible by $\mathbf{2}$? By $\mathbf{3}$? By $\mathbf{5}$? It turns out that the decimal positional notation makes answering these questions extremely easily.

For the case of divisibility by $\mathbf{2}$, we want to investigate whether $\mathbf{1234509876}$ belongs to the set

$$\{\mathbf{0,2,4,6,8,10,\ldots}\}$$

of integers that leave no remainder when divided by $\mathbf{2}$. But we may write

$$\mathbf{1234509876 = 1234509870 + 6 = 123450987 \cdot 10 + 6 = 123450987 \cdot 2 \cdot 5 + 6.}$$

Clearly, $\mathbf{123450987 \cdot 2 \cdot 5}$ is divisible by $\mathbf{2}$. Hence the divisibility of $\mathbf{1234509876}$ hinges on whether its last digit, $\mathbf{6}$ is divisible by $\mathbf{2}$, which is clearly the case. We conclude that thus $\mathbf{1234509876}$ is divisible by $\mathbf{2}$. Notice that we did not need to perform any division. A more general argument can then establish the following result.

97 Theorem A natural number is divisible by $\mathbf{2}$ (even) if and only if its last digit of its decimal expansion is even, that is, it ends in one of $\{\mathbf{0,2,4,6,8}\}$.

For the case of divisibility by $\mathbf{5}$, we want to investigate whether $\mathbf{1234509876}$ belongs to the set

$$\{\mathbf{0,5,10,15,20,25,\ldots}\}$$

of integers that leave no remainder when divided by $\mathbf{5}$. But we may write

$$\mathbf{1234509876 = 1234509870 + 6 = 123450987 \cdot 10 + 6 = 123450987 \cdot 2 \cdot 5 + 6.}$$

Clearly, $\mathbf{123450987 \cdot 2 \cdot 5}$ is divisible by $\mathbf{5}$. Hence the divisibility of $\mathbf{1234509876}$ hinges on whether its last digit, $\mathbf{6}$ is divisible by $\mathbf{5}$, which is clearly not the case. We conclude that thus $\mathbf{1234509876}$ is not divisible by $\mathbf{5}$. Notice that we did not need to perform any division. We state the general result.

98 Theorem A natural number is divisible by $\mathbf{5}$ if and only if its last digit of its decimal expansion ends in one of $\{\mathbf{0,5}\}$.

☞ *In fact, by writing* $\mathbf{1234509876 = 1234509875 + 1}$, *since by the preceding theorem* $\mathbf{1234509875}$ *is evenly divisible by* $\mathbf{5}$, *we see that* $\mathbf{1234509876}$ *leaves remainder* $\mathbf{1}$ *upon division by* $\mathbf{5}$.

Before we introduce a criterion for divisibility by $\mathbf{3}$ or $\mathbf{9}$, we need the concept of *digital sum*. Consider any natural number, say $\mathbf{1234509876}$. Sum its digits successively, and then sum the digits of each successive sum, until only a $\mathbf{1}$-digit integer remains:

$$\mathbf{1234509876 \to 1+2+3+4+5+0+9+8+7+6 = 45 \to 4+5 = 9,}$$

whence the digital sum is $\mathbf{9}$.

Our criterion for divisibility by $\mathbf{3}$ or $\mathbf{9}$ will be linked to the digital sum of the number in question. For the example at hand, observe that

$$\begin{aligned}\mathbf{1234509876} &= \mathbf{1\cdot 10^9 + 2\cdot 10^8 + 3\cdot 10^7 + 4\cdot 10^6 + 5\cdot 10^5 + 0\cdot 10^4 + 9\cdot 10^3 + 8\cdot 10^2 + 7\cdot 10^1 + 6} \\ &= \mathbf{1\cdot 999999999 + 2\cdot 99999999 + 3\cdot 9999999 + 4\cdot 999999 + 5\cdot 99999 + 0\cdot 9999 + 9\cdot 999 + 8\cdot 99 + 7\cdot 9} \\ &\quad \mathbf{+(1+2+3+4+5+0+9+8+7+6),}\end{aligned}$$

Thus whether $\mathbf{1234509876}$ is divisible by $\mathbf{3}$ or $\mathbf{9}$ depends exclusively on whether its digital sum $\mathbf{1+2+3+4+5+0+9+8+7+6 = 45}$ is. In turn $\mathbf{45}$ is divisible by $\mathbf{3}$ or $\mathbf{9}$ if $\mathbf{4+5 = 9}$ is divisible by $\mathbf{3}$ or $\mathbf{9}$. The string of sums make us conclude that $\mathbf{1234509876}$ is divisible by $\mathbf{9}$. In general, we may state the following result.

99 Theorem A natural number is divisible by $\mathbf{3}$ (or $\mathbf{9}$) if and only if its digital sum is divisible by $\mathbf{3}$ (or $\mathbf{9}$).

Homework

Problem 10.1 *You put* **54** *marbles into* **6** *bags, ending up with the same number of marbles in each bag. How many marbles would be in each bag if there were* **6** *bags?*

Problem 10.2 *Find the quotient and the remainder when* **23** *is divided by* **4**.

Problem 10.3 *Find the quotient and the remainder when* **4** *is divided by* **23**.

Problem 10.4 *If today is Thursday, what day will it be* **100** *days from now?*

Problem 10.5 *Bilbo and Frodo have just consumed a plateful of cherries. Each repeats the rhyme 'Tinker, tailor, soldier, sailor, rich man, poor man, beggar man, thief' over and over again as he runs through his own heap of cherry stones. Bilbo finishes on 'sailor', whereas Frodo finishes on 'poor man'. What would they have finished on if they had run through both heaps together?*

Problem 10.6 *A runner ran* **3000** *m in* **8** *minutes. What is his average speed in meters per second?*

Problem 10.7 *Peter and Paul were selling magazines, of which Peter sold* **60** *and Paul* **80**. *Each magazine cost the same. If they received* **$700** *altogether for the magazines, how much money did Peter get?*

Problem 10.8 *Brian is drawing coloured camels in sequence. He draws a yellow camel, a green camel, a blue camel, a red camel, a brown camel, an orange camel, and again, a yellow camel, a green camel, a blue camel, etc. Which colour is the* **2008**-*th camel?*

Problem 10.9 *What is the greatest number of Mondays that can occur in* **45** *consecutive days?*

Problem 10.10 *A club has* **86** *members, and there are* **14** *more girls than boys in the club. How many are there of each?*

Problem 10.11 *When a number is doubled and the result increased by nine, one obtains fifteen. What was the original number?*

Problem 10.12 *When a number is doubled and the result diminished by nine, one obtains fifteen. What was the original number?*

Problem 10.13 *When a number is halved and the result increased by nine, one obtains fifteen. What was the original number?*

Problem 10.14 *When a number is halved and the result diminished by nine, one obtains fifteen. What was the original number?*

Problem 10.15 *There are* **5,064** *marbles that need to be packed in boxes. There are* **6** *boxes. We want to put the same number of marbles in each box. How many marbles will fit into each box?*

Problem 10.16 *The four hundred thirty seven students of a school were called to assembly for a drug raid. If the police line up as many as possible into twenty five equal rows, how many pupils are left out?*

Problem 10.17 *A book publisher sent* **140** *copies of a book to a bookstore. The publisher packed the books in two types of boxes. One type held* **8** *copies of the book and the other held* **12**. *The boxes were all full and there were an equal number of each type of box. How many boxes of each type were sent to the bookstores?*

Problem 10.18 *With what possible values can the digit* a *be replaced so that the four-digit number* $32a2$ *be divisible by* **9**?

Problem 10.19 *With what possible values can the digit* a *be replaced so that the five-digit number* $32aa2$ *be divisible by* **9**?

Problem 10.20 *A book publisher must bind* **4500** *books. One contractor can bind all these books in* **30** *days and another contractor in* **45** *days. How many days would be needed if both contractors are working simultaneously?*

Problem 10.21 *For commencement exercises, the students of a school are arranged in nine rows of twenty eight students per row. How many rows could be made with thirty six students per row?*

Problem 10.22 *For which values of the natural number* n *is* $36 \div n$ *a natural number?*

Problem 10.23 *A two-digit natural number is divided by the sum of its digits. What is the largest possible remainder?*

Problem 10.24 *As a publicity stunt, a camel merchant has decided to pose the following problem: "If one gathers all of my camels into groups of* **4**, **5** *or* **6**, *there will be no remainder. But if one gathers them into groups of* **7** *camels, there will be* **1** *camel left in one group." The number of camels is the smallest positive integer satisfying these properties. How many camels are there?*

Problem 10.25 *What is the least and the largest number of Friday the* 13*th that there can be in a given year?*

Problem 10.26 *A number of bacteria are placed in a glass. One second later each bacterium splits in two, the next second each of the now existing bacteria splits in two, et cetera. After one minute the glass is full. When was the glass half full?*

Problem 10.27 *A lotus flower doubles its surface each day, taking* 32 *days to cover the whole surface of a pond. How long will it take two lotus flowers to cover the same pond?*

Problem 10.28 *A boy and a girl collected* 24 *nuts. The boy collected twice as many nuts as the girl. How many did each collect?*

Problem 10.29 *Prove that the sum of two odd numbers is even.*

Problem 10.30 *Prove that the product of two odd numbers is odd.*

Problem 10.31 *How many integers in the set* $\{100, 101, 102, \ldots, 198, 199\}$ *of* 100 *consecutive integers* **are not** *the sum of four consecutive integers?*

Problem 10.32 *Prove that the square of any natural number either leaves remainder* 0 *when divided by* 4 *or remainder* 1 *when divided by* 4*. That is, prove that the square of any natural number is of the form* $4k$ *or* $4k+1$*.*

Problem 10.33 *Prove that no integer in the sequence*

$$11, 111, 1111, 11111, \ldots$$

is the square of an integer.

Problem 10.34 One Price Shoes *sell all their shoes for the same price, which is an integral number of dollars. Anacleta and Sinforosa go shoe shopping to* One Price Shoes*. Anacleta has $200 and she buys as many pairs of shoes as possible, and there remain $32. Sinforosa has $150 and she buys as many pairs of shoes as possible, and there remain $24. What is the price of a pair of shoes?*

Problem 10.35 *Find the greatest multiple of* 8 *using all ten digits and with all the digits different.*

Problem 10.36 *Find the smallest strictly positive natural number* n *such that* $\frac{n}{2}$ *is a (perfect) square,* $\frac{n}{3}$ *is a (perfect) cube, and* $\frac{n}{5}$ *is a (perfect) fifth power.*

References

The excellent [Aha] is a good guide for adults trying to teach their children the basic of Arithmetic. I have borrowed some of his ideas for these notes.

[Aha] Ron AHARONI, “Arithmetic for Parents”

11 Long Division

Let us now discuss an algorithm for division. The way we display it here is perhaps somewhat different from what you learned in school, but the result will be the same. We will start with simple examples in the hopes of elucidating the general procedure. Before we begin, let us introduce some notation.

100 Definition (Fraction Notation) Let a, b be two natural numbers, with $b \neq 0$. Then we define the *fraction* $\frac{a}{b}$ as

$$\frac{a}{b} = a \div b.$$

Here a is the *numerator* of the fraction and b the *denominator*.

There it is! A fraction is a division that we are too lazy to carry out!

Now, recall that from the distributive law, we have the equality

$$a(b+c) = ab + ac,$$

that is, we are able to "multiply term by term." Suppose now that instead of the factor a we multiplied by $\frac{1}{a}$, with $a \neq 0$. Then we would have

$$\frac{1}{a}(b+c) = \frac{b+c}{a} = \frac{b}{a} + \frac{c}{a},$$

which gives us a "division term by term."

For example, suppose we were considering the task of dividing **9639** by **3**. Taking advantage of the decimal representation of the number could proceed as follows:

$$9639 \div 3 = \frac{9639}{3} = \frac{9000+600+30+9}{3} = \frac{9000}{3} + \frac{600}{3} + \frac{30}{3} + \frac{9}{3} = 3000+200+10+3 = 3213.$$

Notice that we first isolated the thousands, divided them by **3**, then the hundreds, divided them by **3**, etc. We could actually display this "vertically" in the following manner.

```
 _9639 | 3
  9    |______
  --   | 3213
 _06   |
   6   |
  --   |
  _03  |
    3  |
   --  |
   _09 |
     9 |
    -- |
     0 |
```

In the above display, we first begin by the thousands. Divide **9** by **3**, obtain **3**, write this in the space for the quotient. Multiply **3** by **3**, get **9** and write this below the **9** of the dividend. Subtract and obtain **0**. Bring down the next digit from the dividend, which is **6**. Divide **6** by **3**, obtain **2**, write this in the space for the quotient. Multiply **2** by **3**, get **6** and write this below the **6** of the dividend. Subtract and obtain **0**. Bring down the next digit from the dividend, which is **3**. Divide **3** by **3**, obtain **1**, write this in the space for the quotient. Multiply **1** by **3**, get **3** and write this below the **3** of the dividend. Subtract and obtain **0**. Bring down the next digit from the dividend, which is **9**. Divide **9** by **3**, obtain **3**, write this in the space for the quotient. Multiply **3** by **3**, get **9** and write this below the **9** of the dividend. Subtract and obtain **0**. As a check:

$$\mathbf{9639 = 3 \cdot 3213},$$

and so the quotient is **3213** and the remainder **0**.

☞ *You may prefer to display your division using a more Anglo Saxon display:*

$$\begin{array}{r}
\mathbf{3213} \\
3\overline{)\mathbf{9639}} \\
\underline{\mathbf{9000}} \\
639 \\
\underline{\mathbf{600}} \\
39 \\
\underline{\mathbf{30}} \\
9 \\
\underline{\mathbf{9}} \\
0
\end{array}$$

The general algorithm is thus:

1. Find the number of times the divisor is contained into as many of the dividend figures (as read from left to right) an place this number in the place for the quotient.
2. Multiply the divisor by the number found above and place the product under the figures used in the part above from the dividend.
3. Subtract the product found above from the said figures of the dividend. Bring down the next figure from the dividend to the right of the difference just obtained.
4. Rinse and repeat with the remainder just increased by the next figure in the step above.

101 Example Perform the division: $\mathbf{21416 \div 2}$

Solution: ► *We write*

$$\mathbf{\frac{21416}{2} = \frac{20000}{2} + \frac{1000}{2} + \frac{400}{2} + \frac{10}{2} + \frac{6}{2} = 10000 + 500 + 200 + 5 + 3 = 10708.}$$

Using the vertical algorithm for division:

$$\begin{array}{r|l}
-\;2\;1\;4\;1\;6 & 2 \\ \cline{2-2}
\underline{2}\;\;\;\;\;\;\;\; & 1\;0\;7\;0\;8 \\
-\;0\;1\;\;\;\;\;\; & \\
\underline{0}\;\;\;\;\;\; & \\
-\;1\;4\;\;\;\; & \\
\underline{1\;4}\;\;\;\; & \\
-\;0\;1\;\; & \\
\underline{0}\;\; & \\
-\;1\;6 & \\
\underline{1\;6} & \\
0 &
\end{array}$$

This can also be displayed as

$$\begin{array}{r}
\mathbf{10708} \\
2\overline{)\mathbf{21416}} \\
\underline{\mathbf{20000}} \\
1416 \\
\underline{\mathbf{1400}} \\
16 \\
\underline{\mathbf{16}} \\
0
\end{array}$$

Verification: $\mathbf{21416 = 2 \cdot 10708}$, *hence the quotient is* $\mathbf{10708}$ *and the remainder* $\mathbf{0}$. ◄

102 Example Here we display $\mathbf{9634 \div 3}$:

$$
\begin{array}{r|l}
9\,6\,3\,4 & 3 \\ \cline{2-2}
-\underline{9} & 3\,2\,1\,1 \\
0\,6 & \\
-\underline{6} & \\
0\,3 & \\
-\underline{3} & \\
0\,4 & \\
-\underline{3} & \\
1 &
\end{array}
$$

This can also be displayed as

$$
\begin{array}{r}
\mathbf{3211} \\
3\overline{)\mathbf{9634}} \\
\underline{\mathbf{9000}} \\
634 \\
\underline{\mathbf{600}} \\
34 \\
\underline{\mathbf{30}} \\
4 \\
\underline{\mathbf{3}} \\
1
\end{array}
$$

Verification: $\mathbf{9634 = 3 \cdot 3211 + 1}$, hence the quotient is $\mathbf{3211}$ and the remainder $\mathbf{1}$.

103 Example Here we display $\mathbf{1326039 \div 13}$:

$$
\begin{array}{r|l}
1\,3\,2\,6\,0\,3\,9 & 1\,3 \\ \cline{2-2}
-\underline{1\,3} & 1\,0\,2\,0\,0\,3 \\
0\,2 & \\
-\underline{0} & \\
2\,6 & \\
-\underline{2\,6} & \\
0\,0 & \\
-\underline{0} & \\
0\,3 & \\
-\underline{0} & \\
3\,9 & \\
-\underline{3\,9} & \\
0 &
\end{array}
$$

This can also be displayed as

$$
\begin{array}{r}
\mathbf{102003} \\
13\overline{)\mathbf{1326039}} \\
\underline{\mathbf{1300000}} \\
26039 \\
\underline{\mathbf{26000}} \\
39 \\
\underline{\mathbf{39}} \\
0
\end{array}
$$

Verification: $\mathbf{1326039 = 13 \cdot 102003}$, hence the quotient is $\mathbf{10708}$ and the remainder $\mathbf{0}$.

104 Example Here we display $789 \div 123$:

$$\begin{array}{r|l} -\,789 & \underline{123} \\ \underline{738} & 6 \\ 51 & \end{array}$$

This can also be displayed as

$$\begin{array}{r} \mathbf{6} \\ 123\overline{)\mathbf{789}} \\ \underline{\mathbf{738}} \\ 51 \end{array}$$

Verification: $789 = 123 \cdot 6 + 51$, hence the quotient is 6 and the remainder 51.

105 Example Here we display $321123 \div 123$:

$$\begin{array}{r|l} -\,321123 & \underline{123} \\ \underline{246} & 2610 \\ -\,751 & \\ \underline{738} & \\ -\,132 & \\ \underline{123} & \\ -\,93 & \\ \underline{0} & \\ 93 & \end{array}$$

This can also be displayed as

$$\begin{array}{r} \mathbf{2610} \\ 123\overline{)\mathbf{321123}} \\ \underline{\mathbf{246000}} \\ 75123 \\ \underline{\mathbf{73800}} \\ 1323 \\ \underline{\mathbf{1230}} \\ 93 \end{array}$$

Verification: $1326039 = 123 \cdot 2610 + 93$, hence the quotient is 2610 and the remainder 93.

We now discuss algebraic division. Let us start with what is perhaps the easiest case: powers of 10. Observe that

$$\frac{10^5}{10^2} = \frac{100000}{100} = \frac{1000\not{0}\not{0}}{1\not{0}\not{0}} = 1000 = 10^3,$$

that is to perform the division, we cancelled as many zeroes as the smaller power had. But notice what happened to the exponents. We in fact had

$$\frac{10^5}{10^2} = 10^{5-2} = 10^3,$$

that is, to perform division of powers with the same base, we subtracted the exponents. To further our evidence we again observe that

$$\frac{2^6}{2^2} = \frac{64}{4} = 16 = 2^4 \implies \frac{2^6}{2^2} = 2^{6-2} = 2^4.$$

Again,

$$\frac{a^5}{a^2} = \frac{aaaaa}{aa} = \frac{aaa\not{a}\not{a}}{\not{a}\not{a}} = a^3,$$

and, of course, $a^3 = a^{5-2}$. This generalises as follows.

106 Theorem (Second Law of Exponents) Let $a \neq 0$ be a real number and m, n natural numbers, such that $m \geq n$. Then

$$\frac{a^m}{a^n} = a^{m-n}.$$

Proof: *We have*

$$\begin{aligned} \frac{a^m}{a^n} &= \frac{\underbrace{a \cdot a \cdots a}_{m\ a'\mathrm{s}}}{\underbrace{a \cdot a \cdots a}_{n\ a'\mathrm{s}}} \\ &= \underbrace{a \cdot a \cdots a}_{m-n\ a'\mathrm{s}} \\ &= a^{m-n}. \end{aligned}$$

❑

107 Example

$$\frac{2^9}{2^4} = 2^{9-4} = 2^5 = 32.$$

108 Example $\frac{15x^5y^7z^4}{5x^2y^2z^2} = 3x^3y^5z^2.$

In the case when the numerator possesses more than one term, we use the distributive law.

109 Example $\frac{x^3 + x^4}{x^2} = \frac{x^3}{x^2} + \frac{x^4}{x^2} = x + x^2.$

We will postpone the study when the denominator possesses more than one term till an Algebra course.

Homework

Problem 11.1 *Perform the division without a calculator:* $100200300400 \div 25$.

Problem 11.2 *Perform the division without a calculator:* $123123 \div 123$.

Problem 11.3 *Perform the division without a calculator:* $123123 \div 1001$.

Problem 11.4 *Perform the division without a calculator:* $1234567890 \div 90$.

Problem 11.5 *Perform the division without a calculator:* $10111213 \div 9$.

Problem 11.6 *Perform the division without a calculator:* $10111213 \div 14$.

Problem 11.7 *Divide:* $\frac{a^{12}}{a^6}$

Problem 11.8 *Divide:* $\frac{a^8x^6}{a^6x}$

Problem 11.9 *Divide term by term:* $\frac{30x^6 + 9x^9}{3x^3}$

Problem 11.10 *Divide term by term:* $\frac{x^2 + x^3}{x^2}$

Problem 11.11 *Find the exact numerical value of:* $\frac{20^6}{25^5}$.

Problem 11.12 *Calculate:*

$$\frac{100^3 + 100^3 + 100^3 + 100^3 + 100^3 + 100^3}{100^2 + 100^2 + 100^2}.$$

Problem 11.13 *Find the exact numerical value of:*

$$\frac{25^2 + 25^2 + 25^2 + 25^2}{25^4}.$$

Problem 11.14 *If*

$$\left(\frac{4^5+4^5+4^5+4^5}{3^5+3^5+3^5}\right)\left(\frac{6^5+6^5+6^5+6^5+6^5+6^5}{2^5+2^5}\right)=2^n,$$

find n.

Problem 11.15 *If*

$$3^{2001}+3^{2002}+3^{2003}=a3^{2001},$$

find a.

12 Roots and Geometric Progressions

We introduce now the operation of extracting roots. Notice that we will introduce this "new" operation by resorting to the "reverse" of an "old" operation. This is often the case in Mathematics.

110 Definition (Roots) Let m be a natural number greater than or equal to 2, and let a and b be any natural numbers. We write that $\sqrt[m]{a} = b$ if $a = b^m$. In this case we say that b is the *m-th root* of a. The number m is called the *index* of the root.

☞ *In the special case when $m = 2$, we do not write the index. Thus we will write $\sqrt{a}$ rather than $\sqrt[2]{a}$. The number $\sqrt{a}$ is called the* square root of a. *The number $\sqrt[3]{a}$ is called the* cubic root of a.

111 Example We have

$$\sqrt{1} = 1 \quad \text{because} \quad 1^2 = 1,$$

$$\sqrt{4} = 2 \quad \text{because} \quad 2^2 = 4,$$

$$\sqrt{9} = 3 \quad \text{because} \quad 3^2 = 9,$$

$$\sqrt{16} = 4 \quad \text{because} \quad 4^2 = 16,$$

$$\sqrt{25} = 5 \quad \text{because} \quad 5^2 = 25,$$

$$\sqrt{36} = 6 \quad \text{because} \quad 6^2 = 36.$$

112 Example We have

$$\sqrt[10]{1} = 1 \quad \text{because} \quad 1^{10} = 1,$$

$$\sqrt[5]{32} = 2 \quad \text{because} \quad 2^5 = 32,$$

$$\sqrt[3]{27} = 3 \quad \text{because} \quad 3^3 = 27,$$

$$\sqrt[3]{64} = 4 \quad \text{because} \quad 4^3 = 64,$$

$$\sqrt[3]{125} = 5 \quad \text{because} \quad 5^3 = 125.$$

$$\sqrt[10]{1024} = 2 \quad \text{because} \quad 2^{10} = 1024.$$

113 Definition Let a, r be non-zero natural numbers. A *geometric progression* is a sequence of the form

$$a, ar, ar^2, ar^3, \ldots,$$

that is, a sequence where we successively multiply by the same number. Here a is the *first term* of the progression, and r is its *common ratio.*

114 Example What is the next sequence of the geometric progression

$$3, 6, 12, \ldots ?$$

Solution: ► *Observe that each time we are multiplying by* 2*, hence the next term is* $12 \cdot 2 = 24$*.* ◄

115 Example Find a formula for the n-th term of the geometric progression

$$6, 12, 24, 48, \ldots.$$

Solution: ► *We start with* 6*, and then keep multiplying by* 2*, thus*

$$6 = 6 \cdot 2^0, \qquad 12 = 6 \cdot 2^1, \qquad 24 = 6 \cdot 2^2, \qquad 48 = 6 \cdot 2^3, \ldots.$$

The general term is therefore of the form $3 \cdot 2^{n-1}$*, where* $n = 1, 2, 3, \ldots$ *is a natural number.* ◄

Homework

Problem 12.1 *Find* $\sqrt{49}$*.*

Problem 12.2 *Find* $\sqrt{64}$*.*

Problem 12.3 *Find* $\sqrt[5]{243}$*.*

Problem 12.4 *Find* $\sqrt{81} + \sqrt{100}$*.*

Problem 12.5 *Find* $\sqrt[3]{343}$*.*

Problem 12.6 *Find*

$$\sqrt{1 + 2000\sqrt{1 + 2001\sqrt{1 + 2002\sqrt{1 + 2003 \cdot 2005}}}}.$$

Problem 12.7 *What is the closest (integer) perfect square to* 2003*?*

Problem 12.8 *Find the next term of the geometric progression*

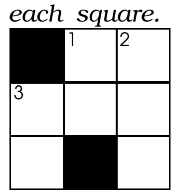
$$2, 6, 18, \ldots.$$

Problem 12.9 *The positive non-squares are ordered in increasing order:*

$$2, \quad 3, \quad 5, \quad 6, \quad 7, \quad 8, \quad 10, \ldots.$$

Find the 100*-th term.*

Problem 12.10 *A crossnumber is like a crossword except that the answers are numbers with one digit in each square. Complete the following crossnumber.*



ACROSS
1 Prime Number
3 Square of 3D.

DOWN
1 Prime Number.
2 Square of 1D.
3 Sqr. root of 3A.

Problem 12.11 *The product of three consecutive even integers is* $87{*}{*}{*}{*}{*}8$*, where the asterisks represent five missing digits. What is the sum of the integers?*

Part IV

Fractions

13 Fractions

I continued to do arithmetic with my father, passing proudly through fractions to decimals. I eventually arrived at the point where so many cows ate so much grass, and tanks filled with water in so many hours I found it quite enthralling. -Agatha CHRISTIE

In this section we review some of the arithmetic pertaining fractions. We have already defined fractions in the lecture on long division, but here is the definition again.

116 Definition A *(positive numerical) fraction* is a number of the form $m \div n = \frac{m}{n}$ where m and n are natural numbers and $n \neq 0$. Here m is the *numerator* or the fraction and n is the *denominator* of the fraction.

Given a natural number $n \neq 0$, we divide the interval between consecutive natural numbers k and $k+1$ into n equal pieces. Figures 13.1 13.2, and 13.3, shew examples with $n = 2$, $n = 3$, and $n = 4$, respectively. Notice that the larger n is, the finer the partition.

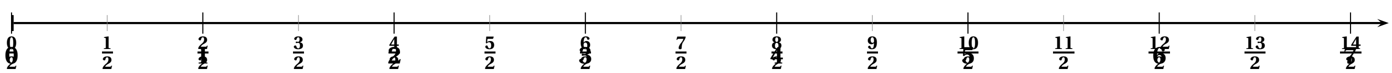


Figure 13.1: Fractions with denominator 2.

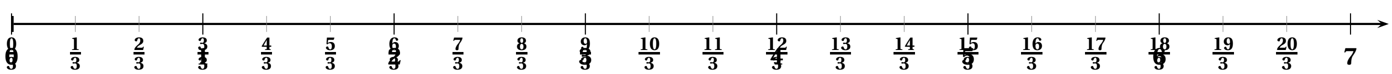


Figure 13.2: Fractions with denominator 3.

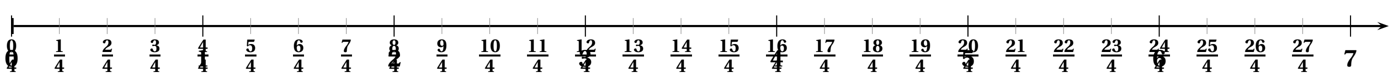


Figure 13.3: Fractions with denominator 4.

You will notice that in many of the fractions written above, the numerator is larger than the denominator. In primary school vocabulary these fractions are commonly called improper, *and perhaps, you were taught to convert them to a* mixed number. *For example, when we write* $\frac{9}{2}$, *your primary school teacher might have preferred to write* $4\frac{1}{2}$. *We will, however, prefer to use improper fractions as this is the common usage in Algebra and in modern computer algebra programmes.*

Note that some divisions of natural numbers are effectively natural numbers. Any fraction whose denominator is a factor of the numerator is a natural number. In particular, if the denominator is 1, or if the numerator is 0, the fraction is indeed a natural number. Let us see some examples.

117 Example Since the divisors of 6 are 1, 2, and 3, all the fractions

$$\frac{6}{1} = 6, \quad \frac{6}{2} = 3, \quad \frac{6}{6} = 1,$$

are natural numbers.

118 Example $\frac{6}{3} = 2$ is a natural number.

119 Example $7 = \frac{7}{1}$ is a natural number.

120 Example $\frac{0}{3} = 0$ is a natural number.

121 Example $\frac{4}{0}$ is undefined.

We now try to understand the meaning of $\frac{m}{n}$ when it is not a natural number. We first consider the following two cases when m and n are non-zero natural numbers with: (i) $m < n$, (ii) $m > n$.

Let us consider the first case.

122 Example A picture representation of the concept of $\frac{3}{4}$ can be obtained as follows. Call a square a *unit*, that is a 1, and divide it into four equal pieces. If we shade three of these four equal squares, we have shaded $\frac{3}{4}$ of the area. There is, of course, nothing special about squares. We may have used pies, line segments, etc. It is clear that $0 < \frac{3}{4} < 1$. See figures 13.4, 13.5, and 13.6.

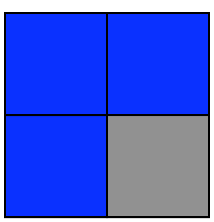


Figure 13.4: $\frac{3}{4}$ of a square.

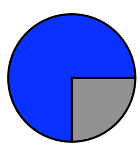


Figure 13.5: $\frac{3}{4}$ of a pie.

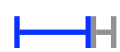


Figure 13.6: $\frac{3}{4}$ of a segment.

Let us consider now the second case.

123 Example A picture representation of the concept of $\frac{5}{2}$ can be obtained as follows. Call a square a *unit*, that is a 1, and divide it into two equal pieces. Since we only have two pieces, it is clear that we need three squares. It follows that $2 < \frac{5}{2} < 3$. If we shade five of these six equal pieces, we have shaded $\frac{5}{2}$ of the area. See figures 13.7, 13.8, and 13.9.

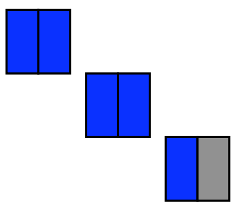


Figure 13.7: $\frac{5}{2}$ squares.

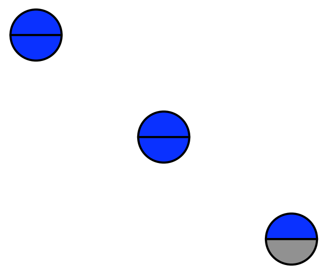


Figure 13.8: $\frac{5}{2}$ pies.

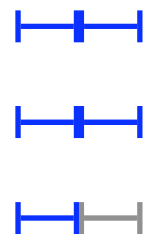


Figure 13.9: $\frac{5}{2}$ segments.

By doing more of the line diagrams above you may have noticed that there are multiple names for, say, the natural number 2. For example

$$2 = \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5} = \frac{12}{6},$$

etc. This observation is a particular case of the following result.

124 Theorem (Cancellation Law) Let m, n, k be natural numbers with $n \neq 0$ and $k \neq 0$. Then

$$\frac{mk}{nk} = \frac{m}{n}.$$

Proof: *We will prove this for $m \leq n$. For $m > n$ the argument is similar. Divide the interval $[0;1]$ into nk pieces. Consider the k-th, $2k$-th, $3k$-th, ..., nk-th markers. Since $\frac{nk}{nk} = 1$, the nk-th marker has to be 1. Thus the n markers k-th, $2k$-th, $3k$-th, ..., nk-th, form a division of $[0;1]$ into n equal spaces. It follows that the k-th marker is $\frac{1}{n}$, that is, $\frac{k}{nk} = \frac{1}{n}$, the $2k$-th marker is $\frac{2}{n}$, that is, $\frac{2k}{nk} = \frac{2}{n}$, etc., and so the mk-th marker is $\frac{m}{n}$, that is, $\frac{mk}{nk} = \frac{m}{n}$, as we wanted to prove.*❑

Thus given a fraction, if the numerator and the denominator have any common factors greater than 1, that is, any non-trivial factors, we may reduce the fraction and get an equal fraction.

125 Definition Two fractions such that $\frac{a}{b} = \frac{x}{y}$ are said to be *equivalent*. If $b < y$, then $\frac{a}{b}$ is said to be a *reduced form* of $\frac{x}{y}$.

☞ *It is possible to prove that any fraction has a unique reduced form with minimal denominator, the so called* equivalent fraction in lowest terms. *This depends on the fact that the natural numbers can be factored uniquely into primes, and hence, though we will accept this result, we will not prove it here.*

126 Example A geometric interpretation of why $\frac{1}{2} = \frac{2}{4}$ is given in figures 13.10 and 13.11.

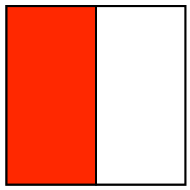


Figure 13.10: $\frac{1}{2}$

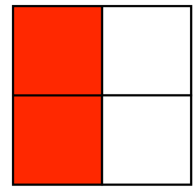


Figure 13.11: $\frac{2}{4}$

127 Example To reduce $\frac{104}{120}$ to lowest terms, observe that

$$\frac{104}{120} = \frac{104 \div 4}{120 \div 4} = \frac{26 \div 2}{30 \div 2} = \frac{13}{15}.$$

Since 13 and 15 do not have a non-trivial factor in common, $\frac{13}{15}$ is the desired reduction.

☞ *The reduction steps above are not unique. For example, if we had seen right away that* 8 *was a common factor, we would have obtained* $\frac{104}{120} = \frac{104 \div 8}{120 \div 8} = \frac{13}{15}$*, obtaining the same result. Hence, no matter how many steps you take, as long as there is valid cancellation to do and you perform them all, you will always obtain the right result.*

128 Example Find a fraction with denominator 120 equivalent to $\frac{11}{24}$.

Solution: ► *Observe that* $120 \div 24 = 5$. *Thus*

$$\frac{11}{24} = \frac{11 \cdot 5}{24 \cdot 5} = \frac{55}{120}.$$

◄

As we mentioned before, we will prefer to use improper fraction to mixed numbers. Here is an example of how to convert from one to the other and viceversa.

129 Example Convert the mixed number $2\frac{3}{4}$ into an improper fraction.

Solution: ► *We have*

$$2\tfrac{3}{4} = 2 + \frac{3}{4} = \frac{8}{4} + \frac{3}{4} = \frac{11}{4}.$$

Some people prefer to write this explanation as $2\frac{3}{4} = \frac{2 \cdot 4 + 3}{4} = \frac{11}{4}$. ◄

130 Example Convert the improper fraction $\frac{19}{4}$ to a mixed number.

Solution: ► *We perform the long division of* $\frac{19}{4} = 19 \div 4$:

$$\begin{array}{r|l} 19 & 4 \\ -\underline{16} & 4 \\ 3 & \end{array}$$

whence $\frac{19}{4} = 4\frac{3}{4}$. ◄

Homework

Problem 13.1 *Locate the following fractions on the number line:* $\frac{3}{4}$, $\frac{7}{4}$, *and* $\frac{11}{4}$.

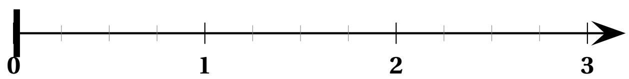


Figure 13.12: Problem 13.1.

Problem 13.2 *Find two fractions equivalent to* $\frac{5}{3}$. *Show all the steps.*

Problem 13.3 *Find two fractions equivalent to* $\frac{3}{5}$. *Show all the steps.*

Problem 13.4 *Find a fraction equivalent to* $\frac{3}{10}$ *with denominator* 50. *Show all the steps.*

Problem 13.5 *Circle* Yes *or* No *depending on whether the pair of fractions are equivalent. Justify your answer.*

1. $\frac{3}{7}$, $\frac{12}{28}$ *Yes* *No*
2. $\frac{2}{3}$, $\frac{5}{6}$ *Yes* *No*
3. $\frac{0}{5}$, $\frac{0}{3}$ *Yes* *No*
4. $\frac{7}{7}$, $\frac{345}{345}$ *Yes* *No*
5. $\frac{14}{2}$, $\frac{21}{3}$ *Yes* *No*
6. $\frac{1}{16}$, $\frac{1}{32}$ *Yes* *No*
7. $\frac{1}{4}$, $\frac{2}{8}$ *Yes* *No*

Problem 13.6 *What fraction of an hour is* 36 *minutes?*

Problem 13.7 *In making a garden fertiliser, a gardener mixes 2 lbs. of nitrate, 3 lbs. of phosphate, and 6 lbs. of potash. What is the ratio of nitrate to the total amount of fertiliser?*

Problem 13.8 *Express* $\frac{102}{210}$ *in least terms.*

Problem 13.9 *Find an equivalent fraction to* $\frac{102}{210}$ *with denominator* 3990.

Problem 13.10 *Arrange in increasing order:* $\frac{2}{3}, \frac{3}{5}, \frac{8}{13}$.

Problem 13.11 *Reduce to lowest terms:* $\frac{116690151}{427863887}$.

Problem 13.12 *Convert the improper fraction to a mixed number:* $\frac{13}{7}$

Problem 13.13 *Convert the improper fraction to a mixed number:* $\frac{23}{5}$

Problem 13.14 *Convert the mixed number to an improper fraction:* $3\frac{1}{3}$

Problem 13.15 *Convert the mixed number to an improper fraction:* $2\frac{4}{7}$

Problem 13.16 *Four singers take part in a musical round of* 4 *equal lines, each finishing after singing the round through* 3 *times. The second singer begins when the first singer begins the second line, the third singer begins when the first singer begins the third line, the fourth singer begins when the first singer begins the fourth line. What is the fraction of the total singing time when all the singers are singing simultaneously?*

14 Addition and Subtraction of Fractions

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.
-Count Lev Nikolgevich TOLSTOY

We now define addition of fractions. We would like this definition to agree with our definition of addition of natural numbers. Recall that we defined addition of natural numbers x and y as the concatenation of two segments of length x and y. We thus define the addition of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ as the concatenation of two segments of length $\frac{a}{b}$ and $\frac{c}{d}$. The properties of associativity, commutativity, etc., stem from this definition. The problem we now have is how to concretely apply this definition to find the desired sum? From this concatenation definition it follows that natural numbers a and $b \neq 0$,

$$\frac{a}{b} = \underbrace{\frac{1}{b} + \cdots + \frac{1}{b}}_{a \text{ times}}. \tag{14.1}$$

It follows from the above property that

$$\frac{x}{b} + \frac{y}{b} = \frac{x+y}{b}. \tag{14.2}$$

131 Example We have

$$\frac{2}{13} + \frac{5}{13} = \frac{7}{13}.$$

We now determine a general formula for adding fractions of different denominators.

132 Theorem (Sum of Fractions) Let a, b, c, d be natural numbers with $b \neq 0$ and $d \neq 0$. Then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}.$$

Proof: *From the Cancellation Law (Theorem 124),*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd},$$

proving the theorem. ❑

The formula obtained in the preceding theorem agrees with that of (14.2) when the denominators are equal. For, using the theorem,

$$\frac{x}{b} + \frac{y}{b} = \frac{xb}{b \cdot b} + \frac{yb}{b \cdot b} = \frac{xb+by}{b \cdot b} = \frac{b(x+y)}{b \cdot b} = \frac{x+y}{b},$$

where we have used the distributive law.

Observe that the trick for adding the fractions in the preceding theorem was to convert them to fractions of the same denominator.

133 Definition To express two fractions in a *common denominator* is to write them in the same denominator. The smallest possible common denominator is called the *least common denominator*.

134 Example Add: $\frac{3}{5}+\frac{4}{7}$.

Solution: ► *A common denominator is* $5\cdot 7=35$. *We thus find*

$$\frac{3}{5}+\frac{4}{7}=\frac{3\cdot 7}{5\cdot 7}+\frac{4\cdot 5}{7\cdot 5}=\frac{21}{35}+\frac{20}{35}=\frac{41}{35}.$$

◄

☞ *In the preceding example,* 35 *is not the only denominator that we may have used. Observe that* $\frac{3}{5}=\frac{42}{70}$ *and* $\frac{4}{7}=\frac{40}{70}$. *Adding,*

$$\frac{3}{5}+\frac{4}{7}=\frac{3\cdot 14}{5\cdot 14}+\frac{4\cdot 10}{7\cdot 10}=\frac{42}{70}+\frac{40}{70}=\frac{82}{70}=\frac{82\div 2}{70\div 2}=\frac{41}{35}.$$

This shews that it is not necessary to find the least common denominator *in order to add fractions, simply a common denominator.*

In fact, let us list the multiples of 5 and of 7 and let us circle the common multiples on these lists:

The multiples of 5 are $5,10,15,20,25,30,\boxed{35},40,45,50,55,60,65,\boxed{70},75,\ldots,$

The multiples of 7 are $7,14,21,28,\boxed{35},42,49,56,63,\boxed{70},77,\ldots.$

The sequence

$$35,70,105,140,\ldots,$$

is a sequence of common denominators for 5 and 7.

135 Example To perform the addition $\frac{2}{7}+\frac{1}{5}+\frac{3}{2}$, observe that $7\cdot 5\cdot 2=70$ is a common denominator. Thus

$$\begin{aligned}\frac{2}{7}+\frac{1}{5}+\frac{3}{2} &= \frac{2\cdot 10}{7\cdot 10}+\frac{1\cdot 14}{5\cdot 14}+\frac{3\cdot 35}{2\cdot 35}\\ &= \frac{20}{70}+\frac{14}{70}+\frac{105}{70}\\ &= \frac{20+14+105}{70}\\ &= \frac{139}{70}.\end{aligned}$$

A few words about subtraction of fractions. Suppose $\frac{a}{b}\geq\frac{c}{d}$. Observe that this means that $ad\geq bc$, which in turns means that $ad-bc\geq 0$. Then from a segment of length $\frac{a}{b}$ we subtract one of $\frac{b}{d}$. In much the same manner of addition of fractions then

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}.$$

Now, since $ad-bc\geq 0$, $ad-bc$ is a natural number and hence $\frac{ad-bc}{bd}$ is a fraction.

136 Example We have

$$\frac{3}{4}-\frac{2}{5}=\frac{15}{20}-\frac{8}{20}=\frac{7}{20}.$$

If one of the terms is a mixed number, it should first be converted to an improper fraction.

137 Example We have

$$3\frac{3}{4}-2\frac{3}{5}=\frac{15}{4}-\frac{13}{10}=\frac{75}{20}-\frac{26}{20}=\frac{49}{20}.$$

138 Example We have

$$2\frac{1}{4}+\frac{1}{3}=\frac{9}{4}+\frac{1}{3}=\frac{27}{12}+\frac{4}{12}=\frac{31}{12}.$$

Homework

Problem 14.1 *Jack and Jill were eating cherries. Jack grabbed* $\frac{1}{3}$ *of the initial amount of cherries and Jill grabbed* $\frac{1}{6}$ *of the initial amount of cherries. What fraction of the original amount of cherries remains?*

Problem 14.2 *Add the fractions and reduce to least terms:* $\frac{1}{7}+\frac{1}{5}$*. If your result is an improper fraction, leave it in this form*

Problem 14.3 *Perform the operations and reduce to least terms:* $\frac{1}{7}+\frac{1}{5}-\frac{2}{35}$*. If your result is an improper fraction, leave it in this form*

Problem 14.4 *Perform the operations and reduce to least terms:* $2\frac{1}{2}-\frac{1}{4}$*. If your result is an improper fraction, leave it in this form*

Problem 14.5 *Perform the operations and reduce to least terms:* $3\frac{5}{9}-\frac{2}{9}$*. If your result is an improper fraction, leave it in this form*

Problem 14.6 *Perform the operations and reduce to least terms:* $3\frac{5}{9}-\frac{8}{9}$*. If your result is an improper fraction, leave it in this form*

Problem 14.7 *Perform the operations and reduce to least terms:* $3\frac{5}{9}+2\frac{8}{9}$*. If your result is an improper fraction, leave it in this form*

Problem 14.8 *Perform the operations and reduce to least terms:* $3\frac{5}{9}-2\frac{8}{9}$*. If your result is an improper fraction, leave it in this form*

15 Multiplication and Division of Fractions

We now approach multiplication of fractions. We saw that a possible interpretation for the product xy of two natural numbers x and y is the area of a rectangle with sides of length x and y. We would like to extend this interpretation in the case when x and y are fractions. That is, given a rectangle of sides of length $x = \frac{a}{b}$ and $y = \frac{c}{d}$, we would like to deduce that $\frac{ac}{bd}$ is the area of this rectangle.

139 Theorem (Multiplication of Fractions) Let a, b, c, d be natural numbers with $b \neq 0$ and $d \neq 0$. Then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Proof: *First consider the case when $a = c = 1$. Start with a unit square and cut it horizontally into b equal segments. Then cut it vertically into d equal segments. We have now bd equal pieces, each one having an area of $\frac{1}{bd}$. Since each piece is in dimension $\frac{1}{b}$ by $\frac{1}{d}$, we have shewn that*

$$\frac{1}{b} \cdot \frac{1}{d} = \frac{1}{bd}.$$

Construct now a rectangle of length $\frac{a}{b}$ and width $\frac{c}{d}$. Such a rectangle is obtained by concatenating along its length a segments of length $\frac{1}{b}$ and along its width c segments of length $\frac{1}{d}$. This partitions the large rectangle into ac sub-rectangles, each of area $\frac{1}{bd}$. Hence the area of the $\frac{a}{b}$ by $\frac{c}{d}$ rectangle is $ab\left(\frac{1}{bd}\right)$, from where

$$\frac{a}{b} \cdot \frac{c}{d} = ac\left(\frac{1}{bd}\right) = \frac{ac}{bd},$$

proving the theorem. ❑

140 Example We have

$$\frac{2}{3} \cdot \frac{3}{7} = \frac{6}{21} = \frac{2}{7}.$$

Alternatively, we could have cancelled the common factors, as follows,

$$\frac{2}{\not{3}} \cdot \frac{\not{3}}{7} = \frac{2}{7}.$$

Again, if one of the terms is a mixed number, it must first be converted to an improper fraction.

141 Example We have

$$(2\tfrac{3}{4}) \cdot \tfrac{2}{11} = \frac{11}{4} \cdot \frac{2}{11} = \frac{2}{4} = \frac{1}{2}.$$

142 Example Find the exact value of the product

$$\left(1 - \frac{2}{5}\right)\left(1 - \frac{2}{7}\right)\left(1 - \frac{2}{9}\right)\cdots\left(1 - \frac{2}{99}\right)\left(1 - \frac{2}{101}\right).$$

Solution: ► *We have,*

$$\left(1-\frac{2}{5}\right)\left(1-\frac{2}{7}\right)\left(1-\frac{2}{9}\right)\cdots\left(1-\frac{2}{99}\right)\left(1-\frac{2}{101}\right) = \frac{3}{5}\cdot\frac{5}{7}\cdot\frac{7}{9}\cdot\frac{9}{11}\cdots\frac{97}{99}\cdot\frac{99}{101} = \frac{3}{101}.$$

◄

We now tackle division of fractions. Recall that we defined division of natural numbers as follows. If $n \neq 0$ and m, x are natural numbers then $m \div n = x$ means that $m = xn$. We would like a definition of fraction division compatible with this definition of natural number division. Hence we give the following definition.

143 Definition Let a, b, c, d be natural numbers with $b \neq 0$, $c \neq 0$, $d \neq 0$. We define the fraction division

$$\frac{a}{b} \div \frac{c}{d} = \frac{x}{y} \iff \frac{a}{b} = \frac{x}{y}\cdot\frac{c}{d}.$$

We would like to know what $\frac{x}{y}$ above is in terms of a, b, c, d. For this purpose we have the following theorem.

144 Theorem (Division of Fractions) Let a, b, c, d be natural numbers with $b \neq 0$, $c \neq 0$, $d \neq 0$. Then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b}\cdot\frac{d}{c} = \frac{ad}{bc},$$

that is, $\frac{x}{y}$ in definition 143 is $\frac{x}{y} = \frac{ad}{bc}$.

Proof: *Let us prove that* $\frac{x}{y} = \frac{ad}{bc}$ *satisfies the definition of division of fractions. Observe that*

$$\frac{x}{y}\cdot\frac{c}{d} = \frac{ad}{bc}\cdot\frac{c}{d} = \frac{adc}{bcd} = \frac{a}{b},$$

and hence $\frac{x}{y}$ *so chosen is the right result for the division of fractions. Could there be another fraction, say* $\frac{x'}{y'}$ *that satisfies definition 143? Suppose*

$$\frac{a}{b} = \frac{x'}{y'}\cdot\frac{c}{d}.$$

Then

$$\frac{x}{y} = \frac{a}{b}\cdot\frac{d}{c} = \frac{x'}{y'}\cdot\frac{c}{d}\cdot\frac{d}{c} = \frac{x'}{y'}.$$

Hence $\frac{x}{y} = \frac{a}{b}\cdot\frac{d}{c}$ *is the only fraction that satisfies the definition of fraction division.* ❑

145 Definition Let $c \neq 0$, $d \neq 0$ be natural numbers. The *reciprocal* of the fraction $\frac{c}{d}$ is the fraction $\frac{d}{c}$.

Theorem 144 says that in order to divide two fractions we must simply multiply the first one by the reciprocal of the other.

146 Example The reciprocal of the fraction $\frac{3}{4}$ is $\frac{4}{3}$.

147 Example The reciprocal of the fraction $\frac{1}{2}$ is $\frac{2}{1}$, that is 2.

148 Example The reciprocal of the mixed number $1\frac{3}{4}$ is the same as the reciprocal of the improper fraction $\frac{7}{4}$, which is is $\frac{4}{7}$.

149 Example We have,

$$\frac{24}{35} \div \frac{20}{7} = \frac{24}{35} \cdot \frac{7}{20} = \frac{4 \cdot 6}{7 \cdot 5} \cdot \frac{7 \cdot 1}{4 \cdot 5} = \frac{6 \cdot 1}{5 \cdot 5} = \frac{6}{25}.$$

150 Example We have,

$$2\tfrac{1}{3} \div 3\tfrac{2}{3} = \frac{7}{3} \div \frac{11}{3} = \frac{7}{3} \cdot \frac{3}{11} = \frac{7}{11}.$$

151 Example Find the value of

$$\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}.$$

Solution: ► *Plainly,*

$$\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}} = \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}} = \frac{1}{1 + \frac{1}{\frac{3}{2}}} = \frac{1}{1 + \frac{2}{3}} = \frac{1}{\frac{5}{3}} = \frac{3}{5}.$$

◄

152 Example Calculate: $\frac{4}{5} \cdot \frac{5}{6} + \frac{4}{5} \div \frac{5}{6}$

Solution: ► *We have*

$$\frac{4}{5} \cdot \frac{5}{6} + \frac{4}{5} \div \frac{5}{6} = \frac{2}{3} + \frac{24}{25} = \frac{122}{75}.$$

◄

153 Example Calculate: $\dfrac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}}$.

Solution: ► *We have*

$$\frac{\frac{1}{2} + \frac{1}{3}}{\frac{1}{2} - \frac{1}{3}} = \frac{\frac{5}{6}}{\frac{1}{6}} = 5.$$

◄

Homework

Problem 15.1 *What is one third of a half of a fifth of a quarter of* 99000*?*

Problem 15.2 *If* 4 *times a number is* 48*, what is* $\frac{1}{3}$ *of the number?*

Problem 15.3 *John had* $240 *of which he lost* $\frac{5}{8}$ *betting at the horses. How much money does he have left?*

Problem 15.4 *Complete the "fraction puzzle" below.*

$\frac{1}{3}$	+		=	2
+		−		
$\frac{3}{4}$	·		=	
=		=		
	÷	$\frac{2}{3}$	=	

Problem 15.5 *Tito was pigging out on cookies, and in the course of three days, he ate* 420 *cookies. On the first day, he ate* $\frac{2}{5}$ *of the cookies. On the second day, he ate* $\frac{5}{6}$ *of the remaining cookies. How many cookies did he eat on the third day?*

Problem 15.6 *David spent* $\frac{2}{5}$ *of his money on a storybook. The storybook cost* $20. *How much money did he have at first?*

Problem 15.7 *Anna and Tina agree to wash* 15 *windows for $90. If Anna washed* 9 *windows and Tina* 6, *and each agrees to receive the same amount per window, how much will each get?*

Problem 15.8 *When the cathedral clock strikes four, it takes* 8 *seconds between the first and the last strokes. How many seconds does it take to strike twelve?*

Problem 15.9 *Find the exact numerical value of*

$$\left(\frac{2}{51}\right) \div \left(\frac{3}{17}\right) \cdot \left(\frac{7}{10}\right).$$

Problem 15.10 *Find the exact numerical value of*

$$\frac{\frac{4}{7}-\frac{2}{5}}{\frac{4}{7}+\frac{2}{5}}.$$

Problem 15.11 *Find the value of*

$$\frac{10+10^2}{\frac{1}{10}+\frac{1}{100}}.$$

Problem 15.12 *Find the exact numerical value of*

$$\frac{1^3+2^3+3^3-3(1)(2)(3)}{(1+2+3)(1^2+2^2+3^2-1\cdot 2-2\cdot 3-3\cdot 1)}.$$

Problem 15.13 *If*

$$\frac{1}{1+\frac{1}{5}}=\frac{a}{b},$$

where the fraction $\frac{a}{b}$ *is in least terms, find* a^2+b^2.

Problem 15.14 *Find the exact numerical value of*

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{99}\right)\left(1-\frac{1}{100}\right).$$

Problem 15.15 *A painter had* 25 *gallons of paint. He used* $2\frac{1}{2}$ *gallons of paint every hour. He finished the job in* $5\frac{1}{2}$ *hours. How much paint did he have left?*

Problem 15.16 *Naomi has* 16 *yards of gift-wrap in order to wrap the gifts for the Festival of Lights at the community centre. Each gift requires* $1\frac{7}{8}$ *yards of paper. How many gifts can she wrap?*

Problem 15.17 *Evaluate*

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}}.$$

Problem 15.18 *Find the value of*

$$1+\frac{1}{2+\frac{1}{3+\frac{1}{4}}}.$$

Problem 15.19 *What is the angle between the hands of a clock at a quarter to five?*

16 Ratio and Proportion

154 Definition Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are said to be *proportional* or *in proportion* if

$$\frac{a}{b} = \frac{c}{d}.$$

Proportions occur in many practical problems and our interest is to be able to provide a framework that helps us solve a variety of proportion problems. For example, suppose a certain recipe demands three eggs for two pints of milk. It does not require much thought to deduce that doubling the recipe requires six eggs for four pints of milk and that trebling the recipe would require nine eggs for six pints of milk. The problem is now to figure out how many eggs are required if we are only given information about the milk, or viceversa, how much milk is required if we are only given information about the eggs. To do this we use a procedure that used to be called the *rule of three*. Suppose that we would like to find out how many eggs are necessary if we had 9 pints of milk. The trick is to form products of fractions that have only eggs as numerator, cancelling all references to milk. The original proportion is

$$\frac{3 \text{ eggs}}{2 \text{ pints}}.$$

In order to cancel the pints from the denominator we must multiply by pints. Since we are interested in how many eggs per 9 pints, we multiply by 9 pints, obtaining,

$$\frac{3 \text{ eggs}}{2 \cancel{\text{pints}}} \cdot (9 \cancel{\text{pints}}) = \frac{27}{2} \text{ eggs},$$

which means we need $13\frac{1}{2}$ eggs. Again, if we had available 20 eggs, we would need

$$\frac{2 \text{ pints}}{3 \cancel{\text{eggs}}} \cdot (20 \cancel{\text{eggs}}) = \frac{40}{3} \text{ pints},$$

that is we would need $13\frac{1}{3}$ pints of milk.

We will now give more examples.

155 Example Yosida can eat 18 hotdogs in 4 minutes. At this rate, how many hotdogs can he eat in 3 minutes?

Solution: ► *The "fundamental ratios" here are*

$$\frac{18 \text{ hotdogs}}{4 \text{ minutes}} \quad \text{or} \quad \frac{4 \text{ minutes}}{18 \text{ hotdogs}}.$$

Since we are interested in the number of hotdogs, we use the one that has hotdogs on the numerator, and cancel the minutes on the denominator. Thus he can eat

$$\frac{18 \text{ hotdogs}}{4 \cancel{\text{minutes}}} \cdot (3 \cancel{\text{minutes}}) = \frac{54}{4} \text{ hotdogs} = \frac{27}{2} \text{ hotdogs},$$

that is, he can eat $13\frac{1}{2}$ hotdogs in 3 minutes. ◄

156 Example Under normal conditions, a car gives 27 miles per gallon of gasoline. Under the same conditions, how many gallons of gasoline are required in order to cover 900 miles?

Solution: ► *The "fundamental ratios" here are*

$$\frac{27 \text{ miles}}{1 \text{ gallon}} \quad \text{or} \quad \frac{1 \text{ gallon}}{27 \text{ miles}}.$$

Since we are interested in the number of gallons, we use the ratio that has gallons on the numerator, and cancel the miles on the denominator. Thus we need

$$\frac{1 \text{ gallon}}{27 \cancel{\text{miles}}} \cdot (900 \cancel{\text{miles}}) = \frac{900}{27} \text{ gallons} = \frac{100}{3} \text{ gallons},$$

that is, we need $33\frac{1}{3}$ *gallons of gasoline in order to ride* 900 *miles.* ◄

The method we used for solving proportion problems can be used in unit conversion problems. Here is an example.

157 Example The current rate of exchange is 1 dollar per 40 Thai Bahts, and 1 euro for 65 Thai baht. How many dollars is 20 euros?

Solution: ► *We want dollars on the numerator. We proceed as follows,*

$$\frac{1 \text{ dollar}}{40 \cancel{\text{baht}}} \cdot \frac{65 \cancel{\text{baht}}}{1 \cancel{\text{euro}}} \cdot (20 \cancel{\text{euros}}) = \frac{1300}{40} \text{ dollars} = \frac{65}{2} \text{ dollars},$$

which is \$32.50. ◄

158 Example Milk, dark, and white chocolate miniature bars are mixed together in a ratio of $5:4:2$ to form 275 lbs of a *Chocoholic-Enticer* mixture. How much of each type of bars was used for this mixture?

Solution: ► *The denominator is* $5+4+2=11$*. One wants* $\frac{5}{11}\cdot 275 = 125$ *lbs of milk chocolate,* $\frac{4}{11}\cdot 275 = 100$ *lbs of dark chocolate, and* $\frac{2}{11}\cdot 275 = 50$ *lbs of white chocolate.* ◄

159 Example John takes 2 hours to paint a room, whereas Bill takes 3 hours to paint the same room. How long would it take if both of them start and are working simultaneously?

Solution: ► *In one hour John does* $\frac{1}{2}$ *of the job. In one hour Bill does* $\frac{1}{3}$*. Thus in one hour they together accomplish* $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ *of the job. Thus in* $\frac{6}{5}$ *of an hour they accomplish the job. Since* $\frac{6}{5}\cdot 60 = 72$*, it takes them* 72 *minutes for them to finish the job.* ◄

Homework

Problem 16.1 *Rahnnya takes 20 minutes to solve 3 maths problems. At this rate, how many minutes will it take her to solve 2880 problems?*

Problem 16.2 *Oscar rides his bike, being able to cover* 6 *miles in* 54 *minutes. At that speed, how long does it take him to cover a mile?*

Problem 16.3 *If there are* 300 *calories in* 100 *g of a certain type of food, how many calories are there in a* 30 *g portion of this food?*

Problem 16.4 *If the ratio of* 7 *to* 13 *is the same as the ratio of* x *to* 52*, what is* x*?*

Problem 16.5 *Brazilian, Colombian, and Kenyan coffee beans are mixed together in ratio* $4:7:2$ *to form* 195 *lbs of a* Ya Betta Drink Me *blend. How much of each type of beans was used for this mixture?*

Problem 16.6 *On a certain map City A is* 4 *inches apart from City B. If the scale is such that* $1\frac{1}{5}$ *inches represent* 30 *miles, find the actual distance, in miles, between City A and City B.*

Problem 16.7 *Nanette is going to make* crêpes. *Her recipe requires* 3 *fl oz of beer for every* 2 *eggs. If she is going to utilise* 8 *eggs, how many fluid ounces of beer does she need?*

Problem 16.8 *A car with five tyres (four road tyres and a spare tyre) travelled* 30,000 *miles. If all five tyres were used equally, how many miles' wear did each tyre receive?*

Problem 16.9 *Andy and Brandy run in opposite directions on a circular track, starting at diametrically opposite points. Each one runs at a constant speed. They first meet after Andy has run* 100 *meters. They next meet after Brandy has run* 150 *meters past their first meeting point. What is the length of the track, in meters?*

Problem 16.10 *Anton, Banton, and Canton are mowing their lawns. Anton's lawn is twice as large as Banton's, and thrice as large as Canton's. Canton's lawn mower cuts half as fast as Banton's, and one third as fast as Anton's. If they start mowing their lawns at the same time, who will finish first.*

Part V

Decimals

17 Decimals

We now consider a set of special fractions.

160 Definition A *decimal fraction* is a fraction whose denominator is a power of 10.

For example, all of the following are decimal fractions:

$$\frac{3}{10}, \quad \frac{7}{1000}, \quad \frac{213}{100}.$$

Decimal fractions are ubiquitous in calculator computations. Since they are so common, we use a shortcut notation for them, which we will call *decimal notation*. The idea is to separate a number from its floor and use a period, called the *decimal point* to separate the integral part from its fractional part. For example, instead of writing

$$123\tfrac{789}{1000},$$

we write—in decimal notation—123.789. Since

$$\frac{789}{1000} = \frac{700}{1000} + \frac{80}{1000} + \frac{9}{1000} = \frac{7}{10} + \frac{8}{100} + \frac{9}{1000},$$

the convention we are making is that on the first place to the right after the decimal point we put the tenths, on the second place we put the hundredths, on the third place we put the thousandths.

Again, suppose we were given the fraction

$$\frac{7813}{100}.$$

We first compute

$$\left\lfloor \frac{7813}{100} \right\rfloor = 78.$$

This means that

$$\frac{7813}{100} = \frac{7800}{100} + \frac{13}{100} = 78 + \frac{13}{100}.$$

Hence, the equivalent decimal notation is

$$\frac{7813}{100} = 78 + \frac{13}{100} = 78.13.$$

Again, we might interpret $\dfrac{7813}{100}$ as $7813 \div 100$. Hence, another way of proceeding would be to perform the long division

$$
\begin{array}{r|l}
7813 & 100 \\ \cline{2-2}
-\ \underline{700} & 78.13 \\
813 & \\
-\ \underline{800} & \\
130 & \\
-\ \underline{100} & \\
300 & \\
-\ \underline{300} & \\
0 &
\end{array}
$$

The procedure exhibited above allows us to convert any decimal fraction to decimal notation, and viceversa. We will see in the examples below that it also allows to convert certain other fractions to decimals, and viceversa.

161 Example Write in decimal notation: $\frac{78}{100}$.

Solution: ► *The denominator is* $100 = 10^2$*, and so we use at most two decimal places. We have*

$$\frac{78}{100} = 0.78.$$

Another way of proceeding would be to observe that

$$\frac{78}{100} = 78 \div 100 = \begin{array}{r|l} 78 & 100 \\ -\ 0 & 0.78 \\ \hline 780 & \\ -\ 700 & \\ \hline 800 & \\ -\ 800 & \\ \hline 0 & \end{array}$$

Notice that we have written a 0 *as the floor of* $\frac{78}{100}$*. It is really not necessary to do this and you will often see the following written:*

$$\frac{78}{100} = .78,$$

even by computer programs such as Maple. We will stick to the former way of writing for easy reading. ◄

162 Example Write in decimal notation: $\frac{78}{10000}$.

Solution: ► *The denominator is* $10000 = 10^4$*, and so we use at most four decimal places. We have*

$$\frac{78}{10000} = 0.0078.$$

Another way of proceeding would be to observe that

$$\frac{78}{10000} = 78 \div 10000 = \begin{array}{r|l} 78 & 10000 \\ -\ 0 & 0.0078 \\ \hline 780 & \\ -\ 0 & \\ \hline 7800 & \\ -\ 0 & \\ \hline 78000 & \\ -\ 70000 & \\ \hline 80000 & \\ -\ 80000 & \\ \hline 0 & \end{array}$$

◄

163 Example Write in decimal notation: $\frac{780}{10000}$.

Solution: ► *The denominator is* $10000 = 10^4$*, and so we use at most four decimal places. We have*

$$\frac{780}{10000} = 0.0780.$$

Now, the last zero in 0.0780 *does not contribute to the magnitude of* 0.0780 *and thus we may drop it. We will write then*

$$\frac{780}{10000} = 0.078,$$

and so adopt the convention that we do not write trailing zeroes. Another way of proceeding would be to observe that

$$\frac{780}{10000} = 780 \div 10000 = \begin{array}{r|l} -\ 780 & 10000 \\ \underline{0} & 0.078 \\ -\ 7800 & \\ \underline{0} & \\ -\ 78000 & \\ \underline{70000} & \\ -\ 80000 & \\ \underline{80000} & \\ 0 & \end{array}$$

◄

So far we have converted decimal fractions to decimal notation. Can we also convert non-decimal fractions? The answer is yes, but we have some caveats. Since $10^n = 2^n 5^n$, it turns out that the only fractions whose denominators we can multiply by a natural number and obtain a power of **10** are those fractions whose denominators are only divisible by **2** or **5**. All other fractions cannot be converted in this manner. But we can always rely on the fact that $\frac{a}{b} = a \div b$, and hence use division in order to convert fractions to decimal.

164 Example Write in decimal notation: $\frac{3}{5}$.

Solution: ► *Here we are given a fraction whose denominator is not a power of ten. Our first task is to write this fraction in an equivalent form as a fraction whose denominator is a power of ten. In this case we easily find*

$$\frac{3}{5} = \frac{6}{10}.$$

Now, we convert $\frac{6}{10}$ *to a decimal, finding*

$$\frac{3}{5} = \frac{6}{10} = 0.6.$$

Another way of proceeding would be to observe that

$$\frac{3}{5} = 3 \div 5 = \begin{array}{r|l} -\ 3 & 5 \\ \underline{0} & 0.6 \\ -\ 30 & \\ \underline{30} & \\ 0 & \end{array}$$

◄

In the above example we said that we need to convert the fraction to an equivalent one whose denominator is a power of ten. Of course, we know that there are infinitely many ways of doing that. Which one to take, then? The beauty of the method is that it works regardless of your choice. For example, you could have written

$$\frac{3}{5} = \frac{600}{1000} = 0.600 = 0.6,$$

following our convention of deleting trailing zeroes.

165 Example Write in decimal notation: $\frac{13}{8}$.

Solution: ► *First observe that* $\mathbf{8 = 2^3}$. *The smallest number we need to multiply* $\mathbf{8}$ *by in order to get a power of ten is* $\mathbf{5^3 = 125}$ *and then* $\mathbf{2^3 5^3 = 1000}$. *Thus*

$$\frac{13}{8} = \frac{13 \cdot 125}{8 \cdot 125} = \frac{1625}{1000} = 1.625.$$

Another way of proceeding would be to observe that

$$\frac{13}{8} = 13 \div 8 \rightsquigarrow \begin{array}{r} 1 \\ 8\overline{)13} \\ \underline{8} \\ 5 \end{array} \rightsquigarrow \begin{array}{r} 1.625 \\ 8\overline{)13.000} \\ \underline{8} \\ 50 \\ \underline{48} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

◄

166 Example Write in decimal notation: $\frac{7}{25}$.

Solution: ► *First observe that* $\mathbf{25 = 5^2}$. *The smallest number we need to multiply* $\mathbf{25}$ *by in order to get a power of ten is* $\mathbf{2^2 = 4}$ *and then* $\mathbf{2^2 5^2 = 100}$. *Thus*

$$\frac{7}{25} = \frac{7 \cdot 4}{25 \cdot 4} = \frac{28}{100} = 0.28.$$

Another way of proceeding would be to observe that

$$\frac{7}{25} = 7 \div 25 = \begin{array}{rl|l} - & 7 & 25 \\ & \underline{0} & 0.28 \\ - & 70 & \\ & \underline{50} & \\ - & 200 & \\ & \underline{200} & \\ & 0 & \end{array}$$

◄

167 Example Write in decimal notation: $\frac{1}{3}$.

Solution: ► *Since the denominator is neither divisible by* $\mathbf{2}$ *nor* $\mathbf{5}$ *only, there is no way we can find an equivalent fraction with power of ten on the denominator equivalent to this fraction. We* *proceed by long division, Another way of proceeding would be to observe thatWe have,*

$$\frac{1}{3} = 1 \div 3 \rightsquigarrow 3\overline{)1} \rightsquigarrow 3\overline{)1.0} \rightsquigarrow 3\overline{)1.00}$$

0	0.3	0.33
3)1	3)1.0	3)1.00
0	0	0
1	10	10
	9	9
	1	10
		9
		1

It is not going to take long to arrive at the conclusion that the remainder will always be **1**, *meaning that there is no way to end the division with a remainder of zero! The meaning of this is that the decimal notation that we are looking for will* ***not*** *be* ***terminating***. *In other words, the digits after the decimal point will never stop. However, there is a pattern that repeats infinitely (in this case the digit* **3**). *For that reason, we say that the decimal expression is* ***repeating***. *In order to indicate the pattern, we will use a horizontal bar on top of it. So, in this case we write,* $\frac{1}{3} = 0.\overline{3}$. *As always, the division may also be displayed as*

```
 _1                 | 3
  0                 |0.333333333
  10                |
 _ 9                |
    10              |
   _ 9              |
      10            |
     _ 9            |
        10          |
       _ 9          |
          10        |
         _ 9        |
            10      |
           _ 9      |
              10    |
             _ 9    |
                10  |
               _ 9  |
                  10|
                 _ 9|
                   1|
```

◂

☞ *Examples 164 and 165 shew two fractions having terminating expansions. Example 167 shews an example of a decimal with a repeating expansion. It is possible (but somewhat hard to prove), that a fraction in reduced form with denominator of the form* 2^m5^n *where* m, n *are natural numbers, will always terminate, while a fraction in reduced form whose denominator does not have this form will always repeat.*

168 Example Convert $\frac{15}{16}$ to decimal.

Solution: ► *Since the denominator is* $16 = 2^4$, *a power of* 2, *the decimal will terminate. In fact, we have*

$$\frac{15}{16} = \frac{15 \cdot 5^4}{16 \cdot 5^4} = \frac{15 \cdot 625}{16 \cdot 625} = \frac{9375}{10000} = 0.9375.$$

Or, performing the long division,

$$\begin{array}{r} 0.9375 \\ 16\overline{)15.0000} \\ \underline{0} \\ 150 \\ \underline{144} \\ 60 \\ \underline{48} \\ 120 \\ \underline{112} \\ 80 \\ \underline{80} \\ 0 \end{array}$$

◄

169 Example Convert $\frac{101}{165}$ into a decimal.

Solution: ► *Again, since the denominator* 165 *is not of the form* 2^m5^n, *the decimal will repeat. Performing the long division,*

$$\begin{array}{l}
\quad\;\; 0.61212 \\
165\overline{)101.00000} \\
\quad\;\underline{0\qquad} \\
\quad\; 1010 \\
\quad\;\underline{\;\,990} \\
\qquad 200 \\
\qquad \underline{165} \\
\qquad\;\; 350 \\
\qquad\;\; \underline{330} \\
\qquad\quad 200 \\
\qquad\quad \underline{165} \\
\qquad\qquad 350 \\
\qquad\qquad \underline{330} \\
\qquad\qquad\;\; 20
\end{array}$$

Since we notice that the remainder will be alternatively 20 *and* 35, *we conclude that this decimal notation is not terminating, it is repeating. The pattern that repeats is* 12. *This means that* $\frac{101}{165} = 0.6\overline{12}$. ◄

The procedure of converting a decimal to a fraction is even easier.

170 Example Convert to a fraction and write the result in lowest terms: 0.125.

Solution: ► *Notice that there are three decimal places. Hence we start with a denominator of* $10^3 = 1000$. *We have*

$$0.125 = \frac{125}{1000} = \frac{1}{8}.$$

◄

171 Example Convert to a fraction and write the result in lowest terms: 0.0028.

Solution: ► *Notice that there are four decimal places. Hence we start with a denominator of* $10^4 = 10000$. *We have*

$$0.0028 = \frac{28}{10000} = \frac{7}{2500}.$$

◄

172 Example Convert to a fraction and write the result in lowest terms: 3.24.

Solution: ► *Notice that there are two decimal places. Hence we start with a denominator of* $10^2 = 100$. *We have*

$$3.24 = \frac{324}{100} = \frac{324}{100} = 3.24.$$

◄

A special kind of decimal fraction is the *percent*. Basically, the notation for percent % means $\div 100$. Thus

$$11\% = \frac{11}{100} = 0.11,$$

for example.

173 Example Convert the percent to a decimal: 55%.

Solution: ► *We have*

$$55\% = \frac{55}{100} = 0.55.$$

◄

174 Example Convert the decimal to a percent: 1.2.

Solution: ► *We have*

$$1.2 = 1 + \frac{2}{10} = \frac{100}{100} + \frac{20}{100} = \frac{120}{100} = 120\%.$$

◄

175 Example From a class of 36, 6 students are absent. Write the percent of students absent in a mixed fraction/percent form.

Solution: ► *This is* $\frac{6}{36} = \frac{1}{6} = \frac{100}{6} \cdot \frac{1}{100} = 16\frac{2}{3}\%$ ◄

We now tackle the problem of comparing decimals. We compare positive decimals *lexicographically*, that is, in much the same manner that we look up words in a dictionary. Let $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ be digits and let P_1 and P_2 be natural numbers. To compare the numbers

$$P_1.a_1a_2a_3\ldots, \qquad P_2.b_1b_2b_3\ldots,$$

we first compare P_1 and P_2. If $P_1 > P_2$, then

$$P_1.a_1a_2a_3\ldots > P_2.b_1b_2b_3\ldots.$$

If $P_1 = P_2$, we now compare a_1 and b_1. If $a_1 > b_1$, then

$$P_1.a_1a_2a_3\ldots > P_2.b_1b_2b_3\ldots.$$

If $P_1 = P_2$ and $a_1 = b_1$, we now compare a_2 and b_2. If $a_2 > b_2$, then

$$P_1.a_1a_2a_3\ldots > P_2.b_1b_2b_3\ldots.$$

If $P_1 = P_2$, $a_1 = b_1$, and $a_2 = b_2$, we now compare a_3 and b_3. If $a_3 > b_3$, then

$$P_1.a_1a_2a_3\ldots > P_2.b_1b_2b_3\ldots,$$

and so on. Thus the procedure compares digits from left to right until we find the first spot where the digits differ.

176 Example $\mathbf{5.1 > 2.99998877}$ since from their integral parts, $\mathbf{5 > 2}$.

177 Example $\mathbf{5.222 > 5.199999}$. Their integral parts are both $\mathbf{5}$, so now we compare the first decimal place. Since $\mathbf{2 > 1}$, we obtain the conclusion.

178 Example $\mathbf{5.212 > 5.209}$. Their integer parts are both $\mathbf{5}$, and their first decimal place are both $\mathbf{2}$, so now we compare the second decimal place. Since $\mathbf{1 > 0}$, we obtain the conclusion.

179 Example $\mathbf{5.02 = 5.02000}$. This is so because $\mathbf{5.02 = 5.02 + 0.00000 = 5.02000}$.

☞ *The preceding example shows that the* trailing zeroes *after the right-most non-zero digit after the decimal point are immaterial.*

180 Example $\mathbf{0.5 < 0.\overline{5}}$.

181 Example $\mathbf{0.\overline{5} < 0.56}$.

182 Example Since $\mathbf{\frac{1}{3} = 0.33333\ldots = 0.\overline{3}}$, it is verified that $\mathbf{0.3 < \frac{1}{3} < 0.34}$.

Homework

Problem 17.1 *Convert into a fraction and express in lowest terms:* $\mathbf{0.204}$

Problem 17.2 *Convert into a decimal:* $\mathbf{\frac{9}{11}}$.

Problem 17.3 *Convert to a decimal:* $\mathbf{\frac{3}{20}}$

Problem 17.4 *Convert to a decimal:* $\mathbf{\frac{3}{200}}$

Problem 17.5 *Convert to a decimal:* $\mathbf{\frac{7}{15}}$

Problem 17.6 *Convert to a decimal:* $\mathbf{\frac{1}{7}}$

Problem 17.7 *Convert to a fraction:* $\mathbf{24.24}$

Problem 17.8 *Convert to a percent:* $\mathbf{0.12}$

Problem 17.9 *Convert to a percent:* $\mathbf{0.012}$

Problem 17.10 *Arrange in increasing order:*

$$\mathbf{0.45, \quad 0.445, \quad 0.4\overline{4}, \quad 0.4\overline{45}, \quad 0.446.}$$

Problem 17.11 *Arrange in increasing order:*

$$\mathbf{\frac{2}{3}, \quad \frac{3}{4}, \quad 0.67, \quad 0.666.}$$

Problem 17.12 *In the infinite repeating decimal*

$$\mathbf{0.567895678956789\ldots = 0.\overline{56789},}$$

what is the 2008th digit after the decimal point?

Problem 17.13 *Add and express your result in fraction form:* $\mathbf{0.2 + \frac{1}{4}}$

Problem 17.14 *Multiply and express your result in fraction form. If your answer is an improper fraction, leave the result in this form:* $\mathbf{\left(\frac{2}{5}\right)(3.3)}$

Problem 17.15 *You start with* $\mathbf{\$100}$. *You give* $\mathbf{20\%}$ *to your friend. But it turns out that you need the* $\mathbf{\$100}$ *after all in order to pay a debt. By what percent should you increase your current amount in order to restore the* $\mathbf{\$100}$*? The answer is* not $\mathbf{20\%}$*!*

18 Addition, Subtraction, and Multiplication

The addition and subtraction algorithms explained in the lesson on integers extend to addition and subtraction of decimals. That is, in order to add or subtract decimals, we line the numbers at the decimal point and proceed as if they were integers.

183 Example Add: $123.456789 + 98.7654321$.

Solution: ► *We line up the numbers at the decimal point and add:*

$$\begin{array}{rr} & {\scriptstyle 1\,1\,1\,1\,1\,1\,1\,1} \\ & 123.456789 \\ + & 98.7654321 \\ \hline & 222.2222211 \end{array}$$

◄

184 Example Subtract: $123.456789 - 98.7654321$.

Solution: ► *We line up the numbers at the decimal point and subtract:*

$$\begin{array}{rr} & 123.4567890 \\ - & 98.7654321 \\ \hline & 24.6913569 \end{array}$$

Notice that we added a zero to the end of 123.456789 *in order to line up with all the decimals of* 98.7654321. ◄

Multiplication of decimals is only slightly more complicated. Suppose we wanted to perform the product $1.1 \cdot 1.23$. Since

$$1.1 \cdot 1.23 = \frac{11}{10} \cdot \frac{123}{100} = \frac{1353}{1000} = 1.353,$$

that is, we simply multiplied the numbers, with no regard of the decimal point, and then, counting from the rightmost digit of the product, we moved the decimal point to the left as many places as there appear in the factors, in our case $1 + 2 = 3$ places.

185 Example Multiply: $12.121 \cdot 34.34$.

Solution: ► *There are* $3+2=5$ *decimal places in the factors. We multiply as we would ordinary integers and then from the rightmost digit of the product we move the decimal point five units to the left. We have*

$$\begin{array}{rr} & 12.121 \\ \times & 34.34 \\ \hline & 48484 \\ & 36363 \\ & 48484 \\ & 36363 \\ \hline & 416.23514 \end{array}$$

◄

We now present some examples involving percents.

186 Example Find 24% of 80.

Solution: ► *Essentially, we interpret "of" as multiplication:*

$$(24\%)(80) = (0.24)(80) = 19.2.$$

◄

Homework

Problem 18.1 *How much would it cost to fill a 12-gallon tank of gasoline at \$3.18 per gallon?*

Problem 18.2 *Compute:* $11\% + 2.1$.

Problem 18.3 *Compute:* $\left(\frac{3}{4}\right)(1.24)$.

Problem 18.4 *Compute:* $(2.1\%)(0.12\%)$.

Problem 18.5 *Compute:* $(0.1)(0.02)(0.003)(0.0004)$.

Problem 18.6 *Compute:* $0.1^2 + 0.3 \cdot 0.4$.

Problem 18.7 *You have* $\$8.00$. *You buy* 2 *oranges and* 3 *juices. Each orange costs* $\$0.35$ *and each juice costs* $\$0.90$. *How much do you have left?*

Problem 18.8 *Find* 30% *of* 810.

Problem 18.9 840 *is* 28% *of what number?*

Problem 18.10 *What percent of* 408 *is* 34*?*

Problem 18.11 *Hillary trades goats for a* 40% *profit. If the original price of a goat is* $\$90$, *find the new selling price.*

Problem 18.12 *After a night of beer and pizza, the bill comes to a total of \$34.20, including sales tax. If you are leaving a 15% tip, how much will you end up paying?*

Problem 18.13 *The area of a living room in a one-bedroom apartment is 60% of the total area (living-room, bedroom, and kitchen). The area of a bedroom is 40% of the living-room area. Find the area of a kitchen, if the living-room area is 468 sq. feet.*

Problem 18.14 *The mass of* 200 *kg cucumbers consists of* 99% *of water. The cucumbers are drying out due to the sun, till the mass consists of* 98% *of water. Determine the weight of the cucumbers now.*

Problem 18.15 *At a certain college* 99% *of the* 100 *students are female, but only* 98% *of the students living on campus are female. If some females live on campus, how many students live off campus?*

Problem 18.16 *Abdullah had* 200 *camels. Eighty died, and all but* 25% *of those remaining ran away. How many were left?*

Problem 18.17 *You mix in a recipient* 3 *litres of water and* 1 *litre of juice. The juice is composed of* 20% *pulp and* 80% *water. In this new* 4*-litre mixture, what percent is water? What percent is pulp?*

Problem 18.18 *What would be the price of a* $5\frac{1}{2}$*-mile trip with the following taxi-cab company?*

SMILING CAMEL TAXI SERVICES	
First $\frac{1}{4}$ *mi*	\$ .85
Additional $\frac{1}{4}$ *mi*	\$ .40

Problem 18.19 *If* a *is 50 % larger than* c, *and* b *is 25 % larger than* c, *then* a *is what percent larger than* b*?*

Problem 18.20 *Consider the two infinite repeating decimals*

$$a = 0.\overline{12345} = 0.1234512345\ldots; \qquad b = 0.\overline{98765} = 0.9876598765\ldots.$$

If the sum $a+b$ *is written as an improper fraction in lowest terms* $\frac{p}{q}$, *find* $p^2 + q^2$.

19 Division

We now consider division of decimals. Let us start with the case where the divisor is a natural number. For example:

$$3.99 \div 3 = \frac{399}{100} \div \frac{3}{1} = \frac{399}{100} \cdot \frac{1}{3} = \frac{399}{100 \cdot 3} = \frac{3 \cdot 133}{100 \cdot 3} = \frac{133}{100} = 1.33.$$

In many cases, however, the numbers may not be "nice" enough to be handled in the manner of the preceding example. We would like to know, therefore, a version of the the long division algorithm allowing us to perform more intricate calculations.

The first thing to do is to divide the natural number before the decimal point by the divisor (in the usual way). This basically means that in the first step of the division we can not go beyond the point. We go on with the usual process. When bringing down the first digit after the decimal point, before dividing, we need to put a decimal point in the quotient and then proceed with the division.

Keep in mind that if necessary, we can always add zeros at the end of the decimal number (it would not change its value), so we can go on with the division until either the remainder is zero, or we determine that the quotient will not be terminating but a repeating decimal.

187 Example Shew that $1.21 \div 2 = 0.605$ by means of the long division algorithm.

Solution: ► *Notice the sequence of divisions:*

```
     0             0.6             0.60             0.605
  ------         ------           ------           -------
 2)1.21   ⇝    2)1.21    ⇝     2)1.21    ⇝      2)1.210
   0              0                0                0
   --             --               --               --
   1              12               12               12
                  12               12               12
                  --               --               --
                   0                01               01
                                     0                0
                                    --               --
                                     1                10
                                                      10
                                                      --
                                                       0
```

We explain the steps as follows.

- *We divide the natural number before the decimal point by the divisor.*
- *Now we bring down the first digit after the decimal point (2), put a point at the quotient and then divide.*
- *Continue with the division bringing down the next digit (1).*

- *Since the remainder is not zero and there are no more digits to bring down, we may add a zero at the end of the dividend and continue with the division.*

◄

We will now provide more examples, but without narrating the explanation.

188 Example Prove that $\mathbf{24.382} \div \mathbf{12} = 2.0318\overline{3}$

Solution: ► *We perform the long division*

$$
\begin{array}{r}
2.031833 \\
\mathbf{12})\overline{24.382000} \\
\underline{24} \\
03 \\
\underline{0} \\
38 \\
\underline{36} \\
22 \\
\underline{12} \\
100 \\
\underline{96} \\
40 \\
\underline{36} \\
40 \\
\underline{36} \\
4
\end{array}
$$

We can see that from this point forward, the remainder will always be **4**. *That means that the decimal expression of the quotient is not terminating and repeating. In this case the pattern that repeats is the digit* **3**. ◄

189 Example Prove that $\mathbf{135.36} \div \mathbf{3} = 45.12$

Solution: ► *We perform the long division*

$$\begin{array}{r} 45.12 \\ 3\overline{)135.36} \\ \underline{12} \\ 15 \\ \underline{15} \\ 03 \\ \underline{3} \\ 06 \\ \underline{6} \\ 0 \end{array}$$

◄

Let us combine a few operations with fractions and decimals.

190 Example Perform the operation and write the result in decimal form:

$$\frac{3}{4} + 1.2\% + 1.2.$$

Solution: ► *We have*

$$\frac{3}{4} + 1.2\% + 1.2 = 0.75 + 0.012 + 1.2 = 1.962.$$

◄

191 Example Write as a fraction: $\dfrac{1}{0.1 + \dfrac{1}{0.2}}$.

Solution: ► *We have*

$$\begin{aligned} \frac{1}{0.1 + \frac{1}{0.2}} &= \frac{1}{0.1 + 5} \\ &= \frac{1}{\frac{1}{10} + 5} \\ &= \frac{1}{\frac{51}{10}} \\ &= \frac{10}{51}. \end{aligned}$$

◄

Here are some percent problems.

192 Example What percent of 72 is equal to 45?

Solution: ► *We know that the multiplication of the decimal corresponding to the percent times* 72 *should be equal to* 45*. But this is the same as saying that* 45 *divided by* 72 *is equal to the decimal corresponding to the percent. Since*

$$45 \div 72 = 0.625,$$

we conclude that 62.5% *of* 72 *is* 45*.* ◄

193 Example 45 is 75% of which number?

Solution: ► *If we multiply* 0.75 *by the given number then we obtain* 45*. Hence, the given number is*

$$45 \div 0.75 = 4500 \div 75 = 60.$$

◄

194 Example If a store offers a 35% discount on a item, and the price after the discount is \$52, what was the original price of the item?

Solution: ► *First we need to understand that if* 35% *was discounted, then the final price is actually* 65% *of the original price. Therefore, we know that* 65% *of the original price is equal to* 52*. This means that* 0.65 *times the original price will be equal to* 52*. Then the original price is the result of dividing* 52 *by* 0.65*:*

$$52 \div 0.65 = 5200 \div 65 = 80,$$

from where it follows that the original price of the item was \$80. ◄

195 Example A company reports that 23% of its employees are smokers. If there are 92 smokers in the company, how many employees in total work there?

Solution: ► *If* x *is the number of employees, we want to solve the proportion*

$$\frac{23x}{100} = 92 \implies x = 92 \cdot \frac{100}{23} = 400,$$

and so 400 *employees work at the company.* ◄

Homework

Problem 19.1 *Calculate:* $1020.4016 \div 637.751$.

Problem 19.2 *Calculate:* $1.23 \div 0.006$.

Problem 19.3 *Calculate:* $1.023 \div 0.15$.

Problem 19.4 *Calculate:* $1.023 \div 0.9$.

Problem 19.5 *Calculate:* $8 \div 7$.

Problem 19.6 *Calculate:* $12 \div 11$.

Problem 19.7 *Calculate:* $\dfrac{1-0.1}{2-0.2}$.

Problem 19.8 *Calculate:* $\dfrac{1-0.01}{2-0.02}$.

Problem 19.9 *Calculate:* $\dfrac{123}{12.3-0.123}$.

Problem 19.10 *Convert to a fraction:* $\dfrac{1}{0.1+\dfrac{1}{0.2+\dfrac{1}{0.3}}}$.

Problem 19.11 *Calculate:* $\frac{0.1+0.3+0.5+0.7+0.9}{0.2+0.4+0.6+0.8}$.

Problem 19.12 *Jill bought* **6** *pounds of apples for* **$1.38**. *How much did each pound cost?*

Problem 19.13 *You bought some furniture for $424.53, price which included a 6% sales tax. What was the price of piece, before sales tax?*

Problem 19.14 *The total weight of a pile of* **500** *of salt crystals is* **6.5** *g. What is the average weight of a salt crystal?*

Problem 19.15 *The only buttons that work on a calculator are* [4], [÷], [×], [+], [−], [(], [)], [!], [.], *and* [=]. *Produce every natural number from* **1** *to* **20** *on the screen of this calculator with the extra constraint that you must press the* [4] **exactly** *four times! For example, to get* **5** *on the screen, you could type*

[(] [4] [×] [4] [+] [4] [)] [÷] [4] [=].

Part VI

Real Numbers

20 Integers: Addition and Subtraction

God created the integers. Everything else is the work of Man. *-Leopold KROENECKER*

The introduction of fractions in previous chapters helped solve the problem that the natural numbers are not closed under division. We now solve the problem that the natural numbers are not closed under subtraction.

196 Definition A natural number not equal to $\mathbf{0}$ is said to be *positive*. The set

$$\{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \ldots\}$$

is called the set of *positive integers*.

197 Definition Given a natural number n, we define its opposite $-n$ as the unique number $-n$ such that

$$n + (-n) = (-n) + n = \mathbf{0}.$$

☞ *Observe that* $-\mathbf{0} = \mathbf{0}$ *is the only integer that equals its opposite.*

The collection

$$\{-\mathbf{1}, -\mathbf{2}, -\mathbf{3}, -\mathbf{4}, -\mathbf{5}, \ldots\}$$

of all the opposites of the natural numbers is called the set of *negative integers*. The collection of natural numbers together with the negative integers is the set of *integers*, which we denote by the symbol[1] $\mathbb{Z}$.

A graphical representation of the integers is given in figure 20.1.

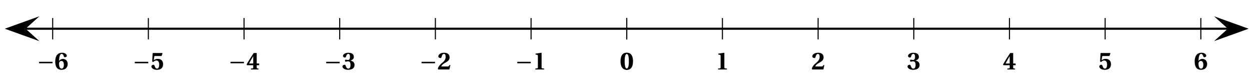


Figure 20.1: The Integers $\mathbb{Z}$.

198 Example If each of the markings in figure 20.2 represents a unit, then we plot and label the numbers $-\mathbf{5}$, $-\mathbf{1}$, and $\mathbf{4}$.

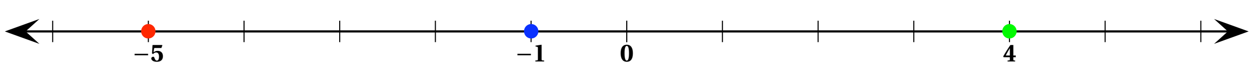


Figure 20.2: Example 198.

There seems to be no evidence of usage of negative numbers by the Babylonians, Pharaonic Egyptians, or the ancient Greeks. It seems that the earliest usage of them came from China and India. In

[1]From the German word for number: *Zählen*.

the 7th Century, negative numbers were used for bookkeeping in India. The Hindu astronomer Brahmagupta, writing around A.D. 630, shews a clear understanding of the usage of negative numbers.

Thus it took humans a few millennia to develop the idea of negative numbers. Since, perhaps, our lives are more complex now, it is not so difficult for us to accept their existence and understand the concept of negative numbers.

Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. If $a > 0$, then $-a < 0$. If $b < 0$, then $-b > 0$. Thus either the number, or its "mirror reflexion" about 0 is positive, and in particular, for any $a \in \mathbb{Z}$, $-(-a) = a$. This leads to the following definition.

199 Definition Let $a \in \mathbb{Z}$. The *absolute value* of a is defined and denoted by

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0, \end{cases}$$

200 Example $|5| = 5$ since $5 > 0$. $|-5| = -(-5) = 5$, since $-5 < 0$.

Letters have no idea of the sign of the numbers they represent. Thus it is a **mistake** *to think, say, that $+x$ is always positive and $-x$ is always negative.*

We would like to define addition, subtraction, multiplication and division in the integers in such a way that these operations are consistent with those operations over the natural numbers and so that they enjoy closure, commutativity, associativity, and distributivity under addition and multiplication.

We start with addition. Recall that we defined addition of two natural numbers and of two fractions as the concatenation of two segments. We would like this definition to extend to the integers, but we are confronted with the need to define what a "negative segment" is. This we will do as follows. If $a < 0$, then $-a > 0$. We associate with a a segment of length $|-a|$, but to the left of 0 on the line, as in figure 20.3.

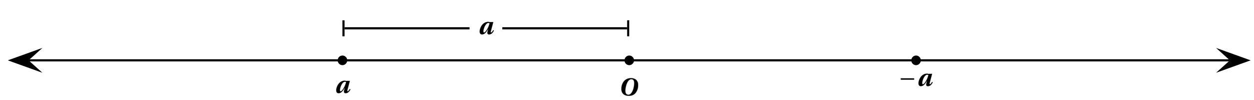


Figure 20.3: A negative segment. Here $a < 0$.

Hence we define the addition of integers a, b, as the concatenation of segments. Depending on the sign of a and b, we have four cases. (We exclude the cases when at least one of a or b is zero, these cases being trivial.)

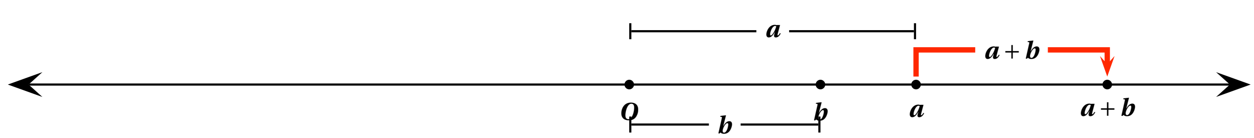


Figure 20.4: $a + b$ with $a > 0$, $b > 0$.

201 Example (Case $a > 0, b > 0$) To add b to a, we first locate a on the line. From there, we move b units right (since $b > 0$), landing at $a + b$. Notice that this case reduces to addition of natural numbers, and hence, we should obtain the same result as for addition of natural numbers. This example is illustrated in figure 20.4. For a numerical example (with $a = 3$, $b = 2$), see figure 20.5.

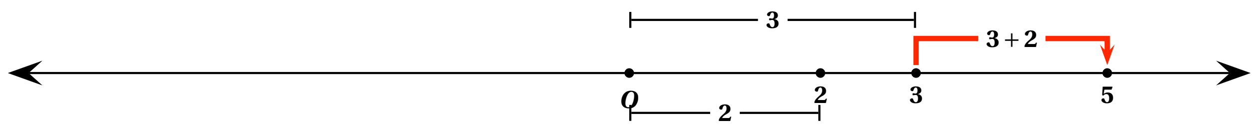


Figure 20.5: $3+2$.

202 Example (Case $a<0, b<0$) To add b to a, we first locate a on the line. From there, we move b units left (since $b<0$), landing at $a+b$. This example is illustrated in figure 20.6. For a numerical example (with $a=-3$, $b=-2$), see figure 20.7.

☞ *Examples 201 and 202 conform to the following intuitive idea. If we associate positive numbers to "gains" and negative numbers to "losses" then a "gain" plus a "gain" is a "larger gain" and a "loss" plus a "loss" is a "larger loss."*

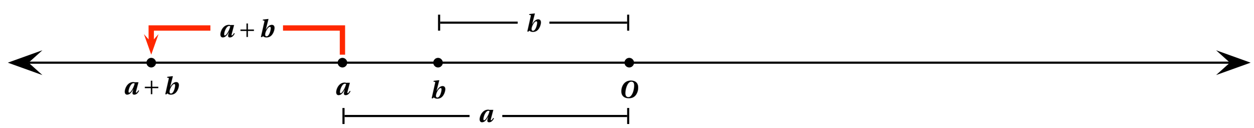


Figure 20.6: $a+b$ with $a<0$, $b<0$.

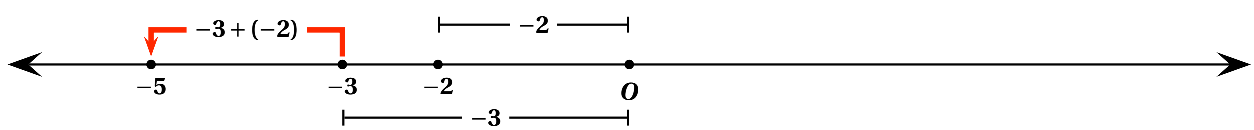


Figure 20.7: $-3+(-2)$.

203 Example We have,

$$(+1)+(+3)+(+5)=+9,$$

since we are adding three gains, and we thus obtain a larger gain.

204 Example We have,

$$(-11)+(-13)+(-15)=-39,$$

since we are adding three losses, and we thus obtain a larger loss.

We now tackle the cases when the summands have opposite signs. In this case, borrowing from the preceding remark, we have a "gain" plus a "loss." In such a case it is impossible to know before hand whether the result is a gain or a loss. The only conclusion we could gather, again, intuitively, is that the result will be in a sense "smaller", that is, we will have a "smaller gain" or a smaller loss." Some more thinking will make us see that if the "gain" is larger than the loss, then the result will be a "smaller gain," and if the "loss" is larger than the "gain" then the result will be a "smaller loss."

205 Example (Case $a<0, b>0$) To add b to a, we first locate a on the line. Since $a<0$, it is located to the left of O. From there, we move b units right (since $b>0$), landing at $a+b$. This example is illustrated in figure 20.8. For a numerical example (with $a=-3$, $b=2$), see figure 20.9. Again, we emphasise, in the sum $(-3)+(+2)$, the "loss" is larger than the "gain." Hence when adding, we expect a "smaller loss", fixing the sign of the result to be minus.

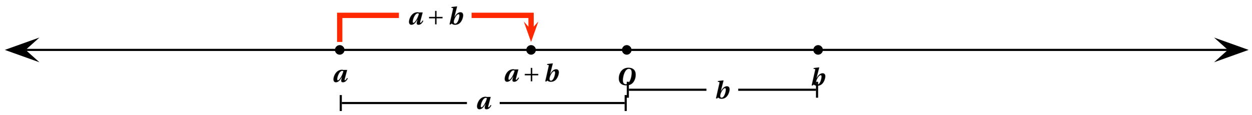


Figure 20.8: $a+b$ with $a<0$, $b>0$.

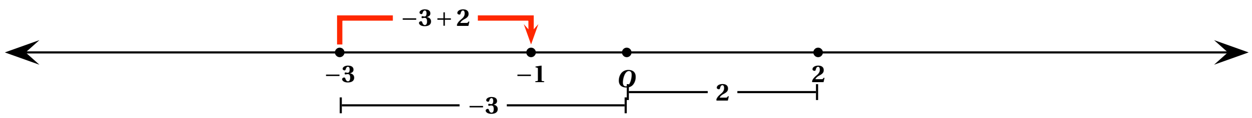


Figure 20.9: $-3+2$.

206 Example (Case $a>0, b<0$) To add b to a, we first locate a on the line. From there, we move b units left (since $b<0$), landing at $a+b$. This example is illustrated in figure 20.10. For a numerical example (with $a=3$, $b=-2$), see figure 20.11.

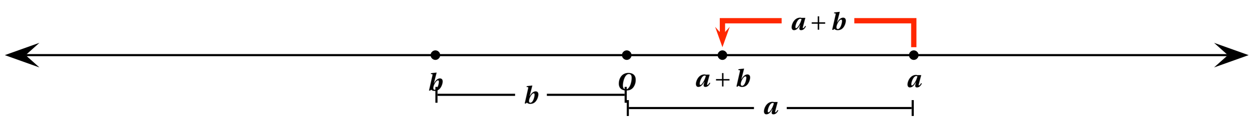


Figure 20.10: $a+b$ with $a>0$, $b<0$.

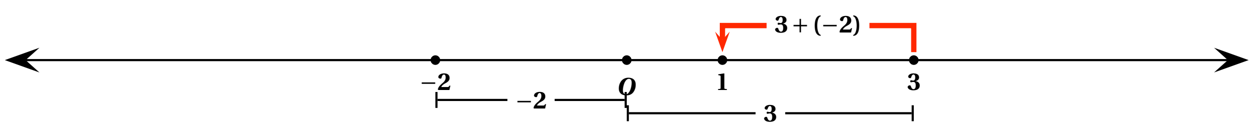


Figure 20.11: $3+(-2)$.

207 Example We have,

$$(+19)+(-21)=-2,$$

since the loss of 21 is larger than the gain of 19 and so we obtain a loss.

208 Example We have,

$$(-100)+(+210)=+110,$$

since the loss of 100 is smaller than the gain of 210 and so we obtain a gain.

We now turn to subtraction. We define subtraction in terms of addition.

209 Definition Subtraction is defined as

$$a-b=a+(-b).$$

210 Example We have, $(+8)-(+5)=(+8)+(-5)=3$.

211 Example We have, $(-8)-(-5)=(-8)+(+5)=-3$.

212 Example We have, $(+8)-(-5)=(+8)+(+5)=13$.

213 Example We have, $(-8)-(+5)=(-8)+(-5)=-13$.

Homework

Problem 20.1 *Perform the following operations mentally.*

1. $(-9)-(-17)$
2. $(-17)-(9)$
3. $(-9)-(17)$
4. $(-1)-(2)-(-3)$
5. $(-100)-(101)+(-102)$
6. $|-2|-|-2|$
7. $|-2|-(-|2|)$
8. $|-100|+(-100)-(-(-100))$

Problem 20.2 $1-2+3-4+5-6+7-8+9-10=$

Problem 20.3 $-2-4-6-8-10+1+3+5+7+9=$

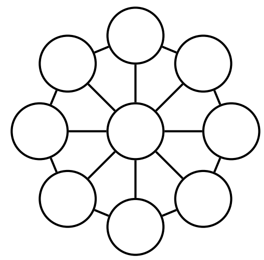
Problem 20.4 $(1-2)-(3-4)+(5-6)-(7-8)+(9-10)=$

Problem 20.5 $a-2a+3a-4a+5a=$

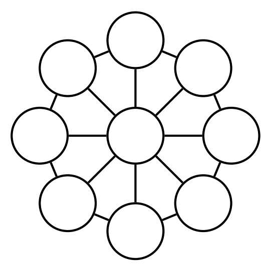
Problem 20.6 *Place the nine integers*

$$\{-4,-3,-2,-1,0,1,2,3,4\}$$

exactly once in the diagram below so that every diagonal sum be the same.



Problem 20.7 *Place the nine integers* $\{-2,-1,0,1,2,3,4,5,6\}$ *exactly once in the diagram below so that every diagonal sum be the same.*



21 Integers: Multiplication and Division

We now explore multiplication of integers. Again, we would like the multiplication rules to be consistent with those we have studied for natural numbers. This entails that, of course, that if we multiply two positive integers, the result will be a positive integer. What happens in other cases? Suppose $a > 0$ and $b < 0$. We will prove that $ab < 0$. Observe that

$$a(b-b) = 0 \implies ab - ab = 0 \implies ab + a(-b) = 0 \implies ab = -a(-b).$$

Since $-b > 0$, $a(-b)$ is the product of two positive integers, and hence positive. Thus $-a(-b)$ is negative, and so $ab = -a(-b) < 0$. We have proved that the product of a positive integer and a negative integer is negative. Using the same trick we can prove that

$$(-x)(-y) = xy.$$

If $x < 0$, $y < 0$, then both $-x > 0$, $-y > 0$, hence the product of two negative integers is the same as the product of two positive integers, and hence positive. We have thus proved the following rules:

$$(+)(+) = (-)(-) = +, \qquad (+)(-) = (-)(+) = -.$$

Intuitively, you may think of a negative sign as a reversal of direction on the real line. Thus the product or quotient of two integers different sign is negative. Two negatives give two reversals, which is to say, no reversal at all, thus the product or quotient of two integers with the same sign is positive. The sign rules for division are obtained from and are identical from those of division.

214 Example We have,

$$(-2)(5) = -10, \qquad (-2)(-5) = +10, \qquad (+2)(-5) = -10, \qquad (+2)(+5) = +10.$$

215 Example We have,

$$(-20) \div (5) = -4, \qquad (-20) \div (-5) = +4, \qquad (+20) \div (-5) = -4, \qquad (+20) \div (+5) = +4.$$

The rules of operator precedence discussed in the section of natural numbers apply.

216 Example We have,

$$\begin{aligned} \frac{(-8)(-12)}{3} + \frac{30}{((-2)(3))} &= \frac{96}{3} + \frac{30}{(-6)} \\ &= 32 + (-5) \\ &= 27. \end{aligned}$$

217 Example We have,

$$\begin{aligned} (5-12)^2 - (-3)^3 &= (-7)^2 - (-27) \\ &= 49 + 27 \\ &= 76. \end{aligned}$$

As a consequence of the rule of signs for multiplication, a product containing an odd number of minus signs will be negative and a product containing an even number of minus signs will be positive.

218 Example

$$(-2)^2 = 4, \qquad (-2)^3 = -8, \qquad (-2)^{10} = 1024.$$

☞ *Notice the difference between, say, $(-a)^2$ and $-a^2$. $(-a)^2$ is the square of $-a$, and hence it is always non-negative. On the other hand, $-a^2$ is the opposite of a^2, and therefore it is always non-positive.*

219 Example We have,

$$5 + (-4)^2 = 5 + 16 = 21,$$
$$5 - 4^2 = 5 - 16 = -11,$$
$$5 - (-4)^2 = 5 - 16 = -11.$$

Homework

Problem 21.1 *What is the value of* $((1 \cdot 2 \div (3 \cdot 4) - 5) \cdot 6 - 7) \div (8 \cdot 9)$*?*

Problem 21.2 *Calculate:* $\dfrac{(10)^3 + (-5)^3 + (1)^3 - 3(10)(-5)(1)}{(10) + (-5) + (1)}$

Problem 21.3 *Calculate:* $(2 \cdot (-3) + 1)^2 - ((-5) \cdot 4 + 2 \cdot 9)^3$.

Problem 21.4 *Calculate:* $(2 \cdot (-3) + 1)^2 - ((-5) \cdot 4 + 2 \cdot 9)^3$.

Problem 21.5 *Calculate:*

$$(2 + 3 + 4)(2^2 + 3^2 + 4^2 - 2 \cdot 3 - 3 \cdot 4 - 4 \cdot 2).$$

Problem 21.6 *Complete the "crossword" puzzle with* 1*'s or* -1*'s.*

-1	$\cdot$		$=$	
$\cdot$	■	$\cdot$	■	$\cdot$
	$\cdot$	-1	$=$	
$=$	■	$=$	■	$=$
	$\cdot$		$=$	-1

22 Rational Numbers

Bridges would not be safer if only people who knew the proper definition of a real number were allowed to design them. *-N. David MERMIN*

220 Definition The set of *negative fractions* is the set

$$\left\{-\frac{a}{b} : a \in \mathbb{N}, a > 0, b \in \mathbb{N}, b > 0\right\}.$$

The set of positive fractions together with the set of negative fractions and the number 0 form the set of *rational numbers*, which we denote by $\mathbb{Q}$.

The rules for operations with rational numbers derive from those of operations with fractions and with integers. Also, the rational numbers are closed under the four operations of addition, subtraction, multiplication, and division. A few examples follow.

221 Example We have

$$\begin{aligned}\frac{2}{5}\cdot\frac{15}{12}-\frac{7}{10}\div\frac{14}{15} &= \frac{2}{5}\cdot\frac{15}{12}-\frac{7}{10}\cdot\frac{15}{14}\\ &= \frac{\not 2}{\not 5}\cdot\frac{\not 5\cdot\not 3}{\not 2\cdot 2\cdot\not 3}-\frac{\not 7}{2\cdot\not 5}\cdot\frac{3\cdot\not 5}{2\cdot\not 7}\\ &= \frac{1}{2}-\frac{3}{4}\\ &= \frac{2}{4}-\frac{3}{4}\\ &= -\frac{1}{4}.\end{aligned}$$

It can be proved that any rational number has a decimal expansion which is either periodic (repeats) or terminates, and that viceversa, any number with either a periodic or a terminating expansion is a rational number. For example, $\frac{1}{4} = 0.25$ has a terminating decimal expansion, and $\frac{1}{11} = 0.0909090909\ldots = 0.\overline{09}$ has a repeating one. By long division you may also obtain

$$\frac{1}{7} = 0.\overline{142857}, \qquad \frac{1}{17} = 0.\overline{0588235294117647},$$

and as you can see, the periods may be longer than what your calculator can handle.

Homework

Problem 22.1 *Find the value of* $\frac{5}{6}-\left(-\frac{5}{6}\right)^2$.

Problem 22.2 *Find the value of*

$$((1\cdot 2\div(3\cdot 4)-5)\cdot 6-7)\div(8\cdot 9).$$

Problem 22.3 *Find*

$$(0.1)(-0.3)(0.5)(-0.7)(0.9)-(-0.2)(-0.4)(-0.6)(-0.8).$$

Problem 22.4 *Find* $\dfrac{\frac{3}{4}-0.7}{0.2-\frac{2}{5}}$ *and convert the result into a fraction.*

Problem 22.5 *Suppose that you know that* $\frac{1}{3} = 0.333333\ldots = 0.\overline{3}$. *What should* $0.1111\ldots = 0.\overline{1}$ *be?*

Problem 22.6 *Find the value of* $121(0.\overline{09})$.

Problem 22.7 *Compute:*

$$\frac{1}{2+\frac{3}{4-\frac{5}{6+\frac{7}{8-\frac{9}{10}}}}}.$$

23 Irrational Numbers

What about numbers whose decimal expansion is infinite and does not repeat? This leads us to the following definition.

222 Definition A number whose decimal expansion is infinite and does not repeat is called an *irrational number.*

From the discussion above, an irrational number is one that cannot be expressed as a fraction of two integers.

223 Example Consider the number

$$0.101001000100001000001000000 1\ldots,$$

where the number of 0's between consecutive 1's grows in sequence: $1,2,3,4,5,\ldots$. Since the number of 0's is progressively growing, this infinite decimal does not have a repeating period and hence must be an irrational number.

Using my computer, when I enter $\sqrt{2}$ I obtain as an answer

$$1.4142135623730950488016887242097.$$

Is this answer exact? Does this decimal repeat? It can be proved that the number $\sqrt{2}$ is irrational, hence the above answer is only an approximation and the decimal does not repeat. The first proof of the irrationality of $\sqrt{2}$ is attributed to Hippasus of Metapontum, one of the disciples of Pythagoras (c 580 BC–c 500 BC).[1] The Greek world view at that time was that all numbers where rational, and hence this discovery was anathema to the Pythagoreans who decided to drown Hipassus for his discovery.

☞ *It can be proved that if n is a natural number that is not a perfect square, then $\sqrt{n}$ is irrational. Hence $\sqrt{2},\sqrt{3},\sqrt{5},\sqrt{6},\sqrt{7},\sqrt{8},\sqrt{10}$, etc., are all irrational.*

In 1760, Johann Heinrich Lambert (1728 - 1777) proved that π is irrational.

☞ *In particular, then, it would be incorrect to write $\pi = 3.14$, or $\pi = \frac{22}{7}$, or $\pi = \frac{355}{113}$, etc., since π is not rational. All of these are simply approximations, and hence we must write $\pi \approx 3.14$, $\pi \approx \frac{22}{7}$, or $\pi \approx \frac{355}{113}$, etc.*

224 Definition The set of *real numbers*, denoted by $\mathbb{R}$, is the collection of rational numbers together with the irrational numbers.

Homework

Problem 23.1 *Use a calculator to round $\sqrt{2}+\sqrt{3}+\sqrt{5}$ to two decimal places.*

Problem 23.2 *Use a calculator to round $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{5}$ to two decimal places.*

Problem 23.3 *Let A be a digit. The four-digit number $12A6$ is a perfect square. What is the value of A?*

[1]The Pythagoreans were akin to religious cults of today. They forbade their members to eat beans, dedicated their lives to Mathematics and Music, and believed that the essence of everything in the world was *number*.

Problem 23.4 *Let a, b be positive real numbers. Is it always true that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$?*

A Multiple-Choice Exam Questions

Ancient Numerals

1. Convert **123** to Egyptian numerals.

   (A) 𓍢𓍢𓍢𓎆𓎆𓏺 (B) 𓍢𓎆𓎆𓏺𓏺𓏺 (C) 𓍢𓎆𓎆𓎆𓏺𓏺 (D) 𓍢𓍢𓎆𓏺𓏺𓏺 (E) none of these

2. Convert **123** to Roman numerals.

   (A) CXXIII (B) CXXXII (C) CCCXII (D) CCXIII (E) none of these

3. Convert the Egyptian numeral 𓍢𓍢𓍢𓎆𓎆𓏺 to decimal.

   (A) **123** (B) **312** (C) **321** (D) **231** (E) none of these

4. Convert the Roman numeral CCXXXI to decimal.

   (A) **123** (B) **312** (C) **321** (D) **231** (E) none of these

5. If two ♡'s are equivalent to one ♣ and three ♣'s are equivalent to two ♠'s, to how many ♠'s is ♡♡♡♡♣♣♣♣?

   (A) **4** (B) **6** (C) **12** (D) **18** (E) none of these

Positional Notation

6. Convert $\mathbf{123_4}$ to decimal.

   (A) **75** (B) **57** (C) **72** (D) **27** (E) none of these

7. Convert **123** to base-**4**.

   (A) $\mathbf{123_4}$ (B) $\mathbf{1331_4}$ (C) $\mathbf{1323_4}$ (D) $\mathbf{3231_4}$ (E) none of these

8. Convert $\boldsymbol{A2B_{16}}$ to decimal.

   (A) **274** (B) **10211** (C) **2588** (D) **2587** (E) none of these

9. $\mathbf{32_4 + 33_4} =$

   (A) $\mathbf{110_4}$ (B) $\mathbf{120_4}$ (C) $\mathbf{131_4}$ (D) $\mathbf{65_4}$ (E) none of these

10. $\mathbf{52_6 - 33_6} =$

    (A) $\mathbf{15_6}$ (B) $\mathbf{19_6}$ (C) $\mathbf{25_6}$ (D) $\mathbf{14_6}$ (E) none of these

11. Which integer follows the hexadecimal integer $\boldsymbol{A2FF_{16}}$? In other words, what is $\boldsymbol{A2FF_{16} + 1_{16}}$?

    (A) $\boldsymbol{A2A00_{16}}$ (B) $\boldsymbol{A300_{16}}$ (C) $\boldsymbol{B300_{16}}$ (D) $\boldsymbol{A2FF1_{16}}$ (E) none of these

12. $11_2 + 10_2 =$

(A) 100_2 (B) 110_2 (C) 111_2 (D) 101_2 (E) none of these

13. Which of the following is the *wrong* description for the '**6**' in **654321**?

(A) **600000** units

(B) **6000** hundreds

(C) **600** thousands

(D) **60** ten thousands

(E) 6 millions

Symbolical Expression

14. Let x be the unknown quantity. Translate into symbols: "the excess of a number over three."

(A) $\frac{x}{3}$ (B) $x-3$ (C) $3-x$ (D) $\frac{3}{x}$ (E) none of these

15. Let s be the unknown quantity. Translate into symbols: "thrice a number is reduced by its square."

(A) s^2+3s (B) s^2-3s (C) $3s-s^2$ (D) $2s-s^2$ (E) none of these

16. If you had a **$20**'s and b **$50**'s, how much money, in dollars, do you have?

(A) $a+b$ (B) $70ab$ (C) $20a+50b$ (D) $70+a+b$ (E) none of these

17. Rod is currently twice as old as Tití. If Tití is currently t years old, how old will Rod be **10** years from now?

(A) $t+10$ (B) $2t+10$ (C) $\frac{t}{2}+10$ (D) $2t+20$ (E) none of these

18. Rod is currently twice as old as Tití. If Rod is currently r years old, how old was Tití **10** years ago?

(A) $r-10$ (B) $2r-10$ (C) $\frac{r}{2}-10$ (D) $2r-20$ (E) none of these

19. Which of the following expressions best represents: "the excess of a number over **3**"?
(A) $\frac{x}{3}$ (B) $x-3$ (C) $3x$ (D) $\frac{3}{x}$ (E) none of these

20. Which of the following expressions best represents: "the sum of a number plus thrice another is being diminished by **3**"?
(A) $\frac{x+3y}{3}$ (B) $x+3y-3$ (C) $x+3(y-3)$ (D) $x+2y-3$ (E) none of these

21. The algebraic expression $x^2 - \frac{1}{x}$ can be translated as:

    (A) "The square of a number is increased by the reciprocal of the number."

    (B) "The square of a number is reduced by the reciprocal of the number."

    (C) "Twice a number is increased by the reciprocal of the number."

    (D) "Twice a number is decreased by the reciprocal of the number."

    (E) none of these

22. You have t \$10 and w \$20 bills. How much money, in dollars, do you have?

    (A) $t+w$ (B) $10t+20w$ (C) $200tw$ (D) $30+t+w$ (E) none of these

23. You have t \$10 and w \$ 20 bills. How many bank notes (bills) do you have?

    (A) $t+w$ (B) $10t+20w$ (C) $200tw$ (D) $30+t+w$ (E) none of these

24. If my age by **31** December 2020 will be s, what was my age by **31** December **2005**?

    (A) $s-15$ (B) $s+15$ (C) $15-s$ (D) $s-14$ (E) none of these

25. If my age by **31** December 2005 was s, what will my age be by **31** December **2020**?

    (A) $s-15$ (B) $s+15$ (C) $15-s$ (D) $s-14$ (E) none of these

26. If a is an even natural number and b is an odd natural number, how many of the following are even numbers?

    $I: a+b \qquad II: ab \qquad III: a+b-1 \qquad IV: ab+2b$

    (A) none (B) exactly one (C) exactly two (D) exactly three (E) all four

27. I am thinking of a rule that converts the number **3** into the number **20**. Which of the following could *not* be my rule?

    (A) add seventeen

    (B) multiply by five, then add five

    (C) add one, then multiply by five

    (D) square, then add eleven

    (E) cube, then add three

Addition and Subtraction of Natural Numbers

28. Collect like terms: $(2a+3b+5c)+(2b+a+2c)$

    (A) $15abc$ (B) $4a+4b+7c$ (C) $2a+6b+10c$ (D) $3a+5b+7c$ (E) none of these

29. Collect like terms: $(x^2+3x+1)+(2x^2+2x+3)$

    (A) $3x^2+5x+4$ (B) $12x^3$ (C) $2x^2+6x+3$ (D) $2x^2+5x+4$ (E) none of these

30. In the addition

$$\begin{array}{r} 1\bullet 8\bullet \\ +\ \bullet 720 \\ \hline 42\bullet 2 \end{array}$$

what is the sum of the missing digits?

(A) **6** (B) **7** (C) **8** (D) **9** (E) none of these

31. How many different sums can we obtain when we add two different numbers from the set $\{1,2,3,4,5\}$?

(A) **5** (B) **6** (C) **7** (D) **8** (E) none of these

32. You start the day with A dollars. Your uncle Bob gives you enough money to double your amount. Your aunt Rita gives you **12** dollars. You have to pay B dollars in fines, and spent **10** dollars fueling your camel with gas. How much money do you have now?

(A) $2A-B-2$ (B) $2A-B+2$ (C) $3A-B-2$ (D) $3A-B+2$ (E) none of these

33. In the **difference**

$$\begin{array}{r} 5\bullet 2\bullet \\ -\ \bullet 321 \\ \hline 38\bullet 2 \end{array}$$

what is the sum of the missing digits?

(A) **4** (B) **5** (C) **6** (D) **7** (E) none of these

34. In the magic square, the three numbers in each row, in each column, and in each diagonal add up to the same number. When the magic square below is completed, which of the following numbers is *not* used?

13		
	10	
9		7

(A) **6** (B) **8** (C) **12** (D) **14** (E) **15**

35. For a given charity, Adam contributes a dollars, Betty contributes b dollars and Carl contributes c dollars. Dwight, Erin, and Frances are **thieves**, and so from the contributions, Dwight steals d dollars, Erin steals e dollars and Frances steals f dollars. What is the available amount of dollars that the charity has?

(A) $a+b+c-d-e-f$ (B) $a+b+c+d+e+f$ (C) $abc-def$ (D) $\frac{abc}{def}$ (E) none of these

Arithmetic progressions

36. Consider the arithmetic progression $2,9,16,23,\ldots$, where **2** is on the first position, **9** in the second position, etc. Which number occupies the **100**th position?

(A) **695** (B) **702** (C) **100** (D) **700** (E) none of these

37. Which of the following numbers does not belong to the arithmetic progression $2,9,16,23,\ldots$?

$I:68$ $II:30$ $III:72$ $IV:702$

(A) **68** (B) **30** (C) **72** (D) **702** (E) all these numbers belong to the progression

38. Consider the arithmetic progression $3, 14, 25, 36, \ldots$, where 3 is on the first position, 14 is on the second, etc. Which number occupies the 101st position?

(A) 101 (B) 1092 (C) 1103 (D) 1114 (E) none of these

Multiplication

39. $2^2 \cdot 3 + 2 \cdot 3^2 =$

(A) 30 (B) 24 (C) 126 (D) 216 (E) none of these

40. Multiply: $(3x)(5x^2)$.

(A) $15x^3$ (B) $8x^3$ (C) $15x^2$ (D) $15x^4$ (E) none of these

41. Multiply: $(2x+1)(x+3)$.

(A) $2x^2+7x+3$ (B) $2x^2+3$ (C) $3x+4$ (D) $2x^2+4$ (E) none of these

42. When writing all the natural numbers from 100 to 500 inclusive, how many digits have I used?

(A) 1203 (B) 1200 (C) 1500 (D) 400 (E) none of these

43. According to experts the first 4 moves in a chess game can be played in 197299 totally different ways. If it takes 30 seconds to make one move, what time, in seconds, would it take one player to try every possible set of 4 moves?

(A) 5918970 (B) 23675880 (C) 120 (D) 789196 (E) none of these

44. Evaluate: $100 - (5+1)^2 + 3 \cdot 2$.
(A) 70 (B) 94 (C) 80 (D) 82 (E) none of these

Primes and Factorisation

45. What is the prime factorisation of 252?

(A) $2^2 \cdot 3^3 \cdot 7$ (B) $2^2 \cdot 3^2 \cdot 7$ (C) $2^2 \cdot 63$ (D) $2 \cdot 3 \cdot 7$ (E) none of these

46. What is the prime factorisation of 360?

(A) $2^3 \cdot 3^2 \cdot 5$ (B) $2^2 \cdot 3^2 \cdot 5$ (C) $2^3 \cdot 45$ (D) $2 \cdot 3 \cdot 5$ (E) none of these

47. Find $\gcd(252, 360)$.

(A) 36 (B) 72 (C) 216 (D) 2520 (E) none of these

48. Find LCM$(252, 360)$.

(A) 2530 (B) 720 (C) 2160 (D) 90720 (E) none of these

49. Find: LCM$(14, 20, 36)$.
(A) 2 (B) 4 (C) 630 (D) 1260 (E) none of these

50. How many primes are there in the set $\{90, 91, 92, 93, 94, 95, 96, 97, 98, 99\}$?
(A) 1 (B) 2 (C) 3 (D) 4 (E) none of these

Division

51. $100 \div 50 \div 2 =$

(A) 1 (B) 4 (C) 2 (D) 8 (E) none of these

52. $100 \div 50 \cdot 2 =$

(A) 4 (B) 1 (C) 8 (D) 10 (E) none of these

53. Evaluate: $80 \div (5-3)^3$.
(A) 8 (B) 10 (C) 13 (D) 11 (E) none of these

54. If today is Monday, what day will it be 300 days from now?

(A) Sunday (B) Saturday (C) Tuesday (D) Wednesday (E) none of these

55. Amanda and Brenda were selling girl scout cookies. Each package they sold was priced the same. Amanda sold 30 packages, Brenda sold 50 packages, and they made $\$\ 320$. How much money did Amanda make?

(A) $\$\ 120$ (B) $\$\ 200$ (C) $\$\ 4$ (D) $\$\ 300$ (E) none of these

56. Which value of the digit d makes the number $1d95$ divisible by 15?

(A) 6 (B) 1 (C) 5 (D) 2 (E) none of these

57. Which of the following numbers is divisible by 6?

$$I:54 \qquad II:56 \qquad III:62 \qquad IV:702$$

(A) 54 (B) 56 (C) 62 (D) 702 (E) none of these

Long Division

58. $102030405060708090 \div 5 =$

(A) 20406081012141618

(B) 24681012141618

(C) 204060810012141618

(D) 20040060081012141618

(E) none of these

59. Divide: $(20x^6) \div (5x^2)$

(A) $4x^4$ (B) $4x^3$ (C) $15x^2$ (D) $15x^4$ (E) none of these

60. Divide: $\frac{6x+2x^2}{2x}$

(A) $3+x$ (B) $3x+x^2$ (C) $3x$ (D) $3+2x^2$ (E) none of these

61. $\frac{3^4}{3^3} =$

(A) 3 (B) 3^{12} (C) $\frac{1}{3}$ (D) 3^7 (E) none of these

62. $(3^3)^4 =$

(A) 3 (B) 3^{12} (C) $\frac{1}{3}$ (D) 3^7 (E) none of these

63. $3^3 \cdot 3^4 =$

(A) 3 (B) 3^{12} (C) $\frac{1}{3}$ (D) 3^7 (E) none of these

64. $\frac{1111^5+1111^5+1111^5+1111^5}{1111^4+1111^4} =$

(A) 1111 (B) 2222 (C) 4444 (D) $\frac{1}{2222}$ (E) none of these

Roots

65. $\sqrt{100}+\sqrt{64}$

(A) 18 (B) 80 (C) 1600 (D) 82 (E) none of these

66. $\sqrt{100}\cdot\sqrt{64}$

(A) 18 (B) 80 (C) 1600 (D) 82 (E) none of these

67. $\sqrt{9}+\sqrt{16}$

(A) $\sqrt{25}$ (B) 7 (C) 12 (D) $12\frac{1}{2}$ (E) none of these

68. $\sqrt{9}\cdot\sqrt{16}$

(A) 36 (B) 7 (C) 12 (D) $12\frac{1}{2}$ (E) none of these

Fractions

69. $\frac{3^3}{3^4} =$

(A) 3 (B) 3^{12} (C) $\frac{1}{3}$ (D) 3^7 (E) none of these

70. How many of the following statements are **true**?

$$I: \frac{3}{4} = \frac{5}{6} \qquad II: \frac{3}{4} < \frac{5}{6} \qquad III: \frac{6}{8} = \frac{15}{20} \qquad IV: 4\tfrac{1}{3} = \frac{4}{3}$$

(A) none (B) exactly one (C) exactly two (D) exactly three (E) all four

71. What fraction of an hour is **18** minutes?

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{3}{10}$ (D) $\frac{5}{6}$ (E) none of these

72. Express $\frac{6}{132}$ in least terms.

(A) $\frac{1}{11}$ (B) $\frac{1}{22}$ (C) $\frac{1}{44}$ (D) $\frac{1}{66}$ (E) none of these

73. Convert the improper fraction to a mixed number: $\frac{43}{8}$

(A) $8\tfrac{5}{8}$ (B) $8\tfrac{3}{5}$ (C) $5\tfrac{3}{8}$ (D) $8\tfrac{3}{8}$ (E) none of these

74. Convert the mixed number to an improper fraction: $2\tfrac{3}{4}$

(A) $\frac{6}{4}$ (B) $\frac{10}{4}$ (C) $\frac{11}{4}$ (D) $\frac{4}{11}$ (E) none of these

75. How many of the following assertions are **false**?

$$I: \frac{1}{2} = \frac{2}{1} \qquad II: \frac{3}{5} < \frac{4}{7} \qquad III: \frac{1}{2} < \frac{1}{3} \qquad IV: \frac{3}{1} < \frac{2}{2}$$

(A) exactly one (B) exactly two (C) exactly three (D) all four (E) none of them

76. How many of the following statements are **true**?

$$I: \frac{3}{4} = \frac{15}{20} \qquad II: \frac{3}{4} > \frac{2}{5} \qquad III: \frac{2}{4} = \frac{3}{6} \qquad IV: 5\tfrac{2}{3} = \frac{17}{3}$$

(A) none (B) exactly one (C) exactly two (D) exactly three (E) all four

77. What fraction of a day is **18** hours?

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{2}{3}$ (D) $\frac{5}{6}$ (E) none of these

78. Express $\frac{104}{140}$ in least terms.

(A) $\frac{26}{35}$ (B) $\frac{52}{70}$ (C) $\frac{13}{7}$ (D) $\frac{10}{14}$ (E) none of these

79. Convert the improper fraction to a mixed number: $\frac{100}{7}$

(A) $14\tfrac{2}{7}$ (B) $13\tfrac{6}{7}$ (C) $\frac{6}{5}$ (D) $\frac{200}{14}$ (E) none of these

80. Convert the mixed number to an improper fraction: $3\tfrac{2}{5}$

(A) $\frac{17}{5}$ (B) $\frac{15}{5}$ (C) $\frac{13}{5}$ (D) $\frac{5}{17}$ (E) none of these

81. How many of the following assertions are **false**?

$$I: \frac{3}{13} = \frac{51}{169} \qquad II: \frac{2}{3} < \frac{4}{5} \qquad III: \frac{1}{15} < \frac{1}{16} \qquad IV: \frac{7}{8} > \frac{1}{2}$$

(A) exactly one (B) exactly two (C) exactly three (D) all four (E) none of them

82. How many of the following expressions are undefined?

$$I: \frac{5}{5-5} \qquad II: \frac{5-5}{5} \qquad III: \frac{1}{1^3 - 2^0} \qquad IV: \frac{0}{0}$$

(A) exactly one (B) exactly two (C) exactly three (D) all four (E) none of them

83. How many of the following expressions are zero?

$$I: \frac{5}{5-5} \qquad II: \frac{5-5}{5} \qquad III: \frac{1}{1^3 - 2^0} \qquad IV: \frac{0}{0}$$

(A) exactly one (B) exactly two (C) exactly three (D) all four (E) none of them

84. Expressed in least terms $\frac{24}{52}$ is

(A) $\frac{8}{13}$ (B) $\frac{6}{13}$ (C) $\frac{2}{13}$ (D) $\frac{12}{26}$ (E) none of these

85. Find the missing numerator: $\frac{7}{12} = \frac{?}{60}$

(A) 7 (B) 5 (C) 35 (D) 12 (E) none of these

86. Expressed in least terms $\frac{221}{247}$ is

(A) $\frac{21}{47}$ (B) $\frac{73}{82}$ (C) $\frac{17}{19}$ (D) $\frac{2}{3}$ (E) none of these

87. Find the missing numerator: $\frac{7}{12} = \frac{?}{60}$

(A) 7 (B) 5 (C) 35 (D) 12 (E) none of these

88. How many of the following fractions are **not** equivalent to $\frac{3}{7}$?

$$I: \frac{9}{21} \qquad II: \frac{15}{35} \qquad III: \frac{2}{5} \qquad IV: \frac{51}{119}$$

(A) exactly one (B) exactly two (C) exactly three (D) all four (E) none of them

89. Three of the following fractions represent the same number. Which is the odd one out?

(A) $\frac{9}{12}$ (B) $\frac{2}{3}$ (C) $\frac{150}{225}$ (D) $\frac{14}{21}$ (E) none of these

90. Change into an improper fraction:

$$2\frac{5}{7}$$

(A) $\frac{17}{7}$ (B) $\frac{19}{7}$ (C) $\frac{25}{7}$ (D) $\frac{14}{7}$ (E) none of these

91. Express as a mixed numeral: $\frac{24}{7}$

(A) $3\frac{3}{7}$ (B) $2\frac{3}{7}$ (C) $\frac{7}{24}$ (D) 3 (E) none of these

92. Write a mixed fraction for $33 \div 4$.

(A) $8\frac{1}{4}$ (B) 8.14 (C) $\frac{33}{4}$ (D) $\frac{4}{33}$ (E) none of these

Addition and Subtraction of Fractions

93. Add the fractions: $\frac{2}{5}+\frac{3}{4}$

(A) $\frac{5}{9}$ (B) $\frac{1}{4}$ (C) $\frac{5}{20}$ (D) $\frac{23}{20}$ (E) none of these

94. At the beginning of the week, a bookstore sold $\frac{3}{7}$ of its inventory of the latest *Hairy Puker* book. In the middle of the week, it sold $\frac{3}{8}$ further. No more such books were sold during the rest of the week. What fraction of the inventory of *Hairy Puker* books remain at the end of the week?

(A) $\frac{45}{56}$ (B) $\frac{11}{56}$ (C) $\frac{3}{56}$ (D) $\frac{6}{15}$ (E) none of these

95. Subtract the fractions: $\frac{11}{20}-\frac{2}{15}$

(A) $\frac{9}{5}$ (B) $\frac{5}{12}$ (C) $\frac{9}{12}$ (D) $\frac{3}{4}$ (E) none of these

96. Calculate: $3\frac{3}{4}-2\frac{1}{2}$.

(A) $\frac{14}{15}$ (B) $1\frac{3}{5}$ (C) $\frac{3}{2}$ (D) $\frac{5}{4}$ (E) none of these

97. Calculate: $\frac{1}{3}-\frac{1}{6}+\frac{1}{9}$.

(A) $\frac{1}{12}$ (B) $\frac{7}{18}$ (C) $\frac{1}{18}$ (D) $\frac{5}{18}$ (E) none of these

98. Add the fractions: $\frac{3}{7}+\frac{1}{4}$

(A) $\frac{19}{28}$ (B) $\frac{4}{11}$ (C) $\frac{3}{28}$ (D) $\frac{4}{28}$ (E) none of these

99. Subtract the fractions: $\frac{5}{6}-\frac{2}{15}$

(A) $\frac{7}{10}$ (B) $\frac{3}{9}$ (C) $\frac{1}{9}$ (D) $\frac{29}{30}$ (E) none of these

100. Calculate: $2\frac{1}{3}-1\frac{2}{5}$.

(A) $\frac{14}{15}$ (B) $1\frac{1}{2}$ (C) $1\frac{1}{14}$ (D) $1\frac{1}{15}$ (E) none of these

101. Calculate: $\frac{2}{5}-\frac{1}{3}+\frac{1}{10}$.

(A) $\frac{1}{6}$ (B) $\frac{1}{12}$ (C) $\frac{1}{75}$ (D) $\frac{1}{3}$ (E) none of these

102. Write as a mixed numeral in lowest terms:

$$4\frac{35}{50}+3\frac{15}{60}$$

(A) $\frac{159}{20}$ (B) $7\frac{190}{200}$ (C) $7\frac{1900}{3000}$ (D) $7\frac{19}{20}$ (E) none of these

103. Write as a mixed numeral in lowest terms:

$$5\tfrac{35}{50} - 2\tfrac{15}{60}$$

(A) $3\tfrac{9}{20}$ (B) $\tfrac{69}{20}$ (C) $3\tfrac{20}{10}$ (D) $3\tfrac{1350}{3000}$ (E) none of these

Multiplication and Division of Fractions

104. Reduce to simplest form: $3\dfrac{1}{3} \div 24$

(A) $\tfrac{1}{80}$ (B) $\tfrac{5}{36}$ (C) $7\tfrac{1}{5}$ (D) $\tfrac{6}{43}$ (E) none of these

105. What is the product of a non-zero fraction and its reciprocal?
(A) 0 (B) 1 (C) $\tfrac{1}{2}$ (D) 2 (E) impossible to determine with the given information

106. Find the reciprocal of $2\tfrac{3}{4}$.
(A) $2\tfrac{4}{3}$ (B) $\tfrac{4}{11}$ (C) $\tfrac{11}{4}$ (D) $3\tfrac{4}{11}$ (E) none of these

107. Simplify as a mixed numeral in lowest terms:

$$\left(2\frac{2}{3}\right) \cdot \left(3\frac{3}{4}\right).$$

(A) 10 (B) $6\tfrac{6}{12}$ (C) $\tfrac{5}{4}$ (D) $8\tfrac{3}{4}$ (E) none of these

108. Calculate: $\dfrac{4}{5} \cdot \dfrac{5}{6} + \dfrac{4}{5} \div \dfrac{5}{6}$

(A) $\dfrac{4}{3}$ (B) $\dfrac{16}{75}$ (C) $\dfrac{49}{15}$ (D) $\dfrac{122}{75}$ (E) none of these

109. Calculate: $\dfrac{\dfrac{1}{2} + \dfrac{1}{3}}{\dfrac{1}{2} - \dfrac{1}{3}}$

(A) 5 (B) $\dfrac{1}{5}$ (C) 1 (D) $\dfrac{2}{5}$ (E) none of these

110. Calculate: $\dfrac{1}{1 + \dfrac{1}{1 + \dfrac{1}{2}}}$

(A) $\dfrac{3}{2}$ (B) $\dfrac{2}{5}$ (C) $\dfrac{5}{3}$ (D) $\dfrac{3}{5}$ (E) none of these

111. Find the reciprocal of $5\tfrac{2}{3}$.
(A) $5\tfrac{3}{2}$ (B) $6\tfrac{1}{2}$ (C) $\tfrac{17}{3}$ (D) $\tfrac{3}{17}$ (E) none of these

112. Simplify as a mixed numeral in lowest terms:

$$\left(8\frac{1}{8}\right) \cdot \left(1\frac{1}{13}\right).$$

(A) $\tfrac{1}{13}$ (B) $8\tfrac{1}{104}$ (C) $\tfrac{35}{4}$ (D) $8\tfrac{3}{4}$ (E) none of these

113. Calculate: $\frac{2}{3} \cdot \frac{3}{5} + \frac{2}{3} \div \frac{5}{4}$

(A) $\frac{14}{15}$ (B) $\frac{16}{75}$ (C) $\frac{37}{30}$ (D) $\frac{377}{180}$ (E) none of these

114. Calculate: $\dfrac{\frac{3}{4} - \frac{2}{3}}{\frac{3}{4} + \frac{2}{3}}$

(A) $\frac{1}{17}$ (B) $\frac{19}{36}$ (C) 1 (D) $\frac{17}{144}$ (E) none of these

115. Calculate: $\dfrac{1}{2 - \dfrac{1}{3 - \dfrac{1}{4}}}$

(A) $\frac{11}{18}$ (B) $\frac{3}{2}$ (C) $\frac{7}{18}$ (D) $\frac{13}{18}$ (E) none of these

Ratio and Proportion

116. Solve the proportion for n

$$\frac{5}{6} = \frac{60}{n}$$

(A) 72 (B) 50 (C) $\frac{1}{2}$ (D) 12 (E) none of these

117. A recipe for five people takes four eggs. Assuming proportional ingredients available, how many people will twenty eggs serve?

(A) 21 (B) 10 (C) 25 (D) 15 (E) none of these

118. John takes **8** hours to fill a hole, Atish takes **4** hours to fill the same hole, and David takes **2** hours to fill the same hole. If the three are working together, how many hours will it take them to fill the hole?

(A) $1\frac{1}{8}$ hours (B) $1\frac{1}{2}$ hours (C) $\frac{7}{8}$ hours (D) $1\frac{1}{7}$ hours (E) none of these

119. 120 ft = ? in

(A) 10 (B) 1440 (C) 144 (D) 100 (E) none of these

120. 15840 ft = ? mi

(A) 3 (B) $\frac{1}{3}$ (C) 9 (D) 27878400 (E) none of these

121. 120 ft = ? yd

(A) 10 (B) 40 (C) 1440 (D) 360 (E) none of these

122. Solve the proportion for n

$$\frac{39}{42} = \frac{13}{n}$$

(A) $12\frac{7}{100}$ (B) 3 (C) 14 (D) 126 (E) none of these

123. A recipe for seven people takes six eggs. Assuming proportional ingredients available, how many people will eighteen eggs serve?

(A) 21 (B) 10 (C) 15 (D) 16 (E) none of these

124. John takes **3** hours drive her crazy and Atish takes **2** hours to drive her crazy. If John and Atish are working together, how long will it take them to drive her crazy, assuming proportional time?

(A) $1\frac{1}{5}$ hours (B) $1\frac{1}{2}$ hours (C) $1\frac{1}{3}$ hours (D) $1\frac{1}{4}$ hours (E) none of these

125. Find n if $\frac{8}{7} = \frac{n}{20}$

(A) 19 (B) 1120 (C) $22\frac{6}{7}$ (D) 17.5 (E) none of these

126. Express the ratio $40:112$ in lowest terms.

(A) $\frac{20}{56}$ (B) $\frac{10}{28}$ (C) $\frac{5}{14}$ (D) $\frac{80}{224}$ (E) none of these

Sets and Inclusion-Exclusion

127. If $X = \{1,2,3,4,5\}$, $Y = \{3,4,6,8,9\}$, what is $X \setminus Y$?

(A) $\{1,2,5\}$ (B) $\{6,8,9\}$ (C) $\{1,2,5,6,8,9\}$ (D) $\varnothing$ (E) none of these

128. If $X = \{1,2,3,4,5\}$, $Y = \{3,4,6,8,9\}$, what is $Y \setminus X$?

(A) $\{1,2,5\}$ (B) $\{6,8,9\}$ (C) $\{1,2,5,6,8,9\}$ (D) $\varnothing$ (E) none of these

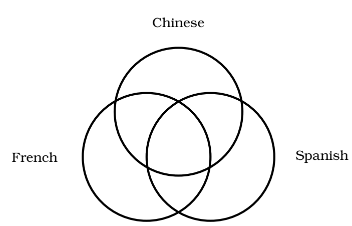
129. Given the sets

$$A = \{a,c,d\},\ B = \{c,d,e,g\},\ C = \{a,b,e,f\},$$

find $A \cap B \cap C$.

(A) $\varnothing$ (B) $\{c,d\}$ (C) $\{a\}$ (D) $\{b,c,e,f,g,h\}$ (E) none of these

Situation: Of the **15** students in a class, **5** speak Chinese, **7** speak Spanish, **7** speak French, **3** speak Spanish and Chinese, **4** speak Spanish and French, and **3** speak Chinese and French. One student speaks all three languages. Questions 130 to 134 refer to this situation, and you may wish to fill in the regions of the diagram below in order to aid your reasoning.



130. How many students speak French and Spanish, but not Chinese?

(A) **1** (B) **2** (C) **3** (D) **4** (E) none of these

131. How many students speak only French?

(A) **1** (B) **2** (C) **3** (D) **4** (E) none of these

132. How many students speak exactly two of these three languages?

(A) **4** (B) **5** (C) **6** (D) **7** (E) none of these

133. How many students speak exactly one of these three languages?

(A) $\mathbf{0}$ (B) $\mathbf{1}$ (C) $\mathbf{2}$ (D) $\mathbf{3}$ (E) none of these

134. How many students speak none of these three languages?

(A) $\mathbf{10}$ (B) $\mathbf{5}$ (C) $\mathbf{2}$ (D) $\mathbf{3}$ (E) none of these

Decimals

135. Convert to a decimal: $\mathbf{2\frac{3}{4}}$.

(A) $\mathbf{2.34}$ (B) $\mathbf{2.25}$ (C) $\mathbf{2.75}$ (D) $\mathbf{2.034}$ (E) none of these

136. Convert to a decimal: $\mathbf{\frac{9}{11}}$.

(A) $\mathbf{0.8\overline{1}}$ (B) $\mathbf{0.\overline{911}}$ (C) $\mathbf{0.81818181818}$ (D) $\mathbf{0.\overline{81}}$ (E) none of these

137. Convert to an improper fraction: $\mathbf{2.34}$.

(A) $\mathbf{\frac{117}{100}}$ (B) $\mathbf{\frac{234}{50}}$ (C) $\mathbf{\frac{3}{2}}$ (D) $\mathbf{\frac{117}{50}}$ (E) none of these

138. Determine the largest number among the following:

$$A: \frac{1}{6} \qquad B: 0.16\overline{7} \qquad C: 0.\overline{17} \qquad D: 0.\overline{16} \qquad E: 0.1\overline{7}$$

(A) (B) (C) (D) (E)

139. Convert to a decimal: $\mathbf{3\frac{3}{5}}$.

(A) $\mathbf{3.6}$ (B) $\mathbf{3.35}$ (C) $\mathbf{3.3}$ (D) $\mathbf{3.06}$ (E) none of these

140. Convert to a decimal: $\mathbf{2\frac{1}{4}\%}$.

(A) $\mathbf{0.0225}$ (B) $\mathbf{2.25}$ (C) $\mathbf{2.14}$ (D) $\mathbf{0.00225}$ (E) none of these

141. Convert to a decimal: $\mathbf{\frac{2}{15}}$.

(A) $\mathbf{0.1\overline{3}}$ (B) $\mathbf{0.\overline{13}}$ (C) $\mathbf{0.1333}$ (D) $\mathbf{0.13333333}$ (E) none of these

142. Convert to a fraction: $\mathbf{1.12}$.

(A) $\mathbf{\frac{28}{25}}$ (B) $\mathbf{\frac{26}{25}}$ (C) $\mathbf{\frac{3}{2}}$ (D) $\mathbf{\frac{6}{5}}$ (E) none of these

143. Determine the largest number among the following:

$$A: \frac{19}{50} \qquad B: 0.3\overline{7} \qquad C: 0.\overline{37} \qquad D: \frac{37}{100} \qquad E: 0.377$$

(A) (B) (C) (D) (E)

144. Which of the following numbers is **One hundred thousand five and three ten thousandths**?
(A) **105003.0010** (B) **105000.0003** (C) **100005.0003** (D) **100005.00003** (E) none of these

145. Write as a decimal: $\frac{3}{16}$.
(A) 0.316 (B) 3.16 (C) 0.1875 (D) $0.\overline{1875}$ (E) none of these

146. Write as a decimal: $\frac{3}{11}$.
(A) 0.311 (B) 3.11 (C) 0.272727273 (D) $0.\overline{27}$ (E) none of these

147. Write **1.03** as a percent.
(A) **1.03%** (B) **10.3%** (C) **0.103%** (D) **103%** (E) none of these

148. Write $\frac{5}{7}$ as a percent.
(A) $\frac{5}{700}\%$ (B) $\frac{5}{70}\%$ (C) $\frac{5}{7}\%$ (D) $71\frac{3}{7}\%$ (E) none of these

149. **12** minutes is what percent of an hour?
(A) **12%** (B) **20%** (C) **1.2%** (D) **5%**

150. What percent of **552** is **24**?
(A) **23%** (B) $\frac{1}{23}\%$ (C) $4\frac{8}{23}\%$ (D) **2300%** (E) none of these

151. Four of these are equal. Which one is the odd one out?

$$I: \frac{1}{3}+\frac{1}{6} \qquad II: 50\% \qquad III: 0.5 \qquad IV: \frac{3}{6} \qquad V: \frac{1}{3}+\frac{2}{3}$$

(A) (B) (C) (D) (E)

152. Write as a decimal: $\frac{3}{16}$.
(A) 0.316 (B) 3.16 (C) 0.1875 (D) $0.\overline{1875}$ (E) none of these

153. Write $\frac{1}{6}$ as a percent.
(A) **.167%** (B) **60%** (C) $16\frac{2}{3}\%$ (D) **600%** (E) none of these

Addition, Subtraction, and Multiplication of Decimals

154. $1.234 + 5.67 =$

(A) **6.904** (B) **6.94** (C) **6.094** (D) **6.0094** (E) none of these

155. $45\% + 9.9\% =$

(A) **0.549** (B) **54.9** (C) **5.49** (D) **0.0549** (E) none of these

156. A janitor found that **8** out of **48** light bulbs had to be replaced. What percent of the lightbulbs needed replacing?
(A) $33\frac{1}{3}\%$ (B) $16\frac{1}{3}\%$ (C) $16\frac{2}{3}\%$ (D) $\frac{1}{6}\%$ (E) none of these

157. What percent of **408** is **34**?
(A) $\frac{1}{12}\%$ (B) $\frac{1}{120}\%$ (C) $\frac{1}{1200}\%$ (D) $8\frac{1}{3}\%$ (E) none of these

158. $9.0909 + 2.0202 =$

(A) **10.1111** (B) **11.0111** (C) **11.111** (D) **11.1111** (E) none of these

159. Convert to a decimal: $11\% + 11\%$.

(A) **0.22** (B) **22.0** (C) **2.2** (D) **0.0022** (E) none of these

160. Convert to a decimal: $(11\%) \cdot (11\%)$.

(A) **0.121** (B) **121.0** (C) **1.21** (D) **0.0121** (E) none of these

161. Convert to a decimal: $\frac{1}{8} + 0.25 =$

(A) **0.43** (B) **0.37** (C) **0.375** (D) **0.5** (E) none of these

162. Convert to a decimal: $(0.1)^2 \cdot 0.2 + 0.1 \cdot (0.2)^2 =$

(A) **0.06** (B) **0.006** (C) **0.0028** (D) **0.6** (E) none of these

163. What percent of **44** is **11**?

(A) **11%** (B) **484%** (C) **400%** (D) **25%** (E) none of these

164. Find **30%** of **810**.

(A) **24.3** (B) **243** (C) **2430** (D) **24300** (E) none of these

165. After a discount of **25%**, a robotic dog sells for **$1500**. What was the original selling price?

(A) **$1875** (B) **$1525** (C) **$2000** (D) **$1750** (E) none of these

166. **840** is **28%** of what number?

(A) **3000** (B) **300** (C) **30** (D) $235\frac{1}{5}$ (E) none of these

167. Find **37%** of **500**.

(A) 18500 (B) 185 (C) $13\frac{19}{37}$ (D) $13\frac{19}{3700}$ (E) none of these

168. Find $5\frac{1}{3}\%$ of **936**.

(A) 4992 (B) 499200 (C) $175\frac{1}{2}$ (D) $49\frac{23}{25}$ (E) none of these

169. Abdullah buys a sandwich for **$2.30**. The sales tax is **8%**. How must tax must he pay?

(A) **$1.84** (B) **$18.40** (C) **$18.40** (D) **$2.48** (E) none of these

170. Adeena's car cost **$32000** brand new. The following year it depreciated to **$22080**. Find the percent of decrease.

(A) **99.20%** (B) **31%** (C) **169%** (D) **69%** (E) none of these

171. Trisha gets a **15%** commission on sales. If she was able to sell **$2040** on Monday, what was her commission?

(A) **$2025** (B) **$2346** (C) **$306** (D) **$1734** (E) none of these

172. Ann Arbor MI, has a **4%** sales tax. Max bought a book at *Dawn Treader* costing **$19.95**. What was his total bill?

(A) **$20.75** (B) **$19.99** (C) **$0.80** (D) **$19.15** (E) none of these

173. Tuition at *Gosh Knows Where College* went from **$320** per credit to **$368** per credit. Find the percent of increase.

(A) **48%** (B) **0.48%** (C) **13.04%** (D) **15%** (E) none of these

174. $\mathbf{0.1^2 + 0.3^4 + 0.5 \cdot 0.6 =}$
Ⓐ **3.0181** Ⓑ **3.018** Ⓒ **3.181** Ⓓ **.00030181%** Ⓔ none of these

175. $\mathbf{0.1 \cdot 0.2 + 0.3 \cdot 0.4 + 0.5 \cdot 0.6 + 0.7 \cdot 0.8 + 0.9 =}$
Ⓐ **1.9** Ⓑ **1.09** Ⓒ **0.19** Ⓓ **19%** Ⓔ none of these

Division of Decimals

176. $\mathbf{0.102 \div 0.03 =}$
Ⓐ **3.4** Ⓑ **34** Ⓒ **0.034** Ⓓ **0.34** Ⓔ none of these

177. $\mathbf{1.1 \div 0.3 =}$
Ⓐ $\mathbf{3.\overline{06}}$ Ⓑ $\mathbf{3.\overline{6}}$ Ⓒ $\mathbf{3.0\overline{6}}$ Ⓓ $\mathbf{3.0\overline{36}}$ Ⓔ none of these

178. **81** is **27%** of what number?
Ⓐ **3000** Ⓑ **300** Ⓒ **30** Ⓓ $\mathbf{10\frac{1}{3}}$ Ⓔ none of these

179. $\mathbf{0.045 \div 0.3 =}$
Ⓐ **0.15** Ⓑ **1.5** Ⓒ **0.015** Ⓓ **0.015** Ⓔ none of these

180. $\mathbf{3.32 \div 0.33 =}$
Ⓐ $\mathbf{10.\overline{03}}$ Ⓑ **10.03** Ⓒ **10.0303** Ⓓ $\mathbf{10.0\overline{3}}$ Ⓔ none of these

Integer Addition and Subtraction

181. If the following numbers were written from smallest to largest, which one would be in the middle?

$\boldsymbol{A: -7 \quad B: -11 \quad C: -18 \quad D: 18 \quad E: 0}$

Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

182. $\mathbf{1 - 2 + 4 - 8 + 16 - 32 =}$
Ⓐ **20** Ⓑ **−20** Ⓒ **21** Ⓓ **−21** Ⓔ none of these

183. $\mathbf{1 - |-3| =}$
Ⓐ **4** Ⓑ **−4** Ⓒ **−2** Ⓓ **2** Ⓔ none of these

184. If the following numbers were written from smallest to largest, which one would be in the middle?

$\boldsymbol{A: 7 \quad B: -3 \quad C: -4 \quad D: 0 \quad E: -1}$

Ⓐ Ⓑ Ⓒ Ⓓ Ⓔ

185. $\mathbf{3 - 9 + 27 - 81 =}$
Ⓐ **60** Ⓑ **−42** Ⓒ **−60** Ⓓ **−114** Ⓔ none of these

186. $\mathbf{-1 + |-2| =}$
Ⓐ **−3** Ⓑ **1** Ⓒ **−1** Ⓓ **3** Ⓔ none of these

187. $\mathbf{3 - (5 - (5 - 3)) =}$
Ⓐ **−10** Ⓑ **−4** Ⓒ **4** Ⓓ **0** Ⓔ none of these

Integer Multiplication and Division

188. $\frac{5^3-(-3)^3}{5^2-(-3)^2}=$

Ⓐ $\frac{49}{2}$ Ⓑ $\frac{3}{2}$ Ⓒ 6 Ⓓ $\frac{19}{2}$ Ⓔ none of these

189. $(-100) \div (10) \div (-10) =$

Ⓐ 1 Ⓑ 10 Ⓒ 100 Ⓓ −1 Ⓔ none of these

190. $\frac{5^3-(-3)^3}{5^2-(-3)^2}=$

Ⓐ $\frac{49}{2}$ Ⓑ $\frac{3}{2}$ Ⓒ 6 Ⓓ $\frac{19}{2}$ Ⓔ none of these

191. $(2)(-8)-(-2)(-2)(3)=$

Ⓐ 4 Ⓑ −4 Ⓒ 28 Ⓓ −28 Ⓔ none of these

192. $5-4^2=$

Ⓐ −11 Ⓑ 11 Ⓒ 21 Ⓓ −21 Ⓔ none of these

193. $1-(-2)^3=$

Ⓐ −7 Ⓑ 9 Ⓒ −9 Ⓓ 7 Ⓔ none of these

194. $2-\left(-\frac{1}{2}\right)^3=$

Ⓐ $\frac{15}{8}$ Ⓑ $\frac{11}{6}$ Ⓒ $\frac{17}{8}$ Ⓓ $\frac{13}{6}$ Ⓔ none of these

195. $\frac{2^3-(-1)^3}{2-(-1)}$

Ⓐ 3 Ⓑ 1 Ⓒ −3 Ⓓ −1 Ⓔ none of these

196. $(2)(-9)-(-1)(-8)(-3)=$

Ⓐ −6 Ⓑ 42 Ⓒ −42 Ⓓ 6 Ⓔ none of these

197. $(-3)^2+(-2)^3=$

Ⓐ −9 Ⓑ −8 Ⓒ 1 Ⓓ −17 Ⓔ none of these

Rational, Irrational, and Real Numbers

198. $\frac{3}{4}-\frac{11}{12}=$

Ⓐ $\frac{1}{6}$ Ⓑ $-\frac{1}{6}$ Ⓒ −1 Ⓓ $\frac{11}{16}$ Ⓔ none of these

199. $-\left(-\frac{2}{3}\right)^2-\left(-\frac{2}{3}\right)^3=$

Ⓐ $-\frac{20}{27}$ Ⓑ $-\frac{4}{27}$ Ⓒ $\frac{20}{27}$ Ⓓ $\frac{4}{27}$ Ⓔ none of these

200. $\frac{\frac{2}{3}-\frac{3}{4}}{-\frac{5}{6}+\frac{7}{8}}=$

Ⓐ 2 Ⓑ −2 Ⓒ $\frac{1}{2}$ Ⓓ $-\frac{1}{2}$ Ⓔ none of these

201. $2+(-0.5)(0.1)^2=$

Ⓐ 1.995 Ⓑ 0.015 Ⓒ 2.005 Ⓓ −1.995 Ⓔ none of these

202. $\dfrac{(-8)(-9)}{-6}+7=$

Ⓐ -19 Ⓑ 5 Ⓒ -5 Ⓓ 19 Ⓔ none of these

203. $\dfrac{14-24}{12-(-5+7)}$

Ⓐ 1 Ⓑ 10 Ⓒ -1 Ⓓ -10 Ⓔ none of these

B Essay Exam Questions

Natural Numbers

1. Convert $\mathbf{123_4}$ to decimal.
2. Convert the decimal number **123** to base-**4**.
3. Compute: $10 \cdot (9 \cdot 7 - 8 \cdot 6)^2 - 2^3 \cdot 3^2$
4. Consider the arithmetic progression $8, 19, 30, \ldots$, where **8** is on the first position, **19** in the second position, etc. Which number occupies the **1001**st position?
5. When writing all the natural numbers from **666** to **999** inclusive, how many digits have I used?
6. Write the prime factorisation of **272** and **170**. Find $\gcd(272, 170)$. Find $\mathrm{LCM}(272, 170)$.
7. If today is Wednesday, what day will it be **1000** days from today?
8. Determine a digit d so that the **3**-digit integer $10d$ be divisible by **9**.
9. Compute: $\dfrac{2^6 + 2^6 + 2^6 + 2^6 + 2^6 + 2^6}{2^3 + 2^3 + 2^3}$.
10. Compute: $\dfrac{1111^5 + 1111^5 + 1111^5 + 1111^5}{1111^4 + 1111^4}$.

Fractions

11. What is one third of a half of a fifth of a quarter of **99000**?
12. Compute and express your answer as an improper fraction: $\dfrac{2}{5} + \dfrac{3}{4} + \dfrac{3}{15}$.
13. Compute and express your answer as a mixed number: $\left(1\frac{2}{3}\right)\left(3\frac{4}{5}\right)$.
14. Compute: $\dfrac{1}{2 + \dfrac{3}{4}}$
15. Compute: $\dfrac{1}{2} \cdot \dfrac{3}{4} + \dfrac{5}{6} \div \dfrac{8}{7}$
16. What fraction of a non-leap year does a **100** days constitute?
17. David spent $\dfrac{2}{5}$ of his money on a storybook. The storybook cost **\$20**. How much money did he have at first?
18. Tito was pigging-out on cookies, and in the course of three days, he ate **420** cookies. On the first day, he ate $\dfrac{2}{5}$ of the cookies. On the second day, he ate $\dfrac{5}{6}$ of the remaining cookies. How many cookies did he eat on the third day?
19. Find n if $\dfrac{2n}{3} = \dfrac{14}{21}$.
20. A recipe for sangría uses **3** measures of lemonade for **2** measures of red wine. How many cups of lemonade and of red wine are required if **40** cups of sangría are desired?

Decimals

21. Compute and write your answer as a decimal: $\frac{1}{2}+3.45+67\%.$

22. Compute and write your answer as a decimal: $\left(\frac{1}{10}\right)\left(\frac{2}{100}\right)\left(\frac{3}{1000}\right)\left(\frac{4}{10000}\right)$.

23. Compute: $(0.1)(0.2)+(0.3)^2$.

24. Convert to a fraction: $\dfrac{1}{0.1+\dfrac{1}{0.1}}$.

25. Back in the good old days of 1999, a euro was worth \$1.05. Now, the value of the dollar has fallen, and a euro is worth **20%** more of what it was in 1999. How much is worth a euro now, in dollars?

26. How much does it cost to fill a **12** gallon tank, if gas costs **\$3.16** a gallon?

27. Divide: $123.123 \div 0.06$. Is the result a periodic or a finite decimal?

28. Divide: $123.123 \div 0.9$. Is the result a periodic or a finite decimal?

29. T-shirts are now at a 20% discount at the local T-shirt store. What is the discounted price of a T-shirt of original price **\$19.90**?

30. The area of an aerobic hall in a tri-complex is 70% of the total area (aerobic hall, tennis court, and pool). The area of a tennis court is 30% of the aerobic hall area. Find the area of a pool, if the aerobic hall area is **476** sq. feet.

Integers

31. Compute: $(1-2+3)(4-5+6)-(-8+9-10)$

32. Compute: $|(-1)^2+(-2)^3-(-4)(-5)|$

33. Calculate: $(6-4^2)(4+(-4)^2)$

34. When a number is halved and to this result we add **5**, we obtain **4**. What was the original number?

35. A number is trebled, and then the result is increased by **10**. One obtains **4** as the final answer. What was the original number?

Rational Numbers

36. Calculate: $2-\left(-\frac{3}{2}\right)^2$.

37. Calculate: $1+(-0.1)^3$.

38. Calculate: $(1-1.1)^3+1$.

39. Calculate: $\dfrac{-\frac{2}{3}\div\frac{3}{4}}{-\frac{2}{3}+\frac{3}{4}}$.

40. Calculate: $\dfrac{(-2008)^1+(-2)^2+(-3)^3}{(-2008)^0+(-2)^1+(-3)^2}$.

C Answers and Hints

1.1 The idea is the following. We use the result of example 6. Now, a subset of $\{a, b, c, d\}$ either contains d or it does not. This means that $\{a, b, c, d\}$ will have $2 \cdot 8 = 16$ subsets. Since the subsets of $\{a, b, c\}$ do not contain d, we simply list all the subsets of $\{a, b, c\}$ and then to each one of them we add d. This gives

$S_1 = \varnothing$	$S_6 = \{b, c\}$	$S_{12} = \{c, d\}$
$S_2 = \{a\}$	$S_7 = \{c, a\}$	$S_{13} = \{a, b, d\}$
$S_3 = \{b\}$	$S_8 = \{a, b, c\}$	$S_{14} = \{b, c, d\}$
$S_4 = \{c\}$	$S_9 = \{d\}$	$S_{15} = \{c, a, d\}$
$S_5 = \{a, b\}$	$S_{10} = \{a, d\}$	$S_{16} = \{a, b, c, d\}$
	$S_{11} = \{b, d\}$	

2.1 52%

2.2 Let A be the set of camels eating wheat, and $|A|$ its number, and B be the set of camels eating barley, and $|B|$ its number. Then

$$90 = 100 - 10 = |A \cup B| = |A| + |B| - |A \cap B| = 46 + 57 - |A \cap B| = 103 - |A \cap B|,$$

whence $|A \cap B| = 13$.

2.3 15

3.1 21

3.2 $1 =$ 𓏺, 12= 𓎆𓏺𓏺 , and $123 =$ 𓍢𓎆𓎆𓏺𓏺𓏺. Egyptian notation has the advantage that you can write the symbols in any order and still get the same number. With our notation, **321** and **123**, for example, have different meanings. On the other hand, performing arithmetic operations like adding, subtracting, multiplying, and dividing, seems more cumbersome. Moreover, for very large numbers, we would have to use too many symbols. For example, if we say that the current population of the USA is around 300 million, we would need to write 𓁨 three hundred times, which is very impractical.

3.3 1914=MCMXIV, 1917=MCMXVII, 1939=MCMXXXIX, 1963=MCMLXIII, 1989=MCMLXXXIX.

3.4 5321

3.5 8888

4.1

1. **4** units
2. **3** tens
3. **2** hundreds
4. **3** ten millions

4.2 669600000

4.3 Yes. The leftmost zero in **010** indicates that it has **0** hundreds.[1] Thus both numbers have **1** ten and **0** units and nothing else, and are therefore equal.

4.4 This is

$$12,000,000 + 12,000 + 1,200 + 12 = 12013212.$$

4.5 Four.

4.6 $11111_2 = 2^4 + 2^3 + 2^2 + 2 + 1 = 31.$

[1]In Spanish, a derogatory term for an inferior, insignificant person is *cero a la izquierda*, that is, a "zero to the left."

4.7 $11111_3 = 3^4 + 3^3 + 3^2 + 3 + 1 = 121.$

4.8 $10101101100111_2.$

4.9 First we convert 123_4 to base 10:

$$123 = 1 \cdot 4^2 + 2 \cdot 4 + 3 = 16 + 8 + 7 = 27.$$

Now we convert 27 to base 5:

5	27	2
5	5	0
5	1	1

so

$$123_4 = 27 = 102_5.$$

4.10 1300

4.11 $11111_3 = 3^4 + 3^3 + 3^2 + 3 + 1 = 121.$

4.12 Writing in base 26: $2007 = 2 \cdot 26^2 + 25 \cdot 26 + 5$. Thus the label is BYE.

4.13 There is no such number with 1 digit. With two digits there are 7:

$$39, 48, 57, 66, 75, 84, 93.$$

With three digits there are 66:

129	138	147	156	165	174	183	192	219	228	237
246	255	264	273	282	291	309	318	327	336	345
354	363	372	381	390	408	417	426	435	444	453
462	471	480	507	516	525	534	543	552	561	570
606	615	624	633	641	651	660	705	714	723	732
741	750	804	813	822	831	840	903	912	921	930.

Thus there is a total of $7 + 66 = 73$ such numbers.

4.14 2002

4.15 Between 9999 and 10101.

4.17 1111_4.

4.18 Yes, you can always guess my number. Every whole number between 0 and 15 is one of the binary numbers

$$0000_2, \quad 0001_2, \quad 0010_2, \quad 0011_2, \quad \ldots, 1111_2.$$

Thus you can ask me:

1. is the first binary digit (from left to right) of your number a zero? Yes or no.
2. is the second binary digit of your number a zero? Yes or no.
3. is the third binary digit of your number a zero? Yes or no.
4. is the fourth binary digit of your number a zero? Yes or no.

In this way you completely identify my number. For example, if I am thinking of $\mathbf{7 = 0111_2}$, my answers to your questions are No, No, No, and Yes, so you deduce my number.

Another equivalent way consists in successively halving the intervals where my number may lie. So, again, if my number is **7** you may ask:

1. Since halfway between **0** and **15** is **8**, you ask: Is your number smaller than **8**? I answer yes.
2. Since halfway between **0** and **7** is **4**, you ask: Is your number smaller than **4**? I answer no.
3. Since halfway between **5** and **7** is **6**, you ask: Is your number smaller than **6**? I answer no.
4. The number must be **7**.

Again, with the method above, if my number were **13**:

1. Since halfway between **0** and **15** is **8**, you ask: Is your number smaller than **8**? I answer no.
2. Since (almost) halfway between **8** and **15** is **11**, you ask: Is your number smaller than **11**? I answer no.
3. Since halfway between **11** and **15** is **13**, you ask: Is your number smaller than **13**? I answer no.
4. Since halfway between **13** and **15** is **14**, you ask: Is your number smaller than **14**? I answer no.

So you want a whole number that is smaller than **14** but is not smaller than **13**. The only option is **13** and you have guessed my number.

5.1 $N-20$.

5.2 $N+20$.

5.3 $3x+10$.

5.4 $3(x+10)$.

5.5 In knitting s scarves, I need $3s$ balls of wool, and hence, at the end of the day I have $b-3s$ balls of wool.

5.6 $4n; 4n+1; 4n+2; 4n+3$, where n is a natural number.

5.7 $a+b$

5.8 $25a+10b$

5.9 $d+1$

6.1

$$\begin{array}{r} \scriptstyle 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ +\quad 9\,8\,7\,6\,5\,4\,3\,2\,1 \\ \hline 1\,1\,1\,1\,1\,1\,1\,1\,1\,0 \end{array}$$

6.2

$$\begin{array}{r} 9\,8\,7\,6\,5\,4\,3\,2\,1 \\ -\ 1\,2\,3\,4\,5\,6\,7\,8\,9 \\ \hline 8\,6\,4\,1\,9\,7\,5\,3\,2 \end{array}$$

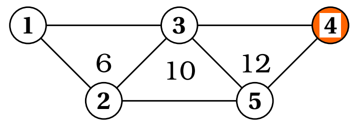
6 **2**

6.4 **24** ft.

6.5 The figure below shows the correct arrangement.

6.6 No. When we add an even integer to another even integer the result is an even integer. Thus the sum of five even integers is even, but **25** is odd.

6.7 **10¢**. The cork costs **10¢** and the bottle **90¢**.



6.8 $686 - 298 = 388$

6.9 81

6.10 Observe that

$$2 = 2 \cdot 1, \quad 4 = 2 \cdot 2, \quad 6 = 2 \cdot 3, \ldots 2006 = 2 \cdot 1003,$$

and so there are 1003 numbers.

The desired quantity is

$$(2-1) + (4-3) + \cdots + (2006 - 2005) = 1003.$$

6.11 $184 - 63 = 121$

6.12 Alice is $69 + 4 = 73$ inches tall and Caroline is $73 - 8 = 65$ inches tall.

6.13 101 years.

6.14

1. 18
2. 10

6.15 3

6.16

1. $3a + 11b$
2. $a + 5b$
3. $4x^2 + 3x + 2$

6.17 Start from 30 and go back:

$$30 + 2 - 4 - 3 + 4 - 5 = 24,$$

so he started on the 24th floor.

6.18 Either $9 + 8 + 7 + 65 + 4 + 3 + 2 + 1 = 99$ or $9 + 8 + 7 + 6 + 5 + 43 + 21 = 99$.

6.19 Cut the bar making pieces of 1, 2, and 4 segments.

6.20

1. $3 = 3 + 0$
2. $5 + 1 \in \mathbb{N}$
3. $1 + 3 = 2 + 2$
4. $64 \neq 604$
5. ✠ $\notin \mathbb{N}$
6. $5 + 6 \neq 7 + 8$

6.21 From the addition table

+	1	3	4	5	7
1	2	4	5	6	8
3	4	6	7	8	10
4	5	7	8	9	11
5	6	8	9	10	12
7	8	10	11	12	14

the different sums belong to the set $\{\mathbf{2,4,5,6,7,8,9,10,11,12,14}\}$, and so there are eleven different sums.

6.22 The frog will escape after seven days. At the end of the sixth day, the frog has leaped **6** feet. Then at the beginning of the seventh day, the frog leaps **5** more feet and is out of the well.

6.23 Let us see what happens in a typical non-leap year, and in a typical leap year.

In a non-leap year

by the end of	she has climbed
31 January	**31** steps
28 February	$\mathbf{31-28=3}$ steps
31 March	$\mathbf{3+31=34}$ steps
30 April	$\mathbf{34-30=4}$ steps
31 May	$\mathbf{31+4=35}$ steps
30 June	$\mathbf{35-30=5}$ steps
31 July	$\mathbf{31+5=36}$ steps
31 August	$\mathbf{36-31=5}$ steps
30 September	$\mathbf{30+5=35}$ steps
31 October	$\mathbf{35-31=4}$ steps
30 November	$\mathbf{30+4=34}$ steps
31 December	$\mathbf{34-31=3}$ steps

In a leap year

by the end of	she has climbed
31 January	**31** steps
29 February	$\mathbf{31-29=2}$ steps
31 March	$\mathbf{2+31=33}$ steps
30 April	$\mathbf{33-30=3}$ steps
31 May	$\mathbf{31+3=34}$ steps
30 June	$\mathbf{34-30=4}$ steps
31 July	$\mathbf{31+4=35}$ steps
31 August	$\mathbf{35-31=4}$ steps
30 September	$\mathbf{30+4=34}$ steps
31 October	$\mathbf{34-31=3}$ steps
30 November	$\mathbf{30+3=33}$ steps
31 December	$\mathbf{33-31=2}$ steps

Now, $\mathbf{100-36=64}$. Let us see how many years it takes her to climb **64** steps. By the end of the four-year range $\mathbf{2001-2004}$, she climbs $\mathbf{3+3+3+2=11}$ steps. By the end of the four-year range $\mathbf{2005-2008}$, she has climbed **22** steps. By the end of the four-year range $\mathbf{2009-20012}$, she has climbed **33** steps. By the end of the four-year range $\mathbf{2013-2016}$, she has climbed **44** steps. By the end of the four-year range $\mathbf{2017-2020}$, she has climbed **55** steps. By the end of the four-year range $\mathbf{2021-2024}$, she has climbed **66** steps. In fact, by 31 December 2023 she has climbed **64** steps, and by 31 July 2024 she has climbed $\mathbf{64+35=99}$ steps. This means that she needs to go into 2025. By 31 March 2025 she has climbed $\mathbf{66+34=100}$ steps and she is now free.

6.24 $\mathbf{98765-10234=88531}$. Why didn't we take **01234** instead of **10234**?

6.25 $\mathbf{50123-49876=247}$.

6.26 Notice that I always choose my number so that when I add it to your number I get **99**, therefore, I end up adding **99** three times and $\mathbf{3\cdot 99=297}$.[2]

6.27 $a+p$

6.28 $m+l+v+c$

6.29 You have $25q+10d$ cents at the beginning of the day, and then you lose $25a+10b$, which leaves you with

$$(25q+10d)-(25a+10b)$$

cents.

6.30 The number of the last page torn must have opposite parity to the first one torn, thus it must be even, hence it must be **314** and hence $\mathbf{314-143=171}$ pages.

[2]Using algebraic language, observe that if you choose x, y, z, then I choose $(99-x), (99-y), (99-z)$. This works because

$$x+(99-x)+y+(99-y)+z+(99-z)=3(99)=297.$$

6.31 The jacket costs $3h$ and the trousers cost $3h-8$. Hence he paid in total

$$h+3h+3h-8=7h-8.$$

6.32 $x+y+z$

6.33 $y+z+s$¢

6.34 $m+x$

6.35 $c-f$ ¢

6.36 $b-e$

6.37 $3679+5283=8962$

6.38 $714+189=903.$

6.39 $5671-1920=3751$

6.40

1. True by commutativity of addition.
2. False.
3. True by associativity of addition.
4. True. First observe that $(547+1250)+3 = 547+(1250+3)$ by associativity and that $(1250+3) = (3+1250)$ by commutativity. Then it follows that $(547+1250)+3 = 547+(3+1250)$ because equals replaced by equals are equal.

6.41 $\boxed{2}75\boxed{3}6-\boxed{2}56\boxed{1}=24\boxed{9}75.$

6.42 Observe that the first and second rows, and the second and third columns add up to 8. Thus $A=4$ works.

6.43 There are multiple solutions. They can be obtained by permuting the entries of one another. Here are two:

8	1	6
3	5	7
4	9	2

4	9	2
3	5	7
8	1	6

6.44 Nine.

6.45 42.

6.46 27.

6.47 6

6.48 If all the magenta, all the yellow, all the white, 14 of the red and 14 of the blue marbles are drawn, then in among these $8+10+12+14+14=58$ there are no 15 marbles of the same color. Thus we need 59 marbles in order to insure that there will be 15 marbles of the same color.

6.49 28

6.50 2

6.51 Yes. Yes.

6.52 No. Yes.

6.53 It is commutative, because for arbitrary a and b we have $a \oplus b = a + b + 10$ and $b \oplus a = b + a + 10$. Since $a + b + 10 = b + a + 10$ by the commutativity of the natural numbers, $\oplus$ is commutative. It is also associative. For

$$a \oplus (b \oplus c) = a \oplus (b + c + 10) = a + (b + c + 10) + 10 = a + b + c + 20.$$

Also

$$(a \oplus b) \oplus c = (a + b + 10) \oplus c + = (a + b + 10) + c + 10 = a + b + c + 20.$$

Since $a \oplus (b \oplus c) = (a \oplus b) \oplus c$, the operation is associative.

6.54 In order to obtain the maximal sum, we must have a chain where every piece is glued to the smallest possible of sides using the smallest numbers. The chain then must have **2003** pieces which are joined on two sides and the two end pieces are joined at only one side. If the **1** and the **2** were shared, the first and second piece must be joined at **1** and the second and third at **2**. But this is impossible because then the first and the third piece would be joined at **3** and **6**, violating domino rules. Hence only **3**'s and **1**'s are shared. The middle **2003** contribute $2 + 4 + 5 + 6 = 17$ each, one of the end pieces contributes $2 + 3 + 4 + 5 + 6 = 20$, and the other end piece contributes $1 + 2 + 4 + 5 + 6 = 18$ hence the maximum sum is $(2003)(17) + 1(20) + 1(18) = 34089.$

7.1 $8 + 3 \cdot 19 = 65.$

7.2 $1 + 9 \cdot 999 = 8992.$

7.3 Observe that to **8** we must add multiples of **3**. Dividing **3005** by **3** we obtain

$$3005 = 3 \cdot 1001 + 2 = 3 \cdot 999 + 3 \cdot 2 + 2 = 3 \cdot 999 + 8,$$

and hence there are $999 + 1 = 1000$ terms.

Adding the **500** pairs

$$8 + 3005 = 3013, \quad 11 + 3002 = 3013, \quad 14 + 2999 = 3013, \quad \ldots, \quad 1505 + 1508 = 3013,$$

we see that the sum is

$$8 + 11 + 14 + \cdots + 3005 = 500 \cdot 3013 = 1506500.$$

7.4 We start with **2**, and then keep adding **5**, thus

$$2 = 2 + 5 \cdot 0, \qquad 7 = 2 + 5 \cdot 1, \qquad 12 = 2 + 5 \cdot 2, \qquad 17 = 2 + 5 \cdot 3, \ldots.$$

The general nth term is therefore of the form $2 + 5(n - 1)$, where $n = 1, 2, 3, \ldots$ is a natural number.

7.5 $1 + 6(n - 1)$ where $n = 1, 2, 3, \ldots.$

7.6 The pattern is as follows,

Position	***Number***		
1	3	=	$3 + 11 \cdot 0$
2	14	=	$3 + 11 \cdot 1$
3	25	=	$3 + 11 \cdot 2$
4	36	=	$3 + 11 \cdot 3$
5	47	=	$3 + 11 \cdot 4$
6	59	=	$3 + 11 \cdot 5$
7	60	=	$3 + 11 \cdot 6$
⋮	⋮	⋮	⋮
100	1092	=	$3 + 11 \cdot 99$

A general formula is $3 + 11(N - 1)$ for $N = 1, 2, 3, \ldots.$

7.7 180

7.8 72

7.9 120

7.10 Upon division by 4, every natural number leaves remainder 0, 1, 2, or 3. Hence we may take

$$\mathbb{N} = \{0,4,8,12,\ldots\} \cup \{1,5,9,13,\ldots\} \cup \{2,6,10,14,\ldots\} \cup \{3,7,11,15,\ldots\}.$$

8.1 Observe that 4 and 5 are on opposite sides, so they never meet. The largest product is $3 \cdot 5 \cdot 6 = 90$.

8.2 56 ¢

8.3 28

8.4 $24 \cdot 60 \cdot 60 + 1 = 86401$

8.5 9240 meters.

8.6 $30 \cdot 197299 = 5918970$ seconds.

8.7 I have written 9 1-digit integers (1 through 9), 90 2-digit integers (from 10 to 99) and 900 3-digit integers (from 100 to 999). Altogether I have used

$$1 \cdot 9 + 2 \cdot 90 + 3 \cdot 900 = 2889$$

digits.

8.8 Pair up the integers from 0 to 999 as

$$(0,999),\ (1,998),\ (2,997),\ (3,996),\ \ldots,(499,500).$$

Each pair has sum of digits 27 and there are 500 such pairs. Adding 1 for the sum of digits of 1000, the required total is

$$27 \cdot 500 + 1 = 13501.$$

8.9 When the number 99 is written down, we have used

$$1 \cdot 9 + 2 \cdot 90 = 189$$

digits. If we were able to write 999, we would have used

$$1 \cdot 9 + 2 \cdot 90 + 3 \cdot 900 = 2889$$

digits, which is more than 1002 digits. The 1002nd digit must be among the three-digit positive integers. We have $1002 - 189 = 813$ digits at our disposal, from which we can make $\lfloor \frac{813}{3} \rfloor = 271$ three-digit integers, from 100 to 370. When the 0 in 370 is written, we have used $189 + 3 \cdot 271 = 1002$ digits. The 1002nd digit is the 0 in 370.

8.10 We have,

$$1^2 = 1, \quad 11^2 = 121, \quad 111^2 = 12321, \quad 1111^2 = 1234321, \quad 11111^2 = 123454321, \quad 111111^2 = 12345654321.$$

8.11 We have,

$$\begin{array}{r} 12 \\ \times\ 45 \\ \hline 60 \\ 48 \\ \hline 540 \end{array}$$

Multiplying,

$$\begin{array}{ccccc} & & x & + & 2 \\ \cdot & & 4x & + & 5 \\ \hline 4x^2 & + & 8x & & \\ & & 5x & + & 10 \\ \hline 4x^2 & + & 13x & + & 10. \end{array}$$

8.12 $120-24=96$

8.13 xyz

8.14 $7\cdot 5=35$

8.15 $668\cdot 669\cdot 670=299417640$.

8.16 $8x^5$.

8.17 $25\times 19=475$.

8.18 28.

8.19 108.

8.21 For $k\geq 10$, $k!$ ends in two or more 0's. Thus the last two digits of the desired sum are the last two digits of

$$1!+2!+3!+4!+5!+6!+7!+8!+9!=409113,$$

thus the desired sum ends in 13.

8.26 $(123^2\cdot 456^2-789)^0+3\cdot 2^3=1+3\cdot 2^3=1+3\cdot 8=1+24=25$.

8.27 We work backwards. He obtained 48 from $48+4=52$. This means that he should have performed $52\div 4=13$.

8.28 We work backwards as follows. We obtained 37 by adding 16 to $37-16=21$. We obtained this 21 by multiplying by 3 the number $21\div 3=7$. Thus the original number was a 7.

8.30 $5671-1920=3751$

8.31 Anna answered 20 questions correctly (She could not answer less than 20, because then her score would have been less than $19\cdot 4=76<77$; She could not answer more than 20, because her score would have been at least $21\cdot 4-3=81>77$). To get exactly 77 points Anna had to answer exactly 3 questions wrong, which means she omitted 2 questions.

8.32 From the distributive law,

$$\begin{aligned}(666)(222)+(1)(333)+(333)(222)&\\ +(666)(333)+(1)(445)+(333)(333)&\\ +(666)(445)+(333)(445)+(1)(222) &= (666+333+1)(445+333+222)\\ &= (1000)(1000)\\ &= 1000000.\end{aligned}$$

8.33 The trick is to use a technique analogous to the one for multiplying, but this time three-digits at a time:

$$321\cdot 654=209934,$$

$$321\cdot 987=316827,$$

$$745\cdot 654=487230,$$

$$745\cdot 987=735315.$$

Thus

$$\begin{array}{rrrr} & & \boxed{987} & \boxed{654} \\ \cdot & & \boxed{745} & \boxed{321} \\ \hline & & 209 & 934 \\ & 316 & 827 & \\ & 487 & 230 & \\ 735 & 315 & & \\ \hline 736 & 119 & 266 & 934 \end{array}$$

8.34 There are **28** digits, since

$$4^{16}5^{25} = 2^{32}5^{25} = 2^{7}2^{25}5^{25} = 128 \cdot 10^{25},$$

which is the **3** digits of **128** followed by **25** zeroes.

8.35 This is asking for the product $(10+11+\cdots+20)(21+22+\cdots+30)$ after all the terms are multiplied. But $10+11+\cdots+20 = 165$ and $21+22+\cdots+30 = 255$. Therefore we want $(165)(255) = 42075$.

9.1 $24 = 2^3 3$

9.2 $36 = 2^2 3^2$

9.3 Three. They are **2**, **11**, **101**.

9.4 Since $24 = 2^3 3$ and $36 = 2^2 3^2$, we have $\gcd(24,36) = 2^2 \cdot 3 = 12$ and $\text{LCM}(24,36) = 2^3 \cdot 3^2 = 72$, from where

$$\gcd(24,36) \cdot \text{LCM}(24,36) = 12 \cdot 72 = 12 \cdot 2 \cdot 36 = 24 \cdot 36.$$

9.5 $36 = 2^2 3^2$, so it has $(2+1)(2+1) = 9$ divisors.

9.6 $38 = 2^1 \cdot 19^1$, so it has $(1+1)(1+1) = 4$ divisors.

9.7 $40 = 2^3 5^1$, so it has $(3+1)(1+1) = 8$ divisors.

9.8 In some cases there may be more than one answer. For example, **20** is $7+13$ and also $3+17$, etc.

$8 = 3+5$	$10 = 3+7$	$12 = 5+7$	$14 = 3+11$	$16 = 5+11$
$18 = 7+11$	$20 = 3+17$	$22 = 3+19$	$24 = 7+17$	$26 = 3+23$
$28 = 11+17$	$30 = 13+17$	$32 = 13+19$	$34 = 11+23$	$36 = 13+23$

9.9 d cannot be even, as then the number would be divisible by **2**. Now, $1+9+7+0+0+0+1+9+d = 27+d$. If $d \in \{3,6,9\}$, then the sum of the digital sum would be divisible by **3**, and so the number would be divisible by **3**. If $d = 5$, the number would be divisible by **5**. If $d = 7$, the number would be divisible by **197**. This leaves $d = 1$ as the only possible digit, and so it must be this one.

9.10 The proper divisors of **496** are

$$\{1,2,4,8,16,31,62,124,248\}$$

and we have

$$1+2+4+8+16+31+62+124+248 = 496.$$

9.11 If p is a prime, then it has only **2** divisors, namely **1** and p.

9.12 We have,

$$\begin{aligned} 111111 &= (111)(1001) \\ &= (3)(37)(11)(91) \\ &= (3)(37)(11)(7)(13) \\ &= (3)(7)(11)(13)(37) \end{aligned}$$

9.13 $\{1, p, p^2\}$

9.14 Notice that consecutive integers are always relatively primes. Any 101 integers is bound to have the pair of one the one hundred sets

$$\{1,2\},\{3,4\},\ldots,\{197,198\},\{199,200\}.$$

9.15 Only perfect squares.

9.16 Number the steps from 0 to N, where N is the last step, and hence there are $N+1$ steps. Notice that the top and the bottom of the stairs are counted as steps. Let us determine first the number of steps. Frodo steps on steps 0, 2, 4, ..., N; Gimli steps on steps 0, 3, 6, ..., N; Legolas steps on steps 0, 4, 8, ..., N; and Aragorn steps on steps 0, 5, 10, ..., N. For this last step to be common, it must be a common multiple of 2, 3, 4, and 5, and hence

$$N = \text{LCM}(2,3,4,5) = 60.$$

Since every step that Legolas covers is also covered by Frodo, we don't consider Legolas in our count. Frodo steps alone on the steps which are multiples of 2, but not multiples of 3 or 5. Hence he steps on the 8 steps

$$2,14,22,26,34,38,46,58.$$

Gimli steps alone on the 8 steps that are multiples of 3 but not multiples of 2 nor 5:

$$3,9,21,27,33,39,51,57.$$

Finally, Aragorn steps alone on the 4 steps that are multiples of 5 but not multiples of 2 or 3:

$$5,25,35,55.$$

Our final count is thus $8+8+4=20$.

9.17 Notice that if d divides n so does $\frac{n}{d}$. Thus we can pair up every the different divisors of n, and have an even number of divisors as long as we do not have $d = \frac{n}{d}$. This means that the integers with an even number of divisors will have all doors open, and those with an odd number of divisors will all all doors closed. This last event happens when $d = \frac{n}{d} \implies n = d^2$, that is, when n is a square. Hence only the ten doors

$$1,4,9,16,25,36,49,64,81,100,$$

are closed.

9.18 Consider the set

$$\{2^n : n \in \mathbb{N}, n \text{ odd}\}.$$

10.1 9

10.2 $23 = 4 \cdot 5 + 3$ and so the quotient is 5 and the remainder 3.

10.3 $4 = 0 \cdot 23 + 4$ and so the quotient is 0 and the remainder 4.

10.4 Since $100 = 7 \cdot 14 + 2$, 98 days from today will be a Thursday, and 100 days from today will be a Saturday.

10.5 This problem is from [Gard]. At 'tailor.'

10.6 $6\frac{1}{4}$ meters per second.

10.7 $\$300$

10.8 Red.

10.9 If one puts Monday at the very beginning of the period one obtains $[\![\frac{45}{7}]\!] = 7$ Mondays.

10.10 There are 36 boys and 50 girls.

10.11 3

10.12 12

10.13 12

10.14 48

10.15 844

10.16 12

10.17 7

10.18 The sum of the digits is $3+2+a+2=7+a$. We want a to be one of $\{0,1,2,3,4,5,6,7,8,9\}$ and also we want $7+a$ to be a multiple of 9. This happens only when $a=2$.

10.19 The sum of the digits is $3+2+a+a+2=7+2a$. We want a to be one of $\{0,1,2,3,4,5,6,7,8,9\}$ and also we want $7+2a$ to be a multiple of 9. This happens only when $a=1$.

10.20 The first contractor binds $4500 \div 30 = 150$ books per day, and the second, $4500 \div 45 = 100$ books per day. Therefore, if they worked simultaneously, they could bind $150+100 = 250$ books per day, and it would take them $4500 \div 250 = 18$ days to bind all books.

10.21 Seven.

10.22 If $36 \div n$ is a natural number, then n must evenly divide 36, which means that n is a divisor of 36. Thus n can be any one of the values in $\{1,2,3,4,6,9,12,18,36\}$.

10.23 15

10.24 The least common multiple of $4,5$ and 6 is 60, hence we want the smallest positive multiple of 60 leaving remainder 1 upon division by 7. This is easily seen to be 120.

10.25 Least: 1. Largest: 3.

10.26 At 59 seconds.

10.27 Thirty one days. Explanation: consider just one lotus flower. On the second day it covers as much as two and thirty one days remain for it to cover the pond.

10.28 We can represent the boy's and girl's amounts by boxes, each box having an equal amount of nuts, and there being two boxes for the boy and one for the girl:

Then clearly, each box must have 8 nuts. The boy has 16 and the girl 8. This problem was taken from [Toom].

10.29 If $2x+1$ and $2y+1$ are the odd numbers, their sum is

$$2x+1+2y+1 = 2x+2y+2 = 2(x+y+1),$$

twice an integer, and hence even.

10.30 If $2x+1$ and $2y+1$ are the odd numbers, their product is

$$(2x+1)(2y+1) = (2x)(2y)+2x(1)+1(2y)+1(1) = 2xy+2x+2y+1 = 2(xy+x+y)+1,$$

twice an integer plus one, and hence odd.

10.31 If $n-1, n, n+1, n+2$ are four consecutive integers, then their sum is $4n+2$, that is, the integers sought do not leave remainder 2 upon division by 4. Since three quarters of the integers leave some other remainder, the answer is $\frac{3}{4} \cdot 100 = 75$.

10.32 By the Division Algorithm, any integer comes in one of two flavours: $2a$ or $2a+1$. Squaring,

$$(2a)^2 = 4a^2, \; (2a+1)^2 = 4(a^2+a)+1$$

and so the assertion follows.

10.33 The previous problem asserts that if a number leaves either remainder 2 or remainder 3 when divided by 4, then the number cannot be a square. Observe that

$$11 = 8+3, 111 = 100+8+3, 1111 = 1100+8+3, 11111 = 11100+8+3, \ldots$$

so every integer in the sequence leaves remainder 3 upon division by 4, and by the preceding problem, they cannot be squares.

10.34 Let A be the number of pairs of shoes Anacleta buys and let S be the number of pairs of shoes Sinforosa buys. Let P be the price, in dollars, of each pair of shoes. Observe that $P > 32$. Then $200 = AP + 32, 150 = SP + 24$, whence

$$AP = 168 = 2^3 \cdot 3 \cdot 7, SP = 126 = 2 \cdot 3^2 \cdot 7.$$

The price of each pair of shoes is a common divisor of 168 and 126, but it must exceed 32. The only such divisor is 42. Hence the shoes cost $\$42$.

10.35 For a number to be divisible by 8, the number formed by its last three digits must the divisible by 8. The largest number that can be formed with all ten digits is 9876543210, but this is not divisible by 8. A permutation of the last three digits to 9876543120 yields a number divisible by 8, and the largest such.

10.36 The integer must be of the form $n = 2^a 3^b 5^c$, with a, b, c positive integers. The conditions imply that $a-1$ is even and that a is divisible by 15; that $b-1$ is divisible by 3 and that b is divisible by 10; and that $c-1$ is divisible by 5 and c divisible by 6. Clearly, the smallest positive integers satisfying those conditions are $a = 15, b = 10, c = 6$. Hence the integer sought is $n = 2^{15} 3^{10} 5^6 = 30233088000000$.

11.1

```
  1 0 0 2 0 0 3 0 0 4 0 0 | 2 5
- 1 0 0                   |--------------------
  -----                   | 4 0 0 8 0 1 2 0 1 6
    0 2
  -   0
    ---
      2 0
    -   0
      ---
      2 0 0
    - 2 0 0
      -----
          0 3
        -   0
          ---
            3 0
          - 2 5
            ---
              5 0
            - 5 0
              ---
                0 4
              -   0
                ---
                  4 0
                - 2 5
                  ---
                  1 5 0
                - 1 5 0
                  -----
                      0
```

11.2

```
         1001
    ________
123) 123123
     123000
     ------
        123
        123
        ---
          0
```

11.3

```
        123
1001)123123
     100100
      23023
      20020
       3003
       3003
          0
```

11.4

```
     13717421
90)1234567890
    900000000
    334567890
    270000000
     64567890
     63000000
      1567890
       900000
       667890
       630000
        37890
        36000
         1890
         1800
           90
           90
            0
```

11.5

```
  1123468
9)10111213
   9000000
   1111213
    900000
    211213
    180000
     31213
     27000
      4213
      3600
       613
       540
        73
        72
         1
```

11.6

$$
\begin{array}{r}
\mathbf{722229} \\
14 \overline{)\,\mathbf{10111213}} \\
\underline{\mathbf{9800000}} \\
311213 \\
\underline{\mathbf{280000}} \\
31213 \\
\underline{\mathbf{28000}} \\
3213 \\
\underline{\mathbf{2800}} \\
413 \\
\underline{\mathbf{280}} \\
133 \\
\underline{\mathbf{126}} \\
7
\end{array}
$$

11.7 a^6

11.8 a^2x^5

11.9 $10x^3 + 3x^6$

11.10 $1 + x$

11.11 $\frac{4096}{625}$

11.12 $\frac{100^3 + 100^3 + 100^3 + 100^3 + 100^3 + 100^3}{100^2 + 100^2 + 100^2} = \frac{6 \cdot 100^3}{3 \cdot 100^2} = 2 \cdot 100 = 200.$

11.13 $\frac{4}{625}$

11.14 The product equals

$$\frac{4(4^5)}{3(3^5)} \cdot \frac{6(6^5)}{2(2^5)} = 4(4^5) = 2^{12},$$

so $n = 12$.

11.15 We have

$$3^{2001} + 3^{2002} + 3^{2003} = 3^{2001}(1 + 3 + 3^2) = (13)3^{2001},$$

whence $a = 13$.

12.1 7

12.2 8

12.3 3

12.4 $9 + 10 = 19$

12.5 7

12.6 2001

12.7 Observe that $\sqrt{2003} > 44$. We see that $44^2 = 1936$ and $45^2 = 2025$. Thus the closest square is 2025.

12.8 54.

12.9 Since there are 10 squares between 1 and 110 inclusive 110 is the 100-th term.

12.11 Let the integers be $n - 2, n, n + 2$. Forcedly, they must end in $2, 4, 6$, since their product ends in 8. The product of the integers is $n^3 - 4n \approx n^3$. Now $n^3 \approx 87000008 \implies n \approx 443$. Inspection gives $n = 444$, and the sum is $442 + 444 + 446 = 1332$.

13.4 $\frac{3}{5} = \frac{3 \cdot 5}{10 \cdot 5} = \frac{15}{50}$

13.6 $\frac{36}{60} = \frac{3}{5}$ of an hour.

13.7 $\frac{2}{2+3+6} = \frac{2}{11}$.

13.8 $\frac{17}{35}$.

13.9 Observe that $3990 = 210 \cdot 19$. Thus $\frac{102}{210} = \frac{102 \cdot 19}{210 \cdot 19} = \frac{1938}{3990}$.

13.10 Observe that $3 \cdot 5 \cdot 13 = 195$ is a common denominator. Now

$$\frac{2}{3} = \frac{130}{195}; \quad \frac{3}{5} = \frac{117}{195}; \quad \frac{8}{13} = \frac{120}{195},$$

whence $\frac{3}{5}$ is the smallest, $\frac{8}{13}$ is in the middle, and $\frac{2}{3}$ is the largest.

13.11 $\frac{116690151}{427863887} = \frac{3 \cdot 38896717}{11 \cdot 38896717} = \frac{3}{11}$.

13.12 $1\frac{6}{7}$

13.13 $4\frac{3}{5}$

13.14 $\frac{10}{3}$

13.15 $\frac{18}{7}$

13.16 Three lines have been sung before the fourth singer starts, and after that he sings 12 more lines. So the total span of lines is 15. They start singing simultaneously from line 4, and the first singer is the first to end, in line 12. Thus $12 - 4 + 1 = 9$ lines out of 15 are sung simultaneously and the fraction sought is $\frac{9}{15} = \frac{3}{5}$.

14.1 $1 - \frac{1}{3} - \frac{1}{6} = \frac{1}{2}$.

14.2 $\frac{12}{35}$

14.3 $\frac{2}{7}$

14.4 $\frac{9}{4}$

14.5 $\frac{10}{3}$

14.6 $\frac{8}{3}$

14.7 $\frac{58}{9}$

14.8 $\frac{2}{3}$

15.1 This is $\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{5} \cdot \frac{1}{4} \cdot 99000 = 825$.

15.2 4

15.3 \$150

15.4 Here are the solutions.

$\frac{1}{3}$	$+$	$\frac{5}{3}$	$=$	2
$+$		$-$		
$\frac{3}{4}$	$\cdot$	1	$=$	$\frac{3}{4}$
$=$		$=$		
$\frac{13}{12}$	$\div$	$\frac{2}{3}$	$=$	$\frac{13}{8}$

15.5 On the first day he ate $\frac{2}{5}\cdot 420 = 168$ cookies, so there remain $420 - 168 = 252$ cookies after the first day. On the second day he ate $\frac{5}{6}\cdot 252 = 210$ cookies, so there remain $252 - 210 = 42$ cookies to be eaten on the third day. Answer: **42** cookies.

15.6 $\frac{5}{2}\cdot 20 = 50$ dollars.

15.7 Anna gets $\frac{9}{15}\cdot 90 = 54$ dollars and Tina $\frac{6}{15}\cdot 9036$ dollars.

15.8 $29\frac{1}{3}$ seconds.

15.9 $\frac{7}{45}$.

15.10 $\frac{3}{17}$.

15.11 We have,

$$\frac{10+10^2}{\frac{1}{10}+\frac{1}{100}} = \frac{10^3+10^4}{10+1} = \frac{11000}{11} = 1000.$$

15.12 1.

15.13 We have,

$$\frac{1}{1+\frac{1}{5}} = \frac{1}{\frac{6}{5}} = \frac{5}{6} = \frac{a}{b},$$

whence $a^2 + b^2 = 5^2 + 6^2 = 61$.

15.14 $\frac{1}{100}$.

15.15 $25 - (2\frac{1}{2})\cdot(5\frac{1}{2}) = \frac{45}{4} = 11\frac{1}{4}$ gallons.

15.16 We have $1\frac{7}{8} = \frac{15}{8}$ and

$$16 \div \frac{15}{8} = \frac{16}{1}\cdot\frac{8}{15} = \frac{128}{15} = 8\frac{8}{15},$$

so she is able to wrap **8** gifts.

15.17 Proceeding from the innermost fraction one easily sees that

$$\frac{1}{2-\frac{1}{2-\frac{1}{2-\frac{1}{2}}}} = \frac{1}{2-\frac{1}{2-\frac{2}{3}}} = \frac{1}{2-\frac{3}{4}} = \frac{4}{5}.$$

15.18 $\frac{43}{30}$

15.19 Measure angles clockwise, with origin $(0°)$, at 12:00. Each minute ran by the minute hands accounts for $\frac{360°}{60} = 6°$, and so, each, when jumping from hour to hour, the hands travel $5(6°) = 30°$. When the minute hand is on the 8, it has travelled $\frac{40}{60} = \frac{2}{3}$ of a circumference, that is

$$\frac{2}{3}360° = 240°$$

and the hour hand has moved $\frac{2}{3}$ of the way from 4 to 5, travelling

$$\left(4 + \frac{2}{3}\right)(30°) = 140°.$$

Hence, angle between the hands is $240° - 140° = 100°$.

16.1 She will take 19200 minutes.

16.2 9 minutes.

16.3 90 calories.

16.4 28

16.5 The denominator is $4 + 7 + 2 = 13$. One uses $\frac{4}{13} \cdot 195 = 60$ lbs of Brazilian, $\frac{7}{13} \cdot 195 = 105$ lbs of Colombian, and $\frac{2}{13} \cdot 195 = 30$ lbs of Kenyan.

16.6 We have $\left(\frac{30 \text{ miles}}{1\frac{1}{5} \text{ inches}}\right)(4 \text{ inches}) = 100$. City A is 100 miles apart from City B.

16.7 She will need 12 fluid ounces of beer.

16.8 Travelling $30{,}000$ miles with 4 tyres is as travelling $120{,}000$ miles on one tyre. The average wear of each of the 5 tyres is thus $120000 \div 5 = 24000$ miles.

16.9 350 meters.

16.10 Banton finishes first. Anton and Canton take the same amount of time.

17.1 $\frac{51}{250}$

17.2 $0.\overline{81}$

17.3 0.15

17.4 0.015

17.5 $0.4\overline{6}$

17.6 $0.\overline{142857}$

17.7 $24\frac{6}{25}$

17.8 12%

17.9 1.2%

17.10 $0.4\overline{4} < 0.445 < 0.4\overline{45} < 0.446 < 0.45$

17.11 $0.666 < \frac{2}{3} < 0.67 < \frac{3}{4}$

17.12 7

17.13 $0.2 + \frac{1}{4} = \frac{2}{10} + \frac{1}{4} = \frac{4}{20} + \frac{5}{20} = \frac{9}{20}$

17.14 $\left(\frac{2}{5}\right)(3.3) = \left(\frac{2}{5}\right)\left(3\frac{3}{10}\right) = \left(\frac{2}{5}\right)\left(\frac{33}{10}\right) = \frac{66}{50} = \frac{33}{25}$

17.15 You need to increase **$20**, and **$20** is **25%** of **$80**, and so you need to increase by **25%**.

18.1 $38.16

18.2 2.21

18.3 0.93

18.4 0.0000252

18.5 0.0000000024

18.6 0.13

18.7 $4.60

18.8 243

18.9 3000

18.10 $\frac{25}{3}\%$

18.11 $126.

18.12 $39.33.

18.13 The area of the apartment is $468 \div 0.6 = 780$ sq feet. The area of the bedroom is $0.4 \cdot 468 = 187.2$ sq feet. The area of the kitchen is thus $780 - 468 - 187.2 = 124.8$ sq feet.

18.15 Of the **100** students, only one is male. He is **2%** of the on-campus population. Thus the whole on-campus population consists of **50** students, so there are $100 - 50 = 50$ off-campus students.

18.16 30

18.17 **95%** water, **5%** pulp.

18.18 Observe that $5\frac{1}{2} = \frac{11}{2} = \frac{22}{4}$. Hence $\frac{21}{4}$ miles are at the rate of **40¢**. The trip costs $\$0.85 + \$.40 \cdot 21 = \$.85 + \$8.40 = \$9.25$.

18.19 One has $a = 1.5c$ and $b = 1.25c$ Thus $a = \frac{1.5b}{1.25} = 1.2b$. Therefore a is **20%** larger than b.

19.1 1.6

19.2 205

19.3 6.82

19.4 $1.13\overline{6}$

19.5 $1.\overline{142857}$

19.6 $1.\overline{09}$

19.7 0.5

19.8 0.5

19.9 $10.\overline{10}$

19.10 $\frac{530}{203}$

19.11 1.25

19.12 **23¢**

19.13 \$400.50

19.14 $\frac{6.5}{500}$ grams, that is, **0.013** grams.

19.15 There is more than one possible answer in some cases.

1 is 4 4 ÷ 4 4 =	11 is 4 ÷ . 4 + 4 ÷ 4 =
2 is 4 ÷ 4 + 4 ÷ 4 =	12 is (4 4 + 4) ÷ 4 =
3 is (4 + 4 + 4) ÷ 4 =	13 is 4 ! − 4 4 ÷ 4 =
4 is 4 + 4 × (4 − 4) =	14 is 4 × (4 − . 4) − . 4 =
5 is (4 × 4 + 4) ÷ 4 =	15 is 4 4 ÷ 4 + 4 =
6 is (4 + 4) ÷ 4 + 4 =	16 is 4 × 4 + 4 − 4 =
7 is 4 4 ÷ 4 − 4 =	17 is 4 × 4 + 4 ÷ 4 =
8 is 4 × 4 − 4 − 4 =	18 is 4 ÷ . 4 + 4 + 4 =
9 is 4 ÷ . 4 − 4 ÷ 4 =	19 is 4 ! − 4 − 4 ÷ 4 =
10 is (4 4 − 4) ÷ 4 =	20 is 4 ÷ . 4 + 4 ÷ . 4 =

20.1

1. **+8**
2. **−25**
3. **−25**
4. **0**
5. **−303**
6. **0**
7. **+4**
8. **−100**

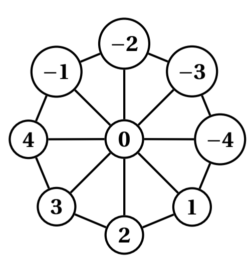
20.2 **−5**

20.3 **−5**

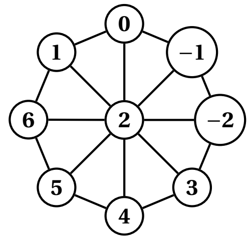
20.4 **−1**

20.5 **3*a***

20.6 Here is one possible answer.



20.7 Here is one possible answer.



21.1 $-\frac{1}{2}$.

21.2 171

21.3 $(2\cdot(-3)+1)^2-((-5)\cdot 4+2\cdot 9)^3=(-6+1)^2-(-20+18)^3=(-5)^2-(-2)^3=25-(-8)=33.$

21.4 $(2\cdot(-3)+1)^2-((-5)\cdot 4+2\cdot 9)^3=(-6+1)^2-(-20+18)^3=(-5)^2-(-2)^3=25-(-8)=33.$

21.5 $(2+3+4)(2^2+3^2+4^2-2\cdot 3-3\cdot 4-4\cdot 2)=(9)(4+9+16-6-12-8)=(9)(3)=27$

21.6 Here are two answers.

−1	·	+1	=	−1
·		·		·
−1	·	−1	=	+1
=		=		=
+1	·	−1	=	−1

−1	·	−1	=	+1
·		·		·
+1	·	−1	=	−1
=		=		=
−1	·	+1	=	−1

22.1 $\frac{5}{36}$

22.2 $-\frac{1}{2}$

22.3 -0.02895

22.4 $-0.25=-\frac{1}{4}$

22.5 $0.1111\ldots=0.\overline{1}=\frac{1}{3}(0.333333\ldots)=\frac{1}{3}\cdot\frac{1}{3}=\frac{1}{9}$.

22.6 From the text, we know that $0.\overline{09}=\frac{1}{11}$, and hence $121(0.\overline{09})=121\cdot\frac{1}{11}=11$.

22.7 $\frac{1411}{4214}$

23.1 $\sqrt{2}+\sqrt{3}+\sqrt{5}\approx 5.38$

23.2 $\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{5}\approx 5.48$

23.3 9

23.4 No, this is not always true. For example, $\sqrt{1}+\sqrt{3}=1+\sqrt{3}\approx 2.73$, but $\sqrt{1+3}=\sqrt{4}=2$. But, is it *ever* true? For $a=0$, $\sqrt{a}+\sqrt{b}=\sqrt{0}+\sqrt{b}=\sqrt{b}$ and $\sqrt{a+b}=\sqrt{0+b}=\sqrt{b}$, so it is true in this occasion.

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-Maldición anónima contra los ladrones de libros en el monasterio de San Pedro, Barcelona.

Updates

The most current version of this book (both the .pdf and the .tex source) can be found here:

`faculty.ccp.edu/faculty/dsantos/lecture_notes.html`

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